MATH 6100 - HOMEWORK ASSIGNMENT 1

DUE TUESDAY, SEPTEMBER 12 BY MIDNIGHT

Exercise 0.1. Let $2^{<\mathbb{N}}$ denote the set of finite sequences in $\{0,1\}$. Prove that $2^{<\mathbb{N}}$ is countable.

Exercise 0.2. Prove that $|\mathbb{R}^{\mathbb{N}}| = |\mathbb{R}|$.

Exercise 0.3. Let X be a countably infinite set. Find an uncountable family $\mathcal{F} \subset 2^X$ so that for any distinct pair $A, B \in \mathcal{F}$ we have that $A \cap B$ is finite. (Hint: It may help to consider the case $X = 2^{<\mathbb{N}}$.)

A complex number α is **algebraic** if it is the solution of a polynomial having integer coefficients. For the following problem you may use without proof the result from algebra that states that a polynomial of degree n can have at most n distinct solutions.

Exercise 0.4. Prove that there exists a non-algebraic real number.

Let X be a set, and $\leq$ be a linear ordering on X. We say that the linear order is **dense** if for all $x < y$ there exists $z \in X$ such that $x < z < y$.

Exercise 0.5 (Cantor's back-and-forth method)**.** Let $(X, \leq)$ and $(Y, \leq)$ be countable dense linear orderings that do not have upper or lower bounds. Enumerate $X = \{x_1, x_2, \ldots\}$, and $Y = \{y_1, y_2, \ldots\}$.

(1) There exist increasing sequences of finite sets $A_n \subset X$, $B_n \subset Y$, and order preserving bijections $f_n : A_n \to B_n$ such that $x_n \in A_n$, $y_n \in B_n$, and $f_{n+1|A_n} = f_n$, for all $n \geq 1$.
(2) There exists an order preserving bijection $f : X \to Y$.

Exercise 0.6. A countable group Γ is called (left) orderable if there exists a linear order $\prec$ on Γ so that left multiplication is order-preserving, i.e., for all $a, b, \gamma \in \Gamma$ we have $a \prec b$ if and only if $\gamma a \prec \gamma b$. Prove that a countable group is orderable if and only if it admits a free action on $\mathbb{Q}$ that preserves the usual order. Optional bonus: Prove that a countable group is orderable if and only if it admits a faithful action on $\mathbb{Q}$ that preserves the usual order.

Exercise 0.7. Let $A \subset \mathbb{R}$ be an uncoutable set. Recall that a point $x \in \mathbb{R}$ is called a cluster point of A if $(x - \varepsilon, x + \varepsilon) \cap A$ is infinite for all $\varepsilon > 0$. Prove that the set of cluster points of A has the cardinality of the continuum. Hint: Prove that there exist disjoint open intervals O_0, O_1 of length 1 such that $O_i \cap A$ is uncountable for each $i = 0, 1$. Now repeat this to produce nested open sets indexed by $2^{<\mathbb{N}}$ and having shrinking diameters. Use this to produce an injection from $2^{\mathbb{N}}$ to the set of accumulation points of A.

Exercise 0.8. Prove that there are continuum many closed subsets of $\mathbb{R}$.

Date: September 10, 2023.

Exercise 0.9. Prove that there exists a subset $E \subset \mathbb{R}$ with the property that for any uncountable closed subset $F \subset \mathbb{R}$ we have $F \cap E \neq \emptyset$ and $(\mathbb{R} \setminus E) \cap F \neq \emptyset$. Hint: Well-order the set of uncountable closed subsets and use induction to collect pairs of elements, each pair having one element coming from $F \cap E$, and the other element coming from $(\mathbb{R} \setminus E) \cap F$.