COMPACT OPERATORS ON BERGMAN SPACES

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ABSTRACT. We prove that a bounded operator S on L_a^p for $p > 1$ is compact if and only if the Berezin transform of S vanishes on the boundary of the unit disk if S satisfies some integrable conditions. Some estimates about the norm and essential norm of Toeplitz operators with symbols in BT are obtained.

1. INTRODUCTION

Let dA denote the normalized Lebesgue area measure on the unit disk D. For $0 < p \leq \infty$, let L^p denote $L^p(D, dA)$ and let $\|u\|_p$ denote the usual L^p norm of u in L^p. The Bergman space L_a^p with $1 \leq p < \infty$ is the Banach space consisting of all analytic functions on D that are also in L^p.

Let P be the projection from L^2 onto its closed subspace L_a^2. P is an integral operator represented by

$$P(h)(z) = \int_D \frac{h(w)}{(1 - z\bar{w})^2} dA(w),$$

for each $z \in D$ and $h \in L^2$. For $f \in L^1$, the Toeplitz operator with symbol f is defined by

$$T_f u(z) = P(fu)(z) = \int_D \frac{f(w)u(w)}{(1 - z\bar{w})^2} dA(w),$$

for any bounded analytic function u on D. Clearly, T_f is densely defined on L_a^p.

For $z \in D$, let φ_z be the analytic map of D onto D defined by

$$\varphi_z(w) = \frac{z - w}{1 - \bar{z}w}.$$

For $z \in D$, let U_z be the operator defined by $U_z f = (f \circ \varphi_z)\varphi_z'$. Clearly, U_z is a unitary operator on L_a^2 and a bounded operator on L_a^p for $p > 1$. For S a bounded operator on L_a^p, define S_z by $S_z = U_z S U_z$. Let $\|S\|_p$ denote the operator norm on L_a^p.

For $z \in D$, let $K_z \in L_a^2$ denote the Bergman reproducing kernel of L_a^2. As is well known,

$$K_z(w) = \frac{1}{(1 - \bar{z}w)^2}.$$

Let k_z denote the normalized reproducing kernel. Thus $k_z = (1 - |z|^2)K_z$ is also in L_a^p for $p \geq 1$. For S a bounded operator on L_a^p for $1 < p < \infty$, the

Berezin transform of S is the function $\tilde{S}$ on D defined by

$$\tilde{S}(z) = \langle Sk_z, k_z \rangle,$$

where

$$\langle u, v \rangle = \int_D u\bar{v}\, dA$$

whenever $u\bar{v} \in L^1$. Let $\tilde{f}$ denote $\widetilde{T_f}$ and let

$$\mathrm{BT} = \{f \in L^1 : \|f\|_{\mathrm{BT}} = \sup_{z\in D} \widetilde{|f|}(z) < \infty\}.$$

On the Hardy space, bounded Toeplitz operators arise from bounded symbols and there are no nontrivial compact Toeplitz operators [5]. In the Bergman space setting, however, there are lots of nontrivial compact Toeplitz operators [12]. In fact, Sarason [12] first constructed a nonzero compact Toeplitz operator T_f such that $f^2 = 1$. Some unbounded symbols induce bounded Toeplitz operators and even compact Toeplitz operators. The problem to determine when a Toeplitz operator is bounded on the Bergman spaces is still open. Axler and the second author [3] showed that a Toeplitz operator with bounded symbol is compact on the Bergman space L^2_a if and only if the Berezin transform of the symbol vanishes on the boundary of the unit disk. Moreover they showed that if S equals a finite sum of finite products of Toeplitz operators with bounded symbols, then S is compact on L^2_a if and only if $\tilde{S}(z) \to 0$ as $z \to \partial D$.

A common intuition is that for operators on the Bergman spaces "closely associated with function theory", compactness is equivalent to having vanishing Berezin transform on the boundary of the unit circle. Our main results will show that this intuition is correct if "closely associated with function theory" is interpreted to integrable conditions on those operators (Theorem 1). Moreover, we will show that the integrable conditions are sharp by examples on the Bergman space L^2_a. As a consequence, we will show that if on the Bergman space L^p_a for $p > 1$, an operator equals a finite sum of finite products of Toeplitz operators with symbols in BT, the operator is compact if and only if the Berezin transform of the operator vanishes on the boundary of the unit disk (Theorem 3). Some estimates about the norm and essential norm of Toeplitz operators with symbols in BT are obtained.

Throughout the paper we use p' to denote the conjugate of p, i.e. $(1/p) + (1/p') = 1$, for $1 < p < \infty$, and use p_1 to denote $\min\{p, p'\}$. The main results of the paper are stated as follows.

Theorem 1. *Suppose $1 < p < \infty$ and S is a bounded operator on L^p_a such that*

$$\sup_{z\in D} \|S_z 1\|_m < \infty, \ \sup_{z\in D} \|S^*_z 1\|_m < \infty$$

for some $m > 3/(p_1 - 1)$. Then S is compact on L^p_a if and only if $\tilde{S}(z) \to 0$ as $z \to \partial D$.

In contrast to Lemma 3.2 of [3], we state a special case of Theorem 1 as the following theorem. We will show that the number 3 in Theorem 2 can not be further reduced in general in Section 3.

Theorem 2. *Suppose S is a bounded operator on L_a^2 such that*

$$\sup_{z\in D}\|S_z 1\|_m < \infty,\ \sup_{z\in D}\|S_z^* 1\|_m < \infty$$

for some $m > 3$. Then S is compact on L_a^2 if and only if $\tilde{S}(z) \to 0$ as $z \to \partial D$.

In this paper, we will show that if f is in BT, then T_f is bounded on the Bergman spaces L_a^p for $p \in (1, \infty)$. The following theorem, which will be shown later as an easy consequence of Theorem 1, extends the main result of [3], where p is assumed to be 2 and all symbols are assumed to be in L^∞. We will provide a concrete example to show that L^∞ is properly contained in BT in Section 3 for reader's convenience.

Theorem 3. *Suppose $1 < p < \infty$ and suppose S is a finite sum of operators of the form $T_{f_1} \cdots T_{f_n}$, where each $f_j \in$ BT. Then S is compact on L_a^p if and only if $\tilde{S}(z) \to 0$ as $z \to \partial D$.*

In particular, for $f \in$ BT, T_f is compact on L_a^2 if and only if the Berezin transform of f vanishes on the unit circle ∂D. In [17] it was obtained that if $f \in \mathrm{BMO}^1$, i.e.,

$$\sup_z \widetilde{|f - \tilde{f}(z)|}(z) < \infty,$$

then T_f is compact on the Bergman space L_a^2 if and only if $\tilde{f}(z)$ vanishes on the unit circle. From the above definition of BMO^1, it is clear that if f is in BMO^1 and $\tilde{f}$ is in L^∞, then f is in BT.

2. Carleson measures and the Berezin transform

The Berezin transform of a bounded operator on the Bergman space L_a^2 contains a lot of information about the operator. It is one of the most useful tools in the study of Toeplitz operators. Another useful tool is Carleson measures on Bergman spaces. The characterization of boundedness and compactness of a positive Toeplitz operator on the Bergman spaces appears in terms of Carleson measures first in [10] and in terms of the Berezin transform first in [16]. For more about Carleson measures, see [2], [8], and [16].

For z, $w \in D$, the distance in the Bergman metric on the unit disk is given by

$$\beta(z, w) = \frac{1}{2}\log\frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}.$$

Let $D(z)$ denote the Bergman metric disk with center z and radius $\frac{1}{2}$. Thus

$$D(z) = \{w \in D : \beta(w, z) < 1/2\}.$$

For $d\mu$ a positive Borel measure on D, let

$$\tilde{\mu}(z) = \int_D |k_z(w)|^2 \, d\mu(w)$$

denote the Berezin transform of $d\mu$. For $\zeta \in \partial D$ and $r \in [0, 1)$, let

$$S(\zeta, r) = \left\{ z \in D : r < |z| < 1, \arg \zeta - \frac{1-r}{2} < \arg z < \arg \zeta + \frac{1-r}{2} \right\}$$

denote the Carleson square.

Throughout the paper we say that two nonnegative quantities Q_1 and Q_2 are equivalent if there are positive constants C_1 and C_2 independent of variables under consideration such that

$$C_1 Q_1 \leq Q_2 \leq C_2 Q_1.$$

We use C to denote a positive constant whose value may change from line to line, but does not depend on variables under consideration.

The following result is well known. See [9] and [16] for example.

Lemma 1. *Suppose $d\mu$ is a positive Borel measure on D and $1 \leq p < \infty$. Then the following four quantities are equivalent:*
(a) $\sup\{\int_D |f|^p \, d\mu / \int_D |f|^p \, dA : f \in L^p_a\}$*;*
(b) $\sup\{\mu(D(z))/A(D(z)) : z \in D\}$*;*
(c) $\sup\{\mu(S(\zeta, r))/A(S(\zeta, r)) : \zeta \in \partial D, r \in [0, 1)\}$*;*
(d) $\sup\{\tilde{\mu}(z) : z \in D\}$*.*
Furthermore, the constants of equivalence depend only on p.

A positive Borel measure $d\mu$ is called a Carleson measure on D if one of (a), (b), (c), and (d) in Lemma 1 is finite.

Lemma 1 implies the following result.

Lemma 2. *Suppose $f \in L^1$. Then $f \in \mathrm{BT}$ if and only if $|f| \, dA$ is a Carleson measure on D.*

Lemma 3. *Suppose $1 < p < \infty$ and $f \in \mathrm{BT}$. Then T_f is bounded on L^p_a and there is a constant C such that $\|T_f\|_p \leq C \|f\|_{\mathrm{BT}}$.*

Proof. It is well known that the dual of L^p_a is $L^{p'}_a$ (see [2]). For $u \in L^p_a$ and $v \in L^{p'}_a$, by Hölder's inequality

$$\begin{aligned} |\langle T_f u, v \rangle| &= |\langle f u, v \rangle| \\ &\leq \int_D |f||u||v| \, dA \\ &\leq \left(\int_D |u|^p |f| \, dA \right)^{1/p} \left(\int_D |v|^{p'} |f| \, dA \right)^{1/p'}. \end{aligned}$$

Thus Lemmas 1 and 2 give

$$|\langle T_f u, v \rangle| \leq C \|f\|_{\mathrm{BT}} \|u\|_p \|v\|_{p'}.$$

This shows that T_f is bounded on L^p_a and $\|T_f\|_p \leq C \|f\|_{\mathrm{BT}}$. $\square$

The following lemma is Proposition 6.1.8 of [15].

Lemma 4. *Suppose $f \in L^1$ and $z \in D$. Then $\widetilde{f \circ \varphi_z} = \tilde{f} \circ \varphi_z$.*

Lemma 5. *Suppose $1 < p < \infty$ and $z \in D$ and suppose $f \in \mathrm{BT}$. Then $T_{f\circ\varphi_z}$ is bounded on L^p_a and there is a constant C independent of z such that $\|T_{f\circ\varphi_z}\|_p \le C\|f\|_{\mathrm{BT}}$.*

Proof. According to Lemma 3, $\|T_{f\circ\varphi_z}\|_p \le C\|f \circ \varphi_z\|_{\mathrm{BT}}$. By Lemma 4

$$\|f \circ \varphi_z\|_{\mathrm{BT}} = \sup_{w\in D} \widetilde{|f \circ \varphi_z|}(w) = \sup_{w\in D} \widetilde{|f|}(\varphi_z(w)) = \|f\|_{\mathrm{BT}}.$$

This finishes the proof of the lemma. □

Lemma 6. *If S is a finite sum of operators of the form $T_{f_1} \cdots T_{f_n}$, where each $f_j \in \mathrm{BT}$, then*

$$\sup_{z\in D} \|S_z 1\|_p < \infty, \ \sup_{z\in D} \|S_z^* 1\|_p < \infty$$

for every $p \in (1, \infty)$.

Proof. Without loss of generality we may assume that $S = T_{f_1} \cdots T_{f_n}$. For $p \in (1, \infty)$, by Lemma 5

$$\|S_z 1\|_p = \|T_{f_1\circ\varphi_z} \cdots T_{f_n\circ\varphi_z} 1\|_p \le C\|f_1\|_{\mathrm{BT}} \cdots \|f_n\|_{\mathrm{BT}}.$$

Clearly each $\bar{f}_j \in \mathrm{BT}$ and $\|\bar{f}_j\|_{\mathrm{BT}} = \|f_j\|_{\mathrm{BT}}$. Thus

$$\|S_z^* 1\|_p = \|T_{\bar{f}_n\circ\varphi_z} \cdots T_{\bar{f}_1\circ\varphi_z} 1\|_p \le C\|f_1\|_{\mathrm{BT}} \cdots \|f_n\|_{\mathrm{BT}}.$$

This finishes the proof of the lemma. □

3. Examples

In this section we will give two concrete examples. The first one will show that L^∞ is properly contained in BT. The second one is more interesting and will show that the hypothesis of Theorem 2 is in a way optimal.

Example 1. We can use a radial function $f(z) = f(|z|)$ for $z \in D$. For $x \in [0, 1)$, $x \in [1 - 1/2^{k-1}, 1 - 1/2^k)$ for some $k = 1, 2, \cdots$, define

$$f(x) = \begin{cases} 2^k, & \text{if } 1 - 1/2^{k-1} \le x \le 1 - 1/2^{k-1} + (1/2^k)^2; \\ 0, & \text{otherwise.} \end{cases}$$

Clearly f is not in L^∞. To show that $f \in \mathrm{BT}$, we will use Lemma 1 (c) and Lemma 2. For $\zeta \in \partial D$ and $r \in [0, 1)$, it is easy to see that

$$A(S(\zeta, r)) = \frac{1}{\pi} \int_r^1 s\, ds \int_{-(1-r)/2}^{(1-r)/2} d\theta \ge \frac{(1-r)^2}{2\pi}.$$

Thus

$$\frac{1}{A(S(\zeta, r))} \int_{S(\zeta,r)} f(z)\, dA(z) \le \frac{2}{1-r} \int_r^1 f(s)\, ds.$$

For $r \in [0,1)$, assume $1 - 1/2^{n-1} \le r < 1 - 1/2^n$ for some $n = 1, 2, \cdots$. Thus

$$\begin{aligned} \int_r^1 f(s)\,ds &\le \int_{1-1/2^{n-1}}^1 f(s)\,ds \\ &= \sum_{k=n}^{\infty} \int_{1-1/2^{k-1}}^{1-1/2^k} f(s)\,ds \\ &= \sum_{k=n}^{\infty} \frac{1}{2^k} = \frac{2}{2^n}. \end{aligned}$$

Therefore

$$\frac{2}{1-r}\int_r^1 f(s)\,ds \le 2^{n+1}\frac{2}{2^n} = 4,$$

showing that $f\,dA$ is a Carleson measure, and hence $f \in \mathrm{BT}$.

Example 2. This example shows that the number 3 in Theorem 2 is sharp. We show that there is a bounded operator S on L_a^2 such that

$$\sup_{z\in D} \|S_z 1\|_3 < \infty, \ \sup_{z\in D} \|S_z^* 1\|_3 < \infty,$$

and $\tilde{S}(z) \to 0$ as $z \to \partial D$, but S is not compact on L_a^2.

The following operator S was constructed in [3] to show that $\tilde{S}(z) \to 0$ as $z \to \partial D$, but S is not compact on L_a^2. Let S be defined on L_a^2 by

$$S\left(\sum_{n=0}^{\infty} a_n w^n\right) = \sum_{n=0}^{\infty} a_{2^n} w^{2^n}.$$

It is clear that S is a self-adjoint projection with infinite-dimensional range. Thus S is not compact on L_a^2. From

$$\begin{aligned} \tilde{S}(z) &= \langle S k_z, k_z\rangle \\ &= \|S k_z\|_2^2 \\ &= (1-|z|^2)^2 \sum_{n=0}^{\infty} (2^n+1)(|z|^2)^{2^n}, \end{aligned}$$

it is easy to see that $\tilde{S}(z) \to 0$ as $z \to \partial D$.

In order to show that

$$\sup_{z\in D} \|S_z 1\|_3 \ < \infty,$$

we need the following well-known result due to Zygmund [18].

Lemma 7. *Suppose $0 < p < \infty$ and $z = re^{i\theta}$ with $r = |z|$. Then the following two quantities are equivalent:*

(a) $\left(\int_0^{2\pi} \left|\sum_{n=0}^{\infty} a_n z^{2^n}\right|^p d\theta\right)^{1/p}$;

(b) $\left(\sum_{n=0}^{\infty} |a_n|^2 r^{2^{n+1}}\right)^{1/2}$.

Furthermore, the constants of equivalence depend only on p.

For $z \in D$, it is easy to see that

$$(U_z 1)(w) = (|z|^2 - 1) \sum_{n=0}^{\infty} (n+1)(\bar{z}w)^n.$$

Thus

$$(SU_z 1)(w) = (|z|^2 - 1) \sum_{n=0}^{\infty} (2^n + 1)(\bar{z}w)^{2^n}.$$

It follows that

$$(S_z 1)(w) = (U_z S U_z 1)(w) = \frac{(1-|z|^2)^2}{(1-\bar{z}w)^2} \sum_{n=0}^{\infty} (2^n + 1)(\bar{z}\varphi_z(w))^{2^n}.$$

Make the substitution $w = \varphi_z(\lambda)$ and use the identities

$$\lambda = \varphi_z(w)$$

$$\frac{1}{1-\bar{z}w} = \frac{1-\bar{z}\lambda}{1-|z|^2}$$

$$dA(w) = |\varphi_z{}'(\lambda)|^2 \, dA(\lambda) = \frac{(1-|z|^2)^2}{|1-\bar{z}\lambda|^4} \, dA(\lambda)$$

to obtain

$$\begin{aligned} \|S_z 1\|_3^3 &= \int_D |(S_z 1)(w)|^3 \, dA(w) \\ &= (1-|z|^2)^2 \int_D |1-\bar{z}\lambda|^2 \left| \sum_{n=0}^{\infty} (2^n+1)(\bar{z}\lambda)^{2^n} \right|^3 dA(\lambda). \end{aligned}$$

Thus

$$\|S_z 1\|_3^3 \le 4(1-|z|^2)^2 \int_0^1 \int_0^{2\pi} \left| \sum_{n=0}^{\infty} (2^n+1)(\bar{z}re^{i\theta})^{2^n} \right|^3 d\theta \, dr.$$

By Lemma 7, there is a constant C such that

$$\|S_z 1\|_3^3 \le C(1-|z|^2)^2 \int_0^1 \left(\sum_{n=0}^{\infty} (2^n+1)^2 (|z|r)^{2^{n+1}} \right)^{3/2} dr.$$

For $x \in [0,1)$, we have

$$\begin{aligned}\frac{1}{(1-x)^2} &= \sum_{k=0}^{\infty}(k+1)x^k \\ &\geq \sum_{n=0}^{\infty}\sum_{k=2^n+1}^{2^{n+1}}(k+1)x^k \\ &\geq \sum_{n=0}^{\infty}2^n(2^n+2)x^{2^{n+1}} \\ &\geq \frac{1}{2}\sum_{n=0}^{\infty}(2^n+1)^2x^{2^{n+1}}.\end{aligned}$$

Thus

$$\|S_z1\|_3^3 \leq 2C(1-|z|^2)^2\int_0^1\frac{dr}{(1-|z|r)^3}.$$

If $|z| \leq 1/2$, then clearly $\|S_z\|_3^3$ is bounded. If $|z| > 1/2$, then

$$\|S_z1\|_3^3 \leq 2C(1-|z|^2)^2\frac{(1-|z|)^{-2}-1}{2|z|} \leq 8C.$$

This shows that $\sup_{z\in D}\|S_z1\|_3 < \infty$.

Since $S_z^* = S_z$, we also have $\sup_{z\in D}\|S_z^*1\|_3 < \infty$.

4. Some estimates

See Lemma 4.2.2 of [15] for the following lemma. Some special cases of the lemma can be found in [1].

Lemma 8. *Suppose $a < 1$ and $a+b < 2$. Then*

$$\sup_{z\in D}\int_D\frac{dA(\lambda)}{(1-|\lambda|^2)^a|1-\bar{z}\lambda|^b} < \infty.$$

The following lemma is an extension of Lemma 3.2 of [3].

Lemma 9. *Suppose $0 < a < 1$ and $1 < s < \min\{1/a, 2/(2-a)\}$. Then there exists a constant C such that if S is a bounded operator on L_a^2, then*

$$\int_D\frac{|(SK_z)(w)|}{(1-|w|^2)^a}\,dA(w) \leq \frac{C\|S_z1\|_{s'}}{(1-|z|^2)^a} \tag{4.1}$$

for all $z \in D$ and

$$\int_D\frac{|(SK_z)(w)|}{(1-|z|^2)^a}\,dA(z) \leq \frac{C\|S_w^*1\|_{s'}}{(1-|w|^2)^a} \tag{4.2}$$

for all $w \in D$.

Proof. To prove (4.1), fix $z \in D$. We have

$$SK_z = \frac{SU_z 1}{|z|^2 - 1} = \frac{U_z S_z 1}{|z|^2 - 1} = \frac{((S_z 1) \circ \varphi_z) {\varphi_z}'}{|z|^2 - 1},$$

where the second equality comes from the definition of S_z, and the third equality comes from the definition of U_z. Thus

$$\int_D \frac{|(SK_z)(w)|}{(1-|w|^2)^a}\, dA(w) = \frac{1}{1-|z|^2} \int_D \frac{|(S_z 1)(\varphi_z(w))|\, |{\varphi_z}'(w)|}{(1-|w|^2)^a}\, dA(w).$$

In the last integral, make the substitution $w = \varphi_z(\lambda)$ to obtain

$$\int_D \frac{|(SK_z)(w)|}{(1-|w|^2)^a}\, dA(w) = \frac{1}{(1-|z|^2)^a} \int_D \frac{|(S_z 1)(\lambda)|}{(1-|\lambda|^2)^a |1-\bar{z}\lambda|^{2-2a}}\, dA(\lambda).$$

Applying Hölder's inequality to the integral on the right-hand side above, we get

$$\int_D \frac{|(SK_z)(w)|}{(1-|w|^2)^a}\, dA(w) \le \frac{\|S_z 1\|_{s'}}{(1-|z|^2)^a} \left(\int_D \frac{dA(\lambda)}{(1-|\lambda|^2)^{as} |1-\bar{z}\lambda|^{2s-2as}} \right)^{1/s}.$$

Thus (4.1) follows from Lemma 8. To prove (4.2), replace S by S^* in (4.1), interchange w and z in (4.1) and then use the equation

$$(S^* K_w)(z) = \langle S^* K_w, K_z \rangle = \langle K_w, SK_z \rangle = \overline{(SK_z)(w)} \tag{4.3}$$

to obtain the desired result. □

The proof of Lemma 9 also implies the following lemma.

Lemma 10. *Suppose* $1 < p < \infty$ *and* $0 < \alpha < \min\{1/p, 1/p'\}$. *Suppose* $s < \min\{1/\alpha p, 2/(2-\alpha p)\}$ *and* $t < \min\{1/\alpha p', 2/(2-\alpha p')\}$. *Then there exists a constant* C *such that if* S *is a bounded operator on* L_a^2, *then*

$$\int_D \frac{|(SK_z)(w)|}{(1-|w|^2)^{\alpha p}}\, dA(w) \le \frac{C \|S_z 1\|_{s'}}{(1-|z|^2)^{\alpha p}}$$

for all $z \in D$ *and*

$$\int_D \frac{|(SK_z)(w)|}{(1-|z|^2)^{\alpha p'}}\, dA(z) \le \frac{C \|S_w^* 1\|_{t'}}{(1-|w|^2)^{\alpha p'}}$$

for all $w \in D$.

If S is a bounded operator on L_a^p for some $p \in (1, \infty)$, then (4.3) still holds. Thus we can replace the assumption that S is a bounded operator on L_a^2 by that S is a bounded operator on L_a^p for some $p \in (1, \infty)$ in Lemmas 9 and 10.

We give a simple application on operator norms. The following Schur's test is well known (see Theorem 3.2.2 of [15]).

Lemma 11. *Suppose $1 < p < \infty$ and $K(z,w)$ is a measurable function on $D \times D$. If there are a nonnegative function $h(z)$ and constants C_1 and C_2 such that*

$$\int_D |K(z,w)| h(z)^p \, dA(z) \le C_1 h(w)^p$$

for almost every $w \in D$ and

$$\int_D |K(z,w)| h(w)^{p'} \, dA(w) \le C_1 h(z)^{p'}$$

for almost every $z \in D$, then the integral operator defined by

$$(Tf)(w) = \int_D f(z) K(z,w) \, dA(z)$$

is bounded on L^p and $\|T\|_p \le (C_1)^{1/p}(C_2)^{1/p'}$.

Proposition 1. *Suppose $1 < p < \infty$ and S is a bounded operator on L^p_a. If*

$$C_1 = \sup_{z \in D} \|S_z 1\|_m < \infty, \ C_2 = \sup_{z \in D} \|S_z^* 1\|_m < \infty$$

for some $m > 3/(p_1 - 1)$. Then there is a constant C such that

$$\|S\|_p \le C (C_1)^{1/p}(C_2)^{1/p'}.$$

Proof. For $f \in L^p_a$ and $w \in D$, we have

$$\begin{aligned}(Sf)(w) &= \langle Sf, K_w \rangle \\ &= \langle f, S^* K_w \rangle \\ &= \int_D f(z) \overline{(S^* K_w)(z)} \, dA(z) \\ &= \int_D f(z) (S K_z)(w) \, dA(z), \end{aligned} \tag{4.4}$$

where the last equation follows from (4.3).

To finish the proof, we just need to find the right test function $h(z)$ and apply Schur's test. Choose $h(z) = 1/(1 - |z|^2)^\alpha$, where

$$\alpha = \frac{2(p_1 - 1)}{3p_1}.$$

It is easy to see that $0 < \alpha < \min\{1/p, 1/p'\}$. It also follows from a simple computation that

$$\min\{1/\alpha p, 2/(2 - \alpha p)\} = \begin{cases} 3/(4-p), & \text{if } p \le 2; \\ 3/2, & \text{if } p > 2. \end{cases}$$

Thus

$$\min\{1/\alpha p, 2/(2 - \alpha p)\} \ge 3/(4 - p_1).$$

Similarly we can show that

$$\min\{1/\alpha p', 2/(2 - \alpha p')\} \ge 3/(4 - p_1).$$

Let $s = m'$. Then $m = s'$. Since $m > 3/(p_1 - 1)$, then $s < 3/(4 - p_1)$. The conclusion of the proposition now follows from Lemmas 10 and 11 (using $s = t = m'$ in Lemma 11). $\square$

5. Proof of main results

In order to prove our main results, we need three more lemmas.

See [14] for the following lemma.

Lemma 12. *Suppose* $1 < p < \infty$. *Then*
(a) $\|K_z\|_p$ *is equivalent to* $(1 - |z|^2)^{-2/p'}$ *for all* $z \in D$.
(b) $K_z/\|K_z\|_p \to 0$ *weakly in* L^p_a *as* $z \to \partial D$.

See Ex. 7 on Page 181 of [4] for the following lemma.

Lemma 13. *Suppose* $1 < p < \infty$ *and* $K(z, w)$ *is a measurable function on* $D \times D$ *such that*

$$\int_D \left(\int_D |K(z, w)|^p \, dA(w) \right)^{p'-1} dA(z) < \infty.$$

Then the integral operator T *defined by*

$$Tf(w) = \int_D f(z) K(z, w) \, dA(z)$$

is compact on L^p.

To write the Berezin transform $\tilde{S}(z)$ precisely we will need a power series formula for the Berezin transform of a bounded operator S on L^2_a. From the definition of the reproducing kernel we get

$$k_z(w) = (1 - |z|^2) \sum_{m=0}^{\infty} (m + 1) \bar{z}^m w^m$$

for $z, w \in D$. To compute $\tilde{S}(z)$, which equals $\langle Sk_z, k_z \rangle$, first compute Sk_z by applying S to both sides of the equation above, and then take the inner product with k_z, again using the equation above, to obtain

$$\tilde{S}(z) = (1 - |z|^2)^2 \sum_{m,n=0}^{\infty} (m + 1)(n + 1) \langle Sw^m, w^n \rangle \bar{z}^m z^n. \tag{5.1}$$

Lemma 14. *Suppose* S *is a bounded operator on* L^p_a *for some* $p \in (1, \infty)$ *such that*

$$\sup_{z \in D} \|S_z 1\|_m < \infty$$

for some $m > 1$. *Then* $\tilde{S}(z) \to 0$ *as* $z \to \partial D$ *if and only if for every* $t \in [1, m)$, $\|S_z 1\|_t \to 0$ *as* $z \to \partial D$.

Proof. Suppose that for every $t \in [1, m)$, $\|S_z 1\|_t \to 0$ as $z \to \partial D$. In particular, $\|S_z 1\|_1 \to 0$ as $z \to \partial D$. Thus

$$|\tilde{S}(z)| = |\langle S_z 1, 1\rangle| \leq \|S_z 1\|_1 \to 0$$

as $z \to \partial D$.

Suppose that $\tilde{S}(z) \to 0$ as $z \to \partial D$. Fix $t \in [1, m)$. We will show that $\|S_z 1\|_t \to 0$ as $z \to \partial D$.

For $z \in D$, $j, m = 0, 1, \cdots$, we have

$$\begin{aligned}
|\langle S_z w^j, w^m\rangle| &= |\langle SU_z w^j, U_z w^m\rangle| \\
&= (1-|z|^2)^2 |\langle S[w^j \circ \varphi_z K_z], w^m \circ \varphi_z K_z\rangle| \\
&\leq (1-|z|^2)^2 \|S\|_p \|w^j \circ \varphi_z K_z\|_p \|w^m \circ \varphi_z K_z\|_{p'} \\
&\leq (1-|z|^2)^2 \|S\|_p \|K_z\|_p \|K_z\|_{p'} \\
&\leq C\|S\|_p,
\end{aligned}$$

where the first inequality comes from Hölder's inequality and the last inequality comes from Lemma 12 (a). The second inequality follows from

$$|w^j \circ \varphi_z| \leq 1$$

and

$$|w^m \circ \varphi_z| \leq 1$$

for all j and m on D.

First we show that $\langle S_z 1, w^n\rangle \to 0$ as $z \to \partial D$ for every nonnegative integer n. If this is not true, then there is a sequence $z_k \in D$ such that

$$\langle S_{z_k} 1, w^n\rangle \to a_{0n}$$

as $|z_k| \to 1$ for some nonzero constant a_{0n} and some $n \geq 1$. We have showed that $|\langle S_z w^j, w^m\rangle|$ is uniformly bounded for $z \in D$ and $j, m = 0, 1, \cdots$. Without loss of generality we may assume that for each j and m

$$\langle S_{z_k} w^j, w^m\rangle \to a_{jm}$$

for some constant a_{jm}.

For $z, \lambda \in D$, we have

$$\tilde{S}(\varphi_z(\lambda)) = \widetilde{S_z}(\lambda) = (1-|\lambda|^2)^2 \sum_{j,m=0}^{\infty} (j+1)(m+1)\langle S_z w^j, w^m\rangle \bar{\lambda}^j \lambda^m, \tag{5.2}$$

where the second equality comes from (5.1).

For each $\lambda \in D$, it is easy to see that $\varphi_{z_k}(\lambda) \to \partial D$ as $z_k \to \partial D$. Thus $\tilde{S}(\varphi_{z_k}(\lambda)) \to 0$ as $z_k \to \partial D$ for each $\lambda \in D$. Replacing z by z_k in (5.2) and taking the limit as $z_k \to \partial D$ for (5.2), we get

$$(1-|\lambda|^2)^2 \sum_{j,m=0}^{\infty} (j+1)(m+1) a_{jm} \bar{\lambda}^j \lambda^m = 0$$

for each $\lambda \in D$ (note that the interchange of limit and infinite sum is justified by the fact that for each fixed $\lambda \in D$, the power series of (5.2) converges uniformly for $z \in D$). Let

$$f(\lambda) = \sum_{j,m=0}^{\infty} (j+1)(m+1)a_{jm}\bar{\lambda}^j \lambda^m.$$

Then $f(\lambda) = 0$ for all $\lambda \in D$. This gives

$$\left[\frac{\partial^m}{\partial \lambda^m} \frac{\partial^j}{\partial \bar{\lambda}^j} f\right](0) = 0$$

for each j and m. On the other hand, we have

$$\left[\frac{\partial^m}{\partial \lambda^m} \frac{\partial^j}{\partial \bar{\lambda}^j} f\right](0) = ((j+1)!(m+1)!)a_{jm}$$

for each j and m. In particular, $a_{0n} = 0$. This is a contradiction. Hence we obtain

$$\lim_{z \to \partial D} \langle S_z 1, w^n \rangle = 0.$$

For $\lambda \in D$, we have

$$(S_z 1)(\lambda) = \sum_{n=0}^{\infty} (n+1) < S_z 1, w^n > \lambda^n.$$

It is clear that for each fixed $\lambda \in D$, the power series above converges uniformly for $z \in D$. This gives

$$\lim_{z \to \partial D} (S_z 1)(\lambda) = 0$$

for each $\lambda \in D$. Thus

$$\lim_{z \to \partial D} |(S_z 1)(\lambda)|^t = 0$$

for each $\lambda \in D$. Let $s = m/t$. Then $s > 1$. Thus

$$\int_D [|(S_z 1)(\lambda)|^t]^s dA(\lambda) = \|S_z 1\|_m^m \le \sup_{z \in D} \|S_z 1\|_m^m < \infty.$$

This implies that $\{|S_z 1|^t\}_{z \in D}$ is uniformly integrable. By Exercise 10 (Vitali's Theorem) or Exercise 11 on pages 133-134 of [11],

$$\lim_{z \to \partial D} \|S_z 1\|_t = 0.$$

This completes the proof of the lemma.

Proof of Theorem 1.

If S is compact on L^p_a, then by Lemma 12 (b),

$$\langle SK_z/\|K_z\|_p, K_z/\|K_z\|_{p'} \rangle \to 0$$

as $z \to \partial D$. By Lemma 12 (a), it is easy to see that $\tilde{S}(z)$ is equivalent to $\langle SK_z/\|K_z\|_p, K_z/\|K_z\|_{p'} \rangle$ for $z \in D$. Thus $\tilde{S}(z) \to 0$ as $z \to \partial D$.

Suppose that $\tilde{S}(z) \to 0$ as $z \to \partial D$. By Lemma 14 we have that $\|S_z 1\|_t \to 0$ as $z \to \partial D$ for every $t \in [1, m)$. We will show that S is compact on L^p_a. Fix t such that $3/(p_1 - 1) < t < m$ in the rest of the proof.

For $f \in L^p_a$ and $w \in D$, we have from (4.4)

$$(Sf)(w) = \int_D f(z)(SK_z)(w)\, dA(z).$$

For $0 < r < 1$, define an operator $S_{[r]}$ on L^p_a by

$$(S_{[r]}f)(w) = \int_{rD} f(z)(SK_z)(w)\, dA(z). \tag{5.3}$$

In other words, $S_{[r]}$ is the integral operator with kernel $(SK_z)(w)\chi_{rD}(z)$. We will use Lemma 13 to show that $S_{[r]}$ is compact on L^p_a. Let

$$I_p(f, r) = \int_D \left(\int_D |(SK_z)(w)\chi_{rD}(z)|^p\, dA(w) \right)^{p'-1} dA(z).$$

By Lemma 12 (a)

$$\begin{aligned} I_p(f, r) &= \int_{rD} \left(\int_D |(SK_z)(w)|^p\, dA(w) \right)^{p'-1} dA(z) \\ &\le \|S\|_p^{p'} \int_{rD} \|K_z\|_p^{p'}\, dA(z) \\ &\le C\|S\|_p^{p'} \int_{rD} \frac{dA(z)}{(1 - |z|^2)^2} \\ &< \infty. \end{aligned}$$

Thus $S_{[r]}$ is compact on L^p_a. Hence to prove that S is compact, we only need show that $\|S - S_{[r]}\|_p \to 0$ as $r \to 1^-$.

If $r \in (0, 1)$, then $S - S_{[r]}$ is the integral operator with kernel

$$(SK_z)(w)\chi_{D \setminus rD}(z),$$

as can be seen from (4.4) and (5.3). The proof of Proposition 1 indicates that $\|S - S_{[r]}\|_p \le C(C_1)^{1/p}(C_2)^{1/p'}$, where

$$C_1 = \sup\{\|S_z 1\|_t : r \le |z| < 1\},\ C_2 = \sup\{\|S_z^* 1\|_t : z \in D\}.$$

We have showed above that $C_1 \to 0$ as $r \to 1^-$. The hypothesis of the theorem gives that $C_2 < \infty$. Thus $\|S - S_{[r]}\|_p \to 0$ as $r \to 1^-$, completing the proof. □

Proof of Theorem 3.

Suppose S is a finite sum of operators of the form $T_{f_1} \cdots T_{f_n}$, where each $f_j \in \mathrm{BT}$. By Lemmas 3 and 6, we have that S is bounded on L^p_a for $1 < p < \infty$, and

$$\sup_{z \in D} \|S_z 1\|_m < \infty,\ \sup_{z \in D} \|S_z^* 1\|_m < \infty$$

for all $0 < m < \infty$. Hence Theorem 3 follows from Theorem 1.

□

6. Norms of Toeplitz operators

In this section, we consider the norm and essential norm of a Toeplitz operator T_f on L_a^2 for $f \in \mathrm{BT}$. For $z \in D$, we have

$$(T_f)_z 1 = P(f \circ \varphi_z), \; (T_f)_z^* 1 = P(\bar{f} \circ \varphi_z)$$

$$\|P(f \circ \varphi_z)\|_2 = \|T_f k_z\|_2, \; \|P(\bar{f} \circ \varphi_z)\|_2 = \|T_{\bar{f}} k_z\|_2.$$

See [13] for the identities above. For a bounded operator S on L_a^2, let $\|S\|$ denote $\|S\|_2$ in this section.

In [6], Englis showed that neither

$$\|T_f\|_e \leq C \limsup_{z \to \partial D} |\widetilde{f}(z)| \qquad \forall f \in L^\infty(D, dA)$$

nor

$$\|T_f\| \leq C \sup_{z \in D} |\widetilde{f}(z)| \qquad \forall f \in L^\infty(D, dA)$$

can hold for any constant C. Here $\|T_f\|_e$ denotes the essential norm of the Toeplitz operator T_f defined by

$$\|T_f\|_e = \inf_{K \in \mathcal{K}} \|T_f - K\|,$$

where $\mathcal{K}$ is the set of compact operators on L_a^2. Later, Nazarov told us that the inequality

$$\|T_f\| \leq C \sup_{z \in D} \|T_f k_z\|_2 \qquad \forall f \in L^\infty(D, dA)$$

cannot hold for any constant C. In this section we will obtain some estimates of the norm and essential norm of Toeplitz operators. To get those estimates we need the Bloch space $\mathcal{B}$ and two lemmas.

The Bloch space $\mathcal{B}$ is defined by

$$\mathcal{B} = \{f \text{ analytic on } D : \sup_{z \in D}(1 - |z|^2)|f'(z)| < \infty\}.$$

The Bloch space can be made into a Banach space by the norm

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in D}(1 - |z|^2)|f'(z)|.$$

The following lemma is a consequence of the Li and Luecking result [7] that the Bergman projection P is bounded from BMO^p for $p \geq 1$ onto the Bloch space B. We present a simple proof here.

Lemma 15. *Suppose $f \in \mathrm{BT}$. Then $P(f) \in \mathcal{B}$. Moreover there is a constant C such that*

$$\|P(f)\|_{\mathcal{B}} \leq C \|f\|_{\mathrm{BT}}$$

for all $f \in \mathrm{BT}$.

Proof. Let $z \in D$. An easy calculation gives

$$[P(f)]''(z) = 6 < f, w^2 K_z^2 > .$$

Thus

$$(1-|z|^2)^2 |[P(f)]''(z)| \le 6 < |f|, |w|^2 |k_z|^2 > \le 6\widetilde{|f|}(z).$$

So

$$\sup_{z\in D}(1-|z|^2)^2 |[P(f)]''(z)| \le 6 \sup_{z\in D} \widetilde{|f|}(z) = 6\|f\|_{\mathrm{BT}}.$$

By Theorem 5.1.5 in [15], $P(f) \in \mathcal{B}$ and $\|P(f)\|_{\mathcal{B}}$ is equivalent to

$$|P(f)(0)| + |[P(f)]'(0)| + \sup_{z\in D}(1-|z|^2)^2 |[P(f)]''(z)|.$$

Note

$$P(f)(0) = \tilde{f}(0),$$

and

$$[P(f)]'(0) = 2\widetilde{wf}(0).$$

Thus the above estimate gives

$$\begin{aligned}\|P(f)\|_{\mathcal{B}} &\le C[|\tilde{f}(0)| + 2\widetilde{|f|}(0) + 6\|f\|_{\mathrm{BT}}] \\ &\le 9C\|f\|_{\mathrm{BT}}\end{aligned}$$

for some constant C, independent of f. This gives the desired result. □

Lemma 16. *Suppose $g \in \mathcal{B}$ and $3 < m < 5$. Then there is a constant C such that*

$$\|g\|_m \le C\|g\|_{\mathcal{B}}^{2-(5/m)} \|g\|_2^{(5/m)-1}.$$

Proof. Write $m = 3+\epsilon$ for some $0 < \epsilon < 2$. Let $s = 2/(2-\epsilon)$. Then $s' = 2/\epsilon$. Hölder's inequality gives

$$\begin{aligned}\int_D |g(w)|^m dA(w) &= \int_D |g(w)|^{(2-\epsilon)+(1+2\epsilon)} dA(w) \\ &\le \left[\int_D |g(w)|^2 dA(w)\right]^{1/s} \left[\int_D |g(w)|^{s'(1+2\epsilon)} dA(w)\right]^{1/s'}.\end{aligned}$$

Since $g \in \mathcal{B}$, then by the proof of Theorem 1 in [1]

$$|g(w) - g(0)| \le \|g\|_{\mathcal{B}} \log \frac{1}{1-|w|}.$$

Thus we have

$$|g(w)| \le \|g\|_{\mathcal{B}} \left[\log \frac{1}{1-|w|} + 1\right].$$

Since $\log(1/1-|w|)$ is in L^p for every $p \in (0,\infty)$, this gives that

$$\left[\int_D |g(w)|^{s'(1+2\epsilon)} dA(w)\right]^{1/s'} \le C\|g\|_{\mathcal{B}}^{1+2\epsilon},$$

where C is independent of g. This leads to

$$\|g\|_m \le C\|g\|_{\mathcal{B}}^{(1+2\epsilon)/m} \|g\|_2^{2/sm} = C\|g\|_{\mathcal{B}}^{2-(5/m)} \|g\|_2^{(5/m)-1}$$

and completes the proof. $\square$

Theorem 4. *For each $t \in (0, 2/3)$, there is a constant C such that*

$$\|T_f\| \le C[\sup_{z\in D} \|T_f k_z\|_2 \sup_{z\in D} \|T_{\bar f} k_z\|_2]^{t/2}$$

and

$$\|T_f\|_e \le C[\limsup_{z\to\partial D} \|T_f k_z\|_2 \limsup_{z\to\partial D} \|T_{\bar f} k_z\|_2]^{t/2}$$

for all $f \in \mathrm{BT}$ with $\|f\|_{\mathrm{BT}} \le 1$.

Proof. For $g \in L^2_a$ and $w \in D$, we have

$$(T_f g)(w) = \int_D g(z)(T_f^* K_z)(w)\, dA(z).$$

For $t \in (0, 2/3)$, let $m = 5/(t+1)$. It is clear that $3 < m < 5$. Proposition 1 gives

$$\|T_f\| \le C[\sup_{z\in D} \|P(f\circ\varphi_z)\|_m \sup_{z\in D} \|P(\bar f\circ\varphi_z)\|_m]^{1/2}.$$

For $0 < r < 1$ and $0 < s < 1$, define an operator $K_{[r]}$ on L^2_a by

$$(K_{[r]} g)(w) = \int_{rD} g(z)(T_f^* K_z)(w)\, dA(z),$$

and an operator $K_{[r],[s]}$ on L^2_a by

$$(K_{[r],[s]} g)(w) = \chi_{sD}(w) \int_{D\setminus rD} g(z)(T_f^* K_z)(w)\, dA(z).$$

As in the proof of Theorem 1, both $K_{[r]}$ and $K_{[r],[s]}$ can be showed to be compact on L^2_a.

If $r,\ s \in (0, 1)$, then $T_f - K_{[r]} - K_{[r],[s]}$ is the integral operator with kernel

$$(T_f^* K_z)(w)\chi_{D\setminus rD}(z)\chi_{D\setminus sD}(w).$$

The proof of Proposition 1 indicates that $\|T_f - K_{[r]} - K_{[r],[s]}\| \le C_m (C_1)^{1/2}(C_2)^{1/2}$, where

$$C_1 = \sup\{\|P(\bar f\circ\varphi_z)\|_m : r \le |z| < 1\},\ C_2 = \sup\{\|P(f\circ\varphi_w)\|_m : s \le |w| < 1\}.$$

We have showed

$$\|T_f\|_e \le C_m[\limsup_{z\to\partial D} \|P(f\circ\varphi_z)\|_m \limsup_{z\to\partial D} \|P(\bar f\circ\varphi_z)\|_m]^{1/2}.$$

To finish the proof it suffices to show that there is a constant C such that

$$\|P(f\circ\varphi_z)\|_m \le C\|T_f k_z\|_2^t,\ \|P(\bar f\circ\varphi_z)\|_m \le C\|T_{\bar f} f k_z\|_2^t$$

for all $f \in \mathrm{BT}$ with $\|f\|_{\mathrm{BT}} \le 1$. For $f \in \mathrm{BT}$, by Lemma 15, $P(f\circ\varphi_z) \in \mathcal{B}$ and

$$\|P(f\circ\varphi_z)\|_{\mathcal{B}} \le C\|f\circ\varphi_z\|_{\mathrm{BT}} = C\|f\|_{\mathrm{BT}}.$$

For $f \in \mathrm{BT}$ with $\|f\|_{\mathrm{BT}} \leq 1$, Lemma 16 gives

$$\begin{aligned}\|P(f\circ\varphi_z)\|_m &\leq C\|P(f\circ\varphi_z)\|_{\mathcal{B}}^{2-(5/m)}\|P(f\circ\varphi_z)\|_2^{(5/m)-1}\\ &\leq C\|f\|_{\mathrm{BT}}^{2-(5/m)}\|P(f\circ\varphi_z)\|_2^{(5/m)-1}\\ &\leq C\|P(f\circ\varphi_z)\|_2^t\\ &= C\|T_f k_z\|_2^t.\end{aligned}$$

Similarly, we have $\|P(\bar{f}\circ\varphi_z)\|_m \leq C\|T_{\bar{f}}k_z\|_2^t$ and the proof is now complete. □

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