COMPACTNESS RESULTS FOR THE YAMABE PROBLEM

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The Yamabe problem consists of finding a constant scalar curvature metric $\tilde{g}$ which is pointwise conformal to a given metric g on an n-dimensional ($n \geq 3$) compact Riemannian manifold M without boundary. This is equivalent to producing a positive solution to the following non-linear elliptic equation

$$\Delta_g u - c(n) R_g u + K u^{\frac{n+2}{n-2}} = 0 \tag{1}$$

where K is a constant and $c(n) = \frac{n-2}{4(n-1)}$. If $u > 0$ is a solution of (1) then the new metric $\tilde{g} = u^{\frac{4}{n-2}} g$ has scalar curvature $c(n)^{-1} K$. This problem was solved in the affirmative through the combined works of Yamabe, Trudinger, Aubin and Schoen.

From an analytic perspective the Yamabe problem has proven to be a rich source of interesting ideas. The complete solution of the problem was the first instance of a satisfactory existence theory for equations involving a critical exponent, where the standard techniques of the calculus of variations fail to apply. When (M, g) has positive Yamabe invariant $Y(M, g)$ (to be defined in the talk), solutions to (1) are not unique, and it is known that the set of solutions can be quite large. Therefore it becomes natural to ask what can be said about the full set of solutions to (1) when $Y(M, g) > 0$. Due to the work of several authors, we now have a solid understanding of the topology of the set of solutions to (1), including its compactness properties.

An obvious extension of such problems is to consider manifolds with boundary. In this case one would like to conformally deform a given metric to one which has not only constant scalar curvature but zero mean curvature as well. This problem is equivalent to showing the existence of a positive solution to the boundary value problem

$$\begin{cases} \Delta_g u - c(n) R_g u + K u^{\frac{n+2}{n-2}} = 0, & \text{in } M, \\ \partial_{\nu_g} u + \frac{n-2}{2} \kappa_g u = 0, & \text{on } \partial M, \end{cases} \tag{2}$$

where ν_g is the unit outer normal and κ_g is the mean curvature. If such a solution exists then the metric $\tilde{g} = u^{\frac{4}{n-2}} g$ has scalar curvature $c(n)^{-1} K$ and minimal boundary. This Yamabe problem on manifolds with boundary was solved affirmatively with contributions from several authors.

In analogy to the case of manifolds without boundary, for positive Yamabe invariant the question of compactness of solutions arises. In this talk, I will discuss some recent results regarding this matter.

The presentation will be divided in three talks. In the first one I will introduce the problem along with the main definitions and state the results. In the second talk some background material will be covered. Finally, in the last presentation I will sketch some of the proofs. Throughout the presentation I will mention several open problems, what may turn the talk specially attractive to graduate students looking for research topics.

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