On uniform K-homology

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Outline

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K-homology and Coarse Baum-Connes conjecture

Uniform K-homology

An application: amenability

Analytic K-homology

"A space" is always assumed to be 2[nd] countable, locally compact, Hausdorff topological space. Let X be a space.

K_0-cycles for X: *Fredholm modules* (H, ϕ, F):

- H — Hilbert space
- $\phi : C_0(X) \to \mathscr{B}(H)$ — $*$-homomorphism
- $F \in \mathscr{B}(H)$, such that $\forall f \in C_0(X)$:
  - (i) $(1 - FF^*)\phi(f) \sim 0$ and $(1 - F^*F)\phi(f) \sim 0$ (Fredholmness)
  - (ii) $[F, \phi(f)] \sim 0$ (pseudolocality)

Notation: $T \sim S$ iff $T - S$ is compact.

Operation: direct sum of Fredholm modules.

To form $K_0(X)$, we mod out by a certain equivalence relation.

K-homology – main example/motivation

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Reformulate the defn ("grading"): $T = \begin{pmatrix} 0 & F^* \\ F & 0 \end{pmatrix} \in \mathscr{B}(H \oplus H)$.
Requirements now: T is odd and selfadjoint;
(i) $(T^2 - 1)\phi(f) \sim 0$; (ii) $[T, \phi(f)] \sim 0$.

Example:

- M a connected Riemannian manifold
- D a essentially self-adjoint, elliptic differential operator acting on sections of a graded vector bundle $S = S^- \oplus S^+$ over M
- $\chi : \mathbb{R} \to \mathbb{R}$ a chopping function (odd, $\chi(\pm\infty) \to \pm 1$)
- $H = L^2(M, S)$, $\phi : C_0(M) \to \mathscr{B}(H)$ by multiplication
- $T = \chi(D) \in \mathscr{B}(H)$
- Then (H, ϕ, T) is a Fredholm module.

K-homology – properties

Analytic K-homology is a generalized homology theory on the category of "spaces" (i.e. it satisfies homotopy axiom and has a long exact sequence).

There are only two distinct K-homology groups $K_0(X)$ and $K_1(X)$ (periodicity).

There is a Chern character homomorphism $\mathrm{ch}: K_*(X) \to \oplus_{\text{even/odd}} H_*^{\text{lf}}(X, \mathbb{R})$.

Interpretation of K-homology as K-theory:

One can fix a suitable $*$-representation $\phi : C_0(X) \to \mathscr{B}(H)$, and represent any $K_0(X)$-element as (H, ϕ, F) for some F (Voiculescu's theorem).

Define

$$\mathscr{D}_\phi = \{T \in \mathscr{B}(H) \mid [T, \phi(f)] \sim 0 \ \forall f\}.$$

One can show that $K_i(X) \cong K_{1-i}(\mathscr{D}_\phi)$.

Coarse geometry

Category of (e.g.) metric spaces X. A function $f : X \to Z$ is *coarse*, if

- for every $r \geq 0$ there is $R \geq 0$, such that $\forall x_1, x_2 \in X$
$$d(x_1, x_2) \leq r \implies d(f(x_1), f(x_2)) \leq R.$$
- $f^{-1}(\{z\})$ is bounded $\forall z \in Z$.

Note f might not be continuous. Any local information is irrelevant.

Two functions $f, g : X \to Z$ are *close*, if there exists $C > 0$, so that $d(f(x), g(x)) \leq C$.

$f : X \to Z$ is a coarse equivalence, if there is a coarse $g : Z \to X$, such that $f \circ g$ and $g \circ f$ are close to the identity maps.

Example: $\mathbb{R}^n$ is coarsely equivalent with $\mathbb{Z}^n$.

Rips complex

• Switching from “discrete” to “continuous” and back in the coarse category:
Let Y be a discrete metric space with bounded geometry.
Rips complex: For $d > 0$, let $P_d(Y)$ be a (locally finite, finite dimensional) simplicial polyhedron as follows:

- the vertex set is Y,
- $(y_0, \ldots, y_q)$ span a simplex iff $\forall i,j : d(y_i, y_j) \leq d$.

Notation from now on: “$Y \subset X$ is a discrete model of X”:

- Y: a uniformly discrete metric space with bounded geometry
- X: a space (with bounded geometry)
- $Y \subset X$: a coarse equivalence

Roe algebras

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An operator $T \in \mathscr{B}(\ell^2 Y \otimes H)$ (a matrix $(t_{yx})_{x,y \in Y}$, $t_{yx} \in \mathscr{B}(H)$) has *propagation at most* R, if $d(x,y) \geq R$ implies $t_{yx} = 0$. Furthermore T is said to be *locally compact*, if $t_{yx} \in \mathscr{K}(H)$.

Define the *Roe algebra*:

$$C^*Y = \overline{\{T \in \mathscr{B}(\ell^2 Y \otimes H) \mid T \text{ is loc.comp. and has fin.prop.}\}}^{\|.\|}$$

If we consider $T = (t_{yx})_{x,y \in Y} \in \mathscr{B}(\ell^2 Y)$ (i.e. the entries $t_{yx} \in \mathbb{C}$), we obtain the *uniform Roe algebra*

$$C_u^*Y = \overline{\{T \in \mathscr{B}(\ell^2 Y) \mid T \text{ has fin.prop.}\}}^{\|.\|}$$

Coarse Baum–Connes conjecture

Coarse assembly map $\mu : K_i(X) \to K_i(C^*X)$ for a space X by example: If D is a "geometric" elliptic operator on a noncompact manifold M, then $\chi(D)$ is invertible modulo C^*M. So, the index $\mu(D)$ can be constructed in $K_*(C^*M)$.

Conjecture (Coarse Baum–Connes conjecture)

If Y is a space with bounded geometry, then

$$\mu : \lim_{d\to\infty} K_*(P_d(Y)) \to K_*(C^*Y)$$

is an isomorphism.

- If true for a discrete group Γ, it implies injectivity of the Baum–Connes map, and hence e.g. the Novikov conjecture on invariance of higher signatures.
- Known to be true for spaces which coarsely embed into a Hilbert space [Yu].
  Known to be false for certain expander graphs [Higson].

Uniform K-homology: motivation

Why uniform Roe algebras instead of Roe algebras?

- smaller, more manageable C*-algebra; can provide insight for Roe algebras
- its C*-algebraic properties relate to coarse geometric properties (e.g. nuclearity, exactness relate to prop. A)
- typically much bigger K-theory (if nonzero)

Plan: construct "uniform K-homology": groups $K_*^u(X)$, with an index map $\mu : K_*^u(X) \to K_*(C_u^*Y)$. Further properties:

- homological properties (e.g. Mayer–Vietoris sequence)
- Chern character to $H_*^{uf}(X)$ of Block and Weinberger (planned)

Benefits:

- a characterization of amenability of X in terms of $K_0^u(X)$
- "easier" to detect higher indices in principle

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Uniform K-homology: preparation

Let X be a metric space. Fix a Hilbert space H with $\phi : C_0(X) \to \mathscr{B}(H)$.

Notation: for $R > 0$, let
$C_R(X) = \{f \in C_0(X) \mid \mathrm{diam}(\mathrm{supp}(f)) \leq R, \|f\|_\infty \leq 1\}$.

An operator $T \in \mathscr{B}(H)$ is (ε, N)-*approximable*, if there is $k \in \mathscr{B}(H)$ with $\mathrm{rank}(k) \leq N$ and $\|T - k\| < \varepsilon$.

An expression $E(f)$ with operators from $\mathscr{B}(H)$ and $\phi(f)$ (for instance $E(f) = [T, \phi(f)]$, or $E(f) = T\phi(f)$) is *uniformly approximable*, if for any $\varepsilon, R > 0$, there exists $N \geq 0$, such that for every $f \in C_R(X)$, $E(f)$ is (ε, N)-approximable. We write $E(f) \sim_{ua} 0$.

Uniform K-homology: definition

(An even) *uniform Fredholm module* for X is a Fredholm module (H, ϕ, F), such that

- $(1 - F^*F)\phi(f) \sim_{ua} 0$ and $(1 - FF^*)\phi(f) \sim_{ua} 0$.
- $[\phi(f), F] \sim_{ua} 0$ (F is uniformly pseudolocal).

Again, to form $K_0^u(X)$, we mod out by the equivalence relation generated by unitary equivalence, (certain) homotopies and adding degenerate modules.

The groups are functorial under continuous, uniformly cobounded maps. ($f : X \to Y$ is uniformly cobounded iff for every $r \geq 0$ there exists $R \geq 0$, such that $\mathrm{diam}(f^{-1}(B(y, r))) \leq R$ for every $y \in Y$.)

Note that for a compact X, $K_*^u(X) = K_*(X)$.

Uniform K-homology: examples

If $X = \mathbb{R}$, $D = i\frac{\mathrm{d}}{\mathrm{d}x}$, the K-homology element associated to it is uniform. (Because it is invariant under translation action, every "piece" has the same approximation properties.)

"The fundamental class": X is an infinite graph (for instance a Cayley graph of a finitely generated group) with vertex set Y. Let $S \in \mathscr{B}(\ell^2\mathbb{N})$ be the unilateral shift, and let $\tilde{S} \in \mathscr{B}(\ell^2 Y \otimes \ell^2\mathbb{N})$ be a "Y-diagonal" operator, with S on the diagonal. Let $C_0(X)$ act on $\ell^2 Y$ by multiplication. Then

$$\mathbf{S} = [(\ell^2 Y \otimes \ell^2\mathbb{N}, \phi, \tilde{S})] \in K_0^u(X).$$

Properties of uniform K-homology

Getting around Voiculescu's theorem:
For fixed H and $\phi : C_0(X) \to \mathscr{B}(H)$, define $K_*^u(X,\phi)$ (then $K_*^u(X) = \lim_\phi K_*^u(X,\phi)$. Similarly, define $\mathscr{D}_\phi^u \subset \mathscr{B}(H)$ as the C*-algebra of uniformly pseudolocal operators.

Proposition

$K_i^u(X,\phi) \cong K_{1-i}(\mathscr{D}_\phi^u)$.

Proposition

K_*^u has a Mayer–Vietoris sequence.

Proposition

If $Y \subset X$ is discrete model of X, then there is a homomorphism $\mu : K_*^u(X) \to K_*(C_u^* Y)$, the assembly map.

Constructing the uniform index/assembly map

Starting with an even uniform Fredholm module (H, ϕ, F) for X, applying the usual formula, we get an idempotent $T \in \mathscr{B}(H \oplus H)$. The task is to prove that we can actually arrange $[T] - [\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)] \in K_0(C_u^* Y)$.

For simplicity, take just $T \in \mathscr{B}(H)$, which is "uniform" (i.e. $T\phi(f) \sim_{ua} 0 \sim_{ua} \phi(f)T$), and also assume that ϕ is "sufficiently big" (the range of ϕ misses $\mathscr{K}(H) \setminus \{0\}$).

Using a suitable decomposition of the space X into the pieces corresponding to Y, we obtain a unitary isomorphism $U : H \cong \ell^2 Y \otimes \ell^2 \mathbb{N}$. The point is to use uniformity to choose U such that for any $\varepsilon > 0$ there is $n \in \mathbb{N}$, so that T is ε-far from an operator in $\mathscr{M}_n(C_u^* Y) \subset \mathscr{M}_n(\mathscr{B}(\ell^2 Y)) \subset \mathscr{B}(\ell^2 Y \otimes \ell^2 \mathbb{N})$.

This places $T \in C_u^* Y \otimes \mathscr{K}(\ell^2 \mathbb{N})$.

Amenability: theorem and related results

Let X be a graph (bounded vertex degree) with vertex set Y. Y is said to be *amenable*, if for any $r, \delta > 0$, there exists a finite $U \subset Y$ with $\frac{\#\{\partial_r U\}}{\#\{U\}} < \delta$. (aka Følner condition. Works well for groups, not so much for general metric spaces.)

The element $\mathbf{S} \in K_0^u(X)$ defined before satisfies $\mu(\mathbf{S}) = \mathbf{1} \in K_0(C_u^* Y)$.

Theorem

Y is amenable iff $\mathbf{S} \neq \mathbf{0} \in K_0^u(X)$.

Theorem (G. Elek, '97)

Y is amenable iff $\mathbf{1} \neq \mathbf{0} \in K_0(C_u^* Y)$.

Theorem (J. Block and S. Weinberger, '92)

Y is amenable iff $H_0^{uf}(Y) \neq \{0\}$.

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