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A Center Manifold Analysis for the Mullins–Sekerka Model

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The Mullins–Sekerka model is a nonlocal evolution model for hypersurfaces, which arises as a singular limit for the Cahn–Hilliard equation. We show that classical solutions exist globally and tend to spheres exponentially fast, provided that they are close to a sphere initially. Our analysis is based on center manifold theory and on maximal regularity. © 1998 Academic Press

Key Words: Mullins–Sekerka model; mean curvature; free boundary problem; generalized motion by mean curvature; center manifold.

1. INTRODUCTION

The Mullins–Sekerka model is a free boundary problem arising from physics [12, 22, 29], which has also been called the Hele–Shaw model [2, 9], or the Hele–Shaw model with surface tension [19, 20]. This model has attracted considerable attention over the last years. Recently it has been shown by Alikakos *et al.* [2] that the two-phase Mullins–Sekerka problem arises as a singular limit of the Cahn–Hilliard equation, rigorously establishing a result that was formally derived by Pego [32]. We mention that the authors in [2] had to include the extra assumption that classical solutions of the Mullins–Sekerka model exist (locally in time). It was not until very recently that existence and regularity of classical solutions was obtained in [19, 20] and, independently, in [10].

The Mullins–Sekerka model can be considered as a nonlocal generalization of the flow by mean curvature. It has some very appealing geometric

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properties, similar to those of motion by mean curvature. Solutions of the Mullins–Sekerka model evolve in such a way that the volume of the region enclosed by the moving hypersurface $\Gamma(t)$ is preserved, while the area of $\Gamma(t)$ shrinks, unless $\Gamma(t)$ is a single sphere or the union of multiple spheres of the same radius. On the other side, there are striking differences to the motion by mean curvature. It is shown in [27] that the one-phase Mullins–Sekerka flow does not preserve convexity, unlike the mean curvature flow [21, 25]. Recently, the same author [28] has proved that the two-phase Mullins–Sekerka flow in $\mathbb{R}^2$ also does not preserve convexity. We also refer to [8] for numerical results in this direction.

Let us introduce the concise model we want to study. We assume that Ω is a bounded domain in $\mathbb{R}^n$, $n \geqslant 2$, with smooth boundary $\partial\Omega$. Let $\Gamma_0 \subset \Omega$ be a compact connected hypersurface which is the boundary of an open set $\Omega_0 \subset \Omega$. For each $t \geqslant 0$, let $\Gamma(t)$ be the position of Γ_0 at time t, and let $V(\cdot, t)$ and $\kappa(\cdot, t)$ be the normal velocity and the mean curvature of $\Gamma(t)$. Here we use the convention that the normal velocity is positive for expanding hypersurfaces and that the mean curvature is positive for uniformly convex hypersurfaces. Let $\Omega^1(t)$ and $\Omega^2(t)$ be the two regions in Ω separated by $\Gamma(t)$, with $\Omega^1(t)$ being the interior region. Moreover, let $n(\cdot, t)$ be the outer unit normal field of $\Gamma(t)$ with respect to $\Omega^1(t)$. Then we let Γ_0 evolve according to the law

$$V = -[\partial_n u_\kappa], \tag{1.1}$$

where the function $u_\kappa = u_\kappa(\cdot, t)$ is the harmonic extension of the mean curvature $\kappa = \kappa(\cdot, t)$ over $\Omega^1(t) \cup \Omega^2(t)$ subject to a homogeneous Neumann boundary condition on $\partial\Omega$, that is, $u_\kappa(\cdot, t)$ is, for each $t \geqslant 0$, the solution of the elliptic boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega^1(t) \cup \Omega^2(t) \\ u = \kappa & \text{on } \Gamma(t) \\ \partial_n u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

where $\partial_n u$ denotes the normal derivative of u on $\partial\Omega$. Let $u^i_\kappa = u^i_\kappa(\cdot, t)$ denote the restriction of $u_\kappa(\cdot, t)$ on $\Omega^i(t)$, $i \in \{1, 2\}$. Then

$$[\partial_n u_\kappa] := \partial_n u^1_\kappa - \partial_n u^2_\kappa \tag{1.3}$$

denotes the jump of the normal derivatives of u_κ across the boundary $\Gamma(t)$. Of course, the position and the regularity of the moving hypersurface $\Gamma(t)$ are a priori unknown and have to be determined as part of the problem. Hence the elliptic problem (1.2) cannot be solved independently without having information on $\Gamma(t)$. On the other hand, Eq. (1.1), governing the motion of $\Gamma(t)$, requires information contained in (1.2). So it is clear that the equations in (1.1) and (1.2) are coupled and have to be solved simultaneously. Having pointed out this fact, we will in the sequel often refer to a solution $\Gamma(t)$, being aware that we actually have to determine the free interface $\Gamma(t)$ and the solution $u_\kappa(\cdot, t)$ of the elliptic equation (1.2).

The equation (1.1) together with (1.2) is called the *two-phase Mullins–Sekerka model.*

If the restriction of u on $\Omega^2(t)$ or on $\Omega^1(t)$ is replaced by a constant while all the other aspects of the problem are left unchanged, then the modified problem is called the *one-phase Mullins–Sekerka model.*

The evolution model (1.1) depends in a nonlocal way upon the mean curvature. It is in this sense that the Mullins–Sekerka model can be considered as a nonlocal generalization of the motion by mean curvature which is governed by the law

$$V = -\kappa.$$

The mathematical analysis of the Mullins–Sekerka model bears considerable difficulties, mainly caused by the nonlocal character of the equations. Existence, uniqueness, and regularity of classical solutions for the one- and two-phase Mullins–Sekerka model was recently obtained by the authors [19, 20]. At about the same time, Chen *et al.* [10] also proved the existence of classical solutions for the two-phase model, using a different approach. It should be mentioned that even weak solutions to the Mullins–Sekerka model were previously not known to exist in higher space dimensions. For the two-phase Mullins–Sekerka model in two dimensions, Chen [9] proved the local existence of weak solutions for arbitrary (smooth) initial curves Γ_0. Still in two dimensions and for a particular geometry, i.e., for strip-like domains, Duchon and Robert [15] established the existence of local solutions for the one-phase model. Also in the two-dimensional case, Constantin and Pugh [11] established global analytic solutions for a related problem, provided the initial curves are small analytic perturbations of circles.

In the case $\Omega = \mathbb{R}^2$, Chen [9] proved that if Γ_0 is close to a circle then there exists a global weak solution for the two-phase problem and the global solution tends to some circle exponentially fast.

In this paper we generalize this result to arbitrary dimensions and, to general domains Ω. We consider both the one- and the two-phase Mullins–Sekerka models and we prove global existence (and uniqueness) of classical solutions if Γ_0 is close to a sphere. Moreover, given $k \in \mathbb{N}$, we show that solutions converge exponentially fast to some sphere in the C^k-topology if Γ_0 is close to a sphere in the $C^{2+\beta}$-topology. The approach relies on our previous paper [20] and uses, as a new ingredient, center manifold theory for quasilinear evolutions equations [33, 34]; see also [14].

In the following we will use the phrasing *Mullins-Sekerka model* interchangeably for the one- or two-phase Mullins–Sekerka model. Let us now formulate our main result.

THEOREM 1. *Assume $\beta \in (0, 1)$ and let $\Gamma_0 \in C^{2+\beta}$ be given. Then*:

(a) *The Mullins–Sekerka model has a unique local classical solution $\Gamma(t)$ on some interval $(0, T)$. Each hypersurface $\Gamma(t)$ is C^∞ for $t \in (0, T)$. In addition, $\Gamma(t)$ depends smoothly on $t \in (0, T)$.*

(b) *If Γ_0 is close to a Euclidean sphere $\mathcal{S}$ in the $C^{2+\beta}$-topology, then the solution of the Mullins–Sekerka model exists globally and converges exponentially fast to some sphere which encloses the same volume as Γ_0. The convergence is in the C^k-topology for every initial hypersurface Γ_0 which is in a sufficiently small $C^{2+\beta}$-neighborhood $V = V(k)$ of $\mathcal{S}$, where $k \in \mathbb{N}$ is a fixed number.*

Proof. (a) We refer to [20], where a more precise statement has been formulated and proved. (b) is a consequence of Theorem 6.5 and Proposition 6.6 below. ∎

Once the existence of classical solutions is established, it is easy to see that the Mullins–Sekerka flow preserves the volume of $\Omega^1(t)$ and decreases the area of $\Gamma(t)$. In order to see this, let $\mathrm{Vol}(t)$ denote the volume of $\Omega^1(t)$ and let $A(t)$ be the area of the moving hypersurface $\Gamma(t)$. Then we can calculate

$$\frac{d}{dt}\,\mathrm{Vol}(t) = \int_{\Gamma(t)} V\,d\sigma = -\int_{\Gamma(t)} [\partial_n u_\kappa]\,d\sigma = -\int_\Omega \Delta u_\kappa\,dx = 0$$

and

$$\frac{1}{n-1}\,\frac{d}{dt}\,A(t) = \int_{\Gamma(t)} \kappa V\,d\sigma = -\int_{\Gamma(t)} u_\kappa[\partial_n u_\kappa]\,d\sigma = -\int_\Omega |\nabla u_\kappa|^2\,dx \leqslant 0,$$

see [20] for more details, and also [9]. Next note that every Euclidean sphere is an equilibrium for the Mullins–Sekerka model and that in every neighborhood of a fixed sphere there is a continuum of further equilibria. In fact, we will show below that the Mullins–Sekerka model admits a stable local center manifold consisting exactly of those equilibria.

To prove Theorem 1 we use the same approach as in [19, 20]; see also [17, 18]. First we transform the original problem to a system of equations on a fixed reference domain. After a natural reduction of the transformed problem we are led to a nonlinear evolution equation for the motion of $\Gamma(t)$. The propagator of this evolution equation turns out to be a nonlinear, nonlocal pseudo-differential operator of third order. In addition, this operator carries a quasilinear structure of parabolic type. We will then establish the existence of a locally invariant, finite-dimensional center manifold, relying on results proved in [33, 34]. We also show that this manifold attracts solutions at an exponential rate. Moreover, we prove that the center manifold is unique, consisting only of equilibria.

2. MOTION OF THE INTERFACE

In this section we introduce the mathematical setting that will allow us to analyze the qualitative behavior of solutions. Let us assume that $\mathscr{S} = \mathscr{S}_R$ is some fixed sphere of radius R which is contained in Ω. Then $\mathscr{S}$ separates Ω in two domains Ω^1 and Ω^2, with Ω^1 being enclosed by $\mathscr{S}$. In the following we will study the asymptotic properties of solutions of the Mullins–Sekerka model that start in a neighborhood of $\mathscr{S}$; that is, we assume that Γ_0 is close to $\mathscr{S}$. Let ν be the outer unit normal field on $\mathscr{S}$ and let

$$X: \mathscr{S} \times (-a_0, a_0) \to \mathbb{R}^n, \qquad X(s, r) := s + r\nu(s).$$

Then X is a smooth diffeomorphism onto its image $\mathscr{R} := \mathrm{im}(X)$; that is,

$$X \in \mathit{Diff}^{\infty}(\mathscr{S} \times (-a_0, a_0), \mathscr{R}),$$

provided $a_0 > 0$ is small enough. It is convenient to decompose the inverse of X into $X^{-1} = (S, \Lambda)$, where

$$S \in C^{\infty}(\mathscr{R}, \mathscr{S}) \qquad \text{and} \qquad \Lambda \in C^{\infty}(\mathscr{R}, (-a_0, a_0)).$$

Note that $S(x)$ is the nearest point on $\mathscr{S}$ to x, and that $\Lambda(x)$ is the signed distance from x to $\mathscr{S}$ (that is, to $S(x)$). Moreover, the neighborhood $\mathscr{R}$ consists of those points with distance less than a_0 to $\mathscr{S}$.

Let $T > 0$ be a fixed number. We assume that $\Gamma(t)$ is a family of hypersurfaces given by

$$\Gamma(t) := \{x \in \mathbb{R}^n;\, x = X(s, \rho(s, t)),\, s \in \mathscr{S}\}, \qquad t \in [0, T],$$

for a function $\rho: \mathscr{S} \times [0, T] \to (-a_0, a_0)$. Note that the hypersurfaces $\Gamma(t)$ are parameterized over $\mathscr{S}$ by the distance function ρ. In addition, $\Gamma(t)$ is the zero-level set of the function

$$\phi: \mathscr{R} \times [0, T] \to \mathbb{R}, \qquad \phi(x, t) := \Lambda(x) - \rho(S(x), t).$$

If ρ is differentiable with respect to the time variable then we can express the normal velocity V of $\Gamma(t)$ at the point $x = X(s, \rho(s, t))$ as

$$V(s, t) = -\frac{\partial_t \phi(x, t)}{|\nabla_x \phi(x, t)|}\bigg|_{x = X(s, \rho(s, t))} = \frac{\partial_t \rho(s, t)}{|\nabla_x \phi(x, t)|}\bigg|_{x = X(s, \rho(s, t))}.$$

Since the outer unit normal field on $\Gamma(t)$ is given by $n(\cdot, t) = \nabla\phi(\cdot, t)/|\nabla\phi(\cdot, t)|$ we conclude that equation (1.1) which governs the motion of $\Gamma(t)$ takes the form

$$\partial_t \rho(s, t) = -(\nabla u_k^1 - \nabla u_k^2\, |\nabla\phi)|_{X(s, \rho(s, t))}, \qquad \rho(s, 0) = \rho_0(s), \tag{2.1}$$

where $\rho_0 \colon \mathscr{S} \to (-a_0, a_0)$ is a given function determined by Γ_0, and where u_κ satisfies the elliptic boundary value problem (1.2). It is now convenient to transform the elliptic problem (1.2) to an equivalent problem on the fixed reference domains Ω^i for $i \in \{1, 2\}$. To simplify the notation we fix $t \in [0, T]$ and suppress it in the following formulas. Let

$$\mathfrak{A} := \{\rho \in C^2(\mathscr{S});\ \|\rho\|_{C^0(\mathscr{S})} < a_0\}$$

denote the set of admissible parameterizations. For convenience we define for each $\rho \in \mathfrak{A}$ the map

$$\theta_\rho \colon \mathscr{S} \to \mathbb{R}^n, \qquad \theta_\rho(s) = s + \rho(s)\, v(s)$$

and we let $\Gamma_\rho := \operatorname{im}(\theta_\rho)$ denote its image. It follows that θ_ρ is a C^2 diffeomorphism between the hypersurfaces $\mathscr{S}$ and Γ_ρ, provided $a_0 > 0$ is chosen sufficiently small. In addition, we assume that $a_0 > 0$ is small enough so that Γ_ρ is contained in Ω for each $\rho \in \mathfrak{A}$. Then Γ_ρ separates Ω into an interior domain Ω_ρ^1 and an exterior domain Ω_ρ^2. Let κ_ρ denote the mean curvature of the hypersurface Γ_ρ and let $\phi_\rho(x) = \Lambda(x) - \rho(S(x))$.

Let us now introduce an extension of the diffeomorphism θ_ρ to $\mathbb{R}^n$. For this we assume that $a \in (0, a_0/4)$ and we fix a $\varphi \in C^\infty(\mathbb{R}, [0, 1])$ such that $\varphi(\lambda) = 1$ if $|\lambda| \leqslant a$, and $\varphi(\lambda) = 0$ if $|\lambda| \geqslant 3a$, and such that $\sup |\varphi'(\lambda)| < 1/a$. Then we define for each $\rho \in \mathfrak{A}$ the map

$$\Theta_\rho(x) := \begin{cases} X(S(x), \Lambda(x) + \varphi(\Lambda(x))\, \rho(S(x))) & \text{if } \ x \in \mathscr{R}, \\ x & \text{if } \ x \notin \mathscr{R}. \end{cases}$$

The function $[\lambda \mapsto \lambda + \varphi(\lambda)\rho]$ is strictly increasing since $|\varphi'(\lambda)\rho| < 1$. It follows that

$$\Theta_\rho \in \mathit{Diff}^{\,2}(\Omega, \Omega) \cap \mathit{Diff}^{\,2}(\Omega^i, \Omega_\rho^i) \qquad \text{and} \qquad \Theta_\rho \,|\, \mathscr{S} = \theta_\rho$$

for $i \in \{1, 2\}$. Moreover, observe that there exists an open neighborhood U of $\partial\Omega$ such that $\Theta_\rho \mid U = id_U$. It should be mentioned that the above diffeomorphism was first introduced by Hanzawa [23] to transform multidimensional Stefan problems to fixed domains. In the following we use the same symbol θ_ρ for both diffeomorphisms θ_ρ and Θ_ρ. Then we define the transformed differential operators

$$\begin{aligned} \mathscr{A}^i(\rho)v^i &:= (\Delta(v^i \circ \theta_\rho{}^{-1})) \circ \theta_\rho, \\ \mathscr{B}^i(\rho)v^i &:= \gamma^i((\nabla(v^i \circ \theta_\rho{}^{-1}) \mid \nabla\phi_\rho) \circ \theta_\rho) \end{aligned} \tag{2.2}$$

for $v^i \in C^2(\Omega^i) \cap C^1(\bar{\Omega}^i)$, $i \in \{1, 2\}$, and $\rho \in \mathfrak{U}$, where γ^i denotes the restriction operator from Ω^i to $\mathscr{S}$. These operators act linearly on the space $C^2(\Omega^i) \cap C^1(\bar{\Omega}^i)$. We set $\mathscr{A}(\rho)v := (\mathscr{A}^1(\rho)v^1, \mathscr{A}^2(\rho)v^2)$ for $v = (v^1, v^2) \in C^2(\Omega^1) \times C^2(\Omega^2)$. We also introduce the transformed mean curvature operator

$$K(\rho) := \kappa_\rho \circ \theta_\rho \qquad \text{on } \mathscr{S}, \quad \rho \in \mathfrak{U}. \tag{2.3}$$

Let $\rho_0 \in \mathfrak{U}$ be given and set $\Gamma_0 = \Gamma_{\rho_0}$. Based on the above transformed operators we can now express the motion equation (2.1) by an evolution equation on $\mathscr{S}$,

$$\partial_t \rho + \mathscr{B}(\rho)\, v(\rho) = 0, \qquad \rho(0) = \rho_0. \tag{2.4}$$

Here $v(\rho)$ is the solution of the transformed elliptic boundary value problem

$$\begin{cases} \mathscr{A}(\rho)v = 0 & \text{in } \Omega^1 \cup \Omega^2 \\ \quad v = K(\rho) & \text{on } \mathscr{S} \\ \partial_n v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

and $\mathscr{B}$ arises as the transform of the right hand side in (2.1)

$$\mathscr{B}(\rho)\, v := \mathscr{B}^1(\rho)v^1 - \mathscr{B}^2(\rho)v^2 \qquad \text{on} \quad \mathscr{S}. \tag{2.6}$$

Observe that $u = v \circ \theta_\rho^{-1}$ is the unique solution of the elliptic problem (1.2) if and only if v is the unique solution of (2.5). We are now left with finding a solution $\rho: \mathscr{S} \times [0, T] \to (-a, a)$ for the evolution equation (2.4) and, simultaneously, with finding a solution $v(\rho)$ for the the elliptic problem (2.5). Again, it should be mentioned that the equations (2.4) and (2.5) are coupled.

3. THE MEAN CURVATURE OPERATOR

In this section we collect some useful properties of the mean curvature operator $K(\rho)$ defined in (2.3). In order to give precise results, let us introduce the following notation. Given an open set Ω of $\mathbb{R}^n$, let $h^s(\Omega)$ denote the little Hölder spaces of order $s>0$; that is, the closure of $BUC^\infty(\Omega)$ in $BUC^s(\Omega)$, the Banach space of all bounded and uniformly Hölder continuous functions of order s. If M is a (sufficiently) smooth submanifold of $\mathbb{R}^n$ then the spaces $h^s(M)$ are defined by means of a smooth atlas for M. Finally, we define $U := h^{2+\alpha}(\mathcal{S}) \cap \mathfrak{A}$ for a fixed $\alpha \in (0, 1)$.

LEMMA 3.1. *There exist functions*

$$P \in C^\infty(U, \mathcal{L}(h^{3+\alpha}(\mathcal{S}), h^{1+\alpha}(\mathcal{S}))) \qquad \text{and} \qquad Q \in C^\infty(U, h^{1+\alpha}(\mathcal{S}))$$

such that

$$K(\rho) = P(\rho)\rho + Q(\rho) \qquad \text{for} \quad \rho \in U \cap h^{3+\alpha}(\mathcal{S}).$$

The derivative of the mean curvature operator K at $\rho = 0$ is given by

$$D := D_{\mathcal{S}} := -\frac{1}{n-1}\left(\frac{n-1}{R^2} + \Delta_{\mathcal{S}}\right),$$

where $\Delta_{\mathcal{S}}$ denotes the Laplace–Beltrami operator on $\mathcal{S}$, cf. [3].

Proof. (a) Let $\rho \in U$ be given. Then the mean curvature $K(\rho)$ of Γ_ρ is given as

$$K(\rho)(s) = \frac{1}{n-1}\operatorname{div}\left(\frac{\nabla\phi_\rho}{|\nabla\phi_\rho|}\right)\Bigg|_{X(s,\rho(s))}, \qquad s \in \mathcal{S}.$$

The diffeomorphism X induces a Riemannian metric g_X on $\mathcal{S} \times (-a, a)$. Let ∇_X, Δ_X, and hess_X, respectively, denote the gradient, the Laplace–Beltrami operator, and the Hessian with respect to $(\mathcal{S} \times (-a, a), g_X)$. Then the mean curvature $K(\rho)$ can be expressed in terms of the differential operators ∇_X, Δ_X, and hess_X as

$$K(\rho)(s) = \frac{1}{(n-1)\,\|\nabla_X\Phi_\rho\|_X}\left(\Delta_X\Phi_\rho - \frac{\operatorname{hess}_X\Phi_\rho(\nabla_X\Phi_\rho, \nabla_X\Phi_\rho)}{\|\nabla_X\Phi_\rho\|_X^2}\right)\Bigg|_{(s,\rho(s))} \tag{3.1}$$

for $s \in \mathcal{S}$, where we use the notation $\Phi_\rho(s, r) := \phi_\rho(X(s, r)) = r - \rho(s)$, and

$$\|\nabla_X\Phi_\rho\|_X := \sqrt{g_X(\nabla_X\Phi_\rho, \nabla_X\Phi_\rho)}\,.$$

(b) Next, we express the mean curvature in local coordinates. To make this precise we need a few notations. Let $\{(U_\ell, \varphi_\ell); 1 \leqslant \ell \leqslant L\}$ be a localization system for the manifold $\mathscr{S}$, that is, $\mathscr{S} = \bigcup_{\ell=1}^{L} U_\ell$ and

$$\varphi_\ell : (-a, a)^{n-1} \to U_\ell, \qquad \ell \in \{1, ..., L\},$$

is a smooth local parameterization of U_ℓ. Let $s = (s_1, \cdots, s_{n-1})$ be the local coordinates of U_ℓ with respect to this parameterization. In addition, let

$$\rho_\ell(s) := \rho(\varphi_\ell(s)), \qquad X_\ell(s, r) := X(\varphi_\ell(s), r), \quad (s, r) \in (-a, a)^n,$$

be the corresponding local representations of the mappings ρ and X. In the following we often employ the same notation for the mappings ρ, X and their local representations ρ_ℓ, X_ℓ. Moreover, we do not always distinguish between the local coordinates $s \in (-a, a)^{n-1}$, and the corresponding points $\varphi_\ell(s)$ on $\mathscr{S}$. We define

$$w_{jk}(\rho)(s) := (\partial_j X \mid \partial_k X)|_{(s, \rho(s))}, \qquad s \in (-a, a)^{n-1},$$

for $j, k \in \{1, ..., n-1\}$. If ρ is sufficiently small then $[w_{jk}(\rho)]$ is invertible and we denote its inverse by $[w^{jk}(\rho)]$. Let

$$\Gamma^i_{jk}(\rho) := w^{im}(\rho)\, (\partial_j \partial_k X \mid \partial_m X)|_{(\cdot, \rho(\cdot))} \qquad \text{on} \quad (-a, a)^{n-1},$$

where $i \in \{1, ..., n-1\}$, $j, k \in \{1, ..., n\}$, and $\Gamma^n_{jk}(\rho) := (\partial_j \partial_k X \mid \partial_n X)|_{(\cdot, \rho(\cdot))}$ for $j, k \in \{1, ..., n-1\}$. Here we use the convention of summation over repeated indices. Finally, we define

$$l_\rho := \sqrt{1 + w^{jk}(\rho)\, \partial_j \rho\, \partial_k \rho}\,.$$

By using well-known representation formulas for ∇_X, Δ_X, and hess_X in local coordinates, and the orthogonality relations

$$(\partial_j X(s, r) \mid \partial_n X(s, r)) = \delta_{jn}, \qquad (s, r) \in (-a, a)^n, \qquad j \in \{1, ..., n\}, \quad (3.2)$$

we find the expression

$$K_\ell(\rho) = \frac{1}{n-1} \left(\sum_{j,k=1}^{n-1} p_{jk}(\rho)\, \partial_j \partial_k \rho + \sum_{i=1}^{n-1} p_i(\rho)\, \partial_i \rho + q(\rho) \right), \quad (3.3)$$

for the mean curvature in local coordinates, where

$$p_{jk}(\rho)=\frac{1}{l_\rho^3}\,(-l_\rho^2 w^{jk}(\rho)+w^{jl}(\rho)\,w^{km}(\rho)\,\partial_l\rho\,\partial_m\rho),$$

$$p_i(\rho)=\frac{1}{l_\rho^3}\,(l_\rho^2 w^{jk}\Gamma^i_{jk}+w^{jl}w^{ki}\Gamma^n_{jk}\partial_l\rho+2w^{km}\Gamma^i_{nk}\partial_m\rho-w^{jl}w^{km}\Gamma^i_{jk}\,\partial_l\rho\,\partial_m\rho),$$

$$q(\rho)=-\frac{1}{l_\rho}\,w^{jk}(\rho)\,\Gamma^n_{jk}(\rho).$$

(c) It follows that $K(\rho)$ admits the decomposition $K(\rho)=P(\rho)\rho+Q(\rho)$, where $P(\rho)$ and $Q(\rho)$ have the following representations in local coordinates

$$P_\ell(\rho):=\frac{1}{(n-1)}\,(p_{jk}(\rho)\,\partial_j\partial_k+p_i(\rho)\partial_i),\qquad Q_\ell(\rho):=\frac{1}{(n-1)}\,q(\rho). \tag{3.4}$$

In addition, we can conclude that the mappings P and Q depend smoothly upon ρ. Hence the operator $[\rho\mapsto K(\rho)]$ is differentiable and the linearization at 0 is given by

$$Dh=P(0)h+\partial Q(0)h\qquad\text{for}\quad h\in h^{3+\alpha}(\mathscr{S}). \tag{3.5}$$

(d) Now we show that $P(0)=-(1/(n-1))\Delta_{\mathscr{S}}$. In order to see this, note that $X_\ell(s,0)=\varphi_\ell(s)$ for $s\in(-a,a)^{n-1}$. Hence the mapping $[s\mapsto X_\ell(s,0)]$ is a parameterization of U_ℓ and

$$P_\ell(0)=-\frac{1}{(n-1)}\,w^{jk}(0)(\partial_j\,\partial_k-\Gamma^i_{jk}(0)\,\partial_i)$$

turns out to be a representation of the Laplace–Beltrami operator $-(n-1)^{-1}\Delta_{\mathscr{S}}$ in local coordinates, with $\Gamma^i_{jk}(0)$ being the Christoffel symbols on $\mathscr{S}$. Finally, we show that $Q_\ell(\rho)=(l_\rho(R+\rho))^{-1}$. Without loss of generality we can assume that $\mathscr{S}$ has its center at the origin of $\mathbb{R}^n$. Consequently, $X_\ell(s,r)=((R+r)/R)\,X_\ell(s,0)$ and $w^{jk}(\rho)=(R/(R+\rho))^2\,w^{jk}(0)$. Moreover

$$\begin{aligned}(\partial_j\,\partial_k X_\ell(s,r)\mid\partial_n X_\ell(s,r)) &= -(\partial_k X_\ell(s,r)\mid\partial_j\,\partial_n X_\ell(s,r))\\ &= -((R+r)/R^2)\,w_{jk}(0)(s)\end{aligned}$$

where we used the orthogonality relations (3.2). Therefore,

$$-w^{jk}(\rho)\,\Gamma^n_{jk}(\rho)=(R+\rho)^{-1}\,w^{jk}(0)\,w_{jk}(0)=(n-1)(R+\rho)^{-1}.$$

Since the derivative of l_ρ vanishes at $\rho = 0$ and since $l_0 = 1$ we conclude that $\partial Q_\ell(0)\, h = -R^{-2} h$. This completes the proof of Lemma 3.1. ∎

Remarks 3.2. (a) Observe that the local representations (3.3), (3.4) are valid for any smooth hypersurface $\mathscr{S}$ in $\mathbb{R}^n$, see also Lemmas 3.2 and 3.3 in [20].

(b) If $\mathscr{S}$ is an arbitrary, smooth surface in $\mathbb{R}^3$ then it has been shown in [3], Appendix, that $D = -(\Delta_{\mathscr{S}} + (\kappa_1^2 + \kappa_2^2))/2$, where κ_1 and κ_2 are the principal curvatures of $\mathscr{S}$. This result contains the second part of Lemma 3.1 as a special case, at least if $n = 3$. We also refer to [1, 24, 30] for spectral information relevant to the present work.

4. THE REDUCED EQUATION

In this section we reduce the coupled equations (2.4) and (2.5) to a single evolution equation for the distance function ρ only. For the reader's convenience we state some relevant results which are proved in [20]. For the following lemma we refer to (2.2) where the definition of the operators $\mathscr{A}^i(\rho)$ and $\mathscr{B}^i(\rho)$ is given.

Lemma 4.1. *Let $\rho \in U$ be given. Then the elliptic boundary value problem*

$$\begin{cases} \mathscr{A}^i(\rho)\, v^i = 0 & \text{in } \Omega^i \\ v^i = g & \text{on } \mathscr{S} \\ \partial_n v^i = 0 & \text{on } \partial\Omega \cap \bar{\Omega}^i \end{cases} \tag{4.1}$$

has a unique solution $v^i = T^i(\rho)\, g \in h^{1+\alpha}(\Omega^i)$ for each $g \in h^{1+\alpha}(\mathscr{S})$ and

$$[\rho \mapsto T^i(\rho)] \in C^\omega(U, \mathscr{L}(h^{1+\alpha}(\mathscr{S}), h^{1+\alpha}(\Omega^i))).$$

Moreover,

$$[\rho \mapsto \mathscr{B}^i(\rho)] \in C^\omega(U, \mathscr{L}(h^{1+\alpha}(\Omega^i), h^\alpha(\mathscr{S}))).$$

Proof. We refer to Lemmas 2.2 and 2.3 in [20]. ∎

After this preparation we set

$$\mathscr{B}(\rho)\, T(\rho)\, g := \mathscr{B}^1(\rho)\, T^1(\rho)\, g - \mathscr{B}^2(\rho)\, T^2(\rho)\, g \tag{4.2}$$

for $\rho \in U$ and $g \in h^{1+\alpha}(\mathscr{S})$. We mention that $\mathscr{B}^i(\rho)\, T^i(\rho)$ is a pseudo-differential operator of first order for $i = 1, 2$, which is called the generalized

Dirichlet–Neumann operator, see [16]. We can now introduce the mapping

$$H(\rho) := \mathscr{B}(\rho)\, T(\rho)\, K(\rho) \qquad \text{for} \quad \rho \in U_1, \tag{4.3}$$

where $U_1 := U \cap h^{3+\alpha}(\mathscr{S})$. Note that H is a nonlinear, nonlocal operator of third order, mapping an open subset of $h^{3+\alpha}(\mathscr{S})$ into $h^{\alpha}(\mathscr{S})$. The coupled set of equations (2.4), (2.5) can now be merged into a single evolution equation,

$$\partial_t \rho + H(\rho) = 0, \qquad \rho(0) = \rho_0, \tag{4.4}$$

as follows from Lemma 4.1. It has been shown in Section 3 that the mean curvature K has a quasilinear structure. As a consequence, the mapping $[\rho \mapsto H(\rho)]$ defined in (4.3) inherits a quasilinear structure as well, and the evolution equation (4.4) can be rewritten as

$$\partial_t \rho + \mathscr{B}(\rho)\, T(\rho)\, P(\rho)\rho = -\mathscr{B}(\rho)\, T(\rho)\, Q(\rho), \qquad \rho(0) = \rho_0. \tag{4.5}$$

To investigate the evolution equation (4.5) we can use the theory of abstract quasilinear evolution equations of parabolic type developed by Amann [5, 6]; see also [7]. A thorough knowledge of the linear part $\mathscr{B}(\rho)\, T(\rho)\, P(\rho)$ is essential in order to apply this theory. For this, let E_0 and E_1 be Banach spaces such that E_1 is densely injected in E_0 and let $\mathscr{H}(E_1, E_0)$ denote the set of all $A \in \mathscr{L}(E_1, E_0)$ such that $-A$ is the generator of a strongly continuous analytic semigroup on E_0. We can now state the main result of this section.

THEOREM 4.2. *$\mathscr{B}(\rho)\, T(\rho)\, P(\rho) \in \mathscr{H}(h^{3+\alpha}(\mathscr{S}), h^{\alpha}(\mathscr{S}))$ for $\rho \in U$.*

Proof. This is a special case of a more general result obtained in [20], where an arbitrary smooth hypersurface $\mathscr{S}$ was considered. ∎

Let us state the following local existence, uniqueness, and regularity result for the evolution equation (4.5).

THEOREM 4.3. *Let $\beta \in (\alpha, 1)$ be fixed and let ρ_0 in $V := U \cap h^{2+\beta}(\mathscr{S})$ be given. Then the evolution equation* (4.5) *has a unique maximal solution*

$$\rho \in C([0, t^+(\rho_0)), V) \cap C^{\infty}(\mathscr{S} \times (0, t^+(\rho_0))),$$

where $[0, t^+(\rho_0))$ denotes the maximal interval of existence. The map $(t, \rho_0) \mapsto \rho(t, \rho_0)$ defines a smooth semiflow on V.

Proof. Let $\rho_0 \in V$ be given. The existence of a unique maximal solution

$$\rho \in C([0, t^+(\rho_0)), V) \cap C^1((0, t^+(\rho_0)), h^\alpha(\mathcal{S})) \cap C((0, t^+(\rho_0)), h^{3+\alpha}(\mathcal{S}))$$

to problem (4.5) follows from Theorem 12.1 in [5]. Moreover, the results in [5, Sect. 12] also show that (4.5) generates a smooth semiflow on V. The fact that the solution is smooth in space and time is based on a bootstrapping argument in the scale $h^{l+\alpha}(\mathcal{S})$, $l \in \mathbb{N}$. We refer to [20, Sect. 4] where the details are provided. ∎

The unique solution of the evolution equation (4.5) constitutes the unique solution of the Mullins–Sekerka model.

5. THE LINEARIZATION

In order to further analyze the long-time behavior of solutions we will now study the mapping H introduced in (4.3) in more detail. Recall that $[\rho \mapsto H(\rho)] \in C^\infty(U_1, h^\alpha(\mathcal{S}))$, where $U_1 = U \cap h^{3+\alpha}(\mathcal{S})$. Hence

$$L := \partial H(0) \in \mathcal{L}(h^{3+\alpha}(\mathcal{S}), h^\alpha(\mathcal{S})) \tag{5.1}$$

is well defined. For convenience we will always hereafter use the notation

$$\mathcal{B}^i := \mathcal{B}^i(0) \qquad \text{and} \qquad T^i := T^i(0), \qquad i = 1, 2.$$

We recall that $\mathcal{B}^i(0)$ and $T^i(0)$ were introduced in (2.2) and in Lemma 4.1, respectively.

LEMMA 5.1. *We have $L = \mathcal{B}TD$, where D is defined in Lemma* 3.1. *In addition, L belongs to $\mathcal{H}(h^{3+\alpha}(\mathcal{S}), h^\alpha(\mathcal{S}))$.*

Proof. It follows from (4.3) and from Lemmas 3.1 and 4.1 that

$$Lh = \partial(\mathcal{B}(\rho)\, T(\rho))|_{\rho=0}\, [h, K(0)] + \mathcal{B}T\, Dh$$

for $h \in h^{3+\alpha}(\mathcal{S})$. Observe that $K(0)$ is the mean curvature of the sphere $\mathcal{S}_R$, and hence $K(0) = R^{-1}$. We conclude that

$$\mathcal{B}(\rho)\, T(\rho)\, K(0) = 0 \qquad \text{for all} \quad \rho \in U.$$

Indeed, let $v^i := R^{-1}$ on Ω^i. Then we see that v^i is a solution of the elliptic boundary value problem (4.1) with $g = R^{-1}$, and it follows from Lemma 4.1 that $v^i = T^i(\rho)\, K(0)$. It is then readily seen that $\mathcal{B}^i(\rho) v^i = 0$. Hence,

$$\partial(\mathcal{B}(\rho) T(\rho))|_{\rho=0}\,[h, K(0)] = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{B}(\varepsilon h)\, T(\varepsilon h)\, K(0) = 0,$$

showing that $Lh = \mathcal{B}T\, Dh$. (6.6) below yields $h^{2+\alpha}(\mathcal{S}) = (h^{\alpha}(\mathcal{S}), h^{3+\alpha}(\mathcal{S}))_{2/3}$, where $(\cdot, \cdot)_\theta$ with $\theta \in (0, 1)$ is the the continuous interpolation functor. The remaining assertion is now a consequence of (3.5), Lemma 3.1, Theorem 4.2, [6, Sect. 2.4.4 and Eq. (2.2.2)], and a well-known perturbation result, e.g., [31, Theorem 3.2.1]. ▮

Remarks 5.2. (a) Observe that $\mathcal{B}^i v^i$ is the derivative of v^i in direction of v at $\mathcal{S}$. Moreover, $v^i = T^i g \in h^{1+\alpha}(\Omega^i)$ is the harmonic extension of g on Ω^i, that is, the solution of the elliptic problem

$$\begin{cases} \Delta v^i = 0 & \text{in } \Omega^i \\ v^i = g & \text{on } \mathcal{S} \\ \partial_n v^i = 0 & \text{on } \partial\Omega \cap \bar{\Omega}^i \end{cases} \tag{5.2}$$

where $g \in h^{1+\alpha}(\mathcal{S})$ is a given function.

(b) 0 is an eigenvalue of $\mathcal{B}T$ and $\ker(\mathcal{B}T) = \operatorname{span}\{\mathbf{1}\}$, where $\mathbf{1}(x) := 1$ for $x \in \mathcal{S}$.

Proof. Suppose that $\mathcal{B}Tg = 0$ for some $g \in h^{1+\alpha}(\mathcal{S})$. Let $v := Tg$ be the harmonic extension of g on $\Omega^1 \cup \Omega^2$, see part (a). Then we obtain, after multiplying this identity with $\bar{g}$ and using the divergence theorem,

$$(\mathcal{B}Tg \mid g) = \int_{\Omega^1 \cup \Omega^2} |\nabla v|^2\, dx = 0. \tag{5.3}$$

Observe that (5.3) can be established by first replacing g with a smooth function, using the divergence theorem, and then passing to the limit. Hence v, and therefore $g = v \mid \mathcal{S}$ are constant. Clearly, $\mathcal{B}T\mathbf{1} = 0$. Consequently, 0 is an eigenvalue with eigenspace spanned by $\mathbf{1}$. ▮

Given $r > 0$, let

$$h_0^r(\mathcal{S}) := \left\{ g \in h^r(\mathcal{S});\ \int_{\mathcal{S}} g\, d\sigma = 0 \right\}$$

denote the space of all functions in $h^r(\mathcal{S})$ having zero average.

In the sequel we will always employ the natural complexification in connection with spectral theory without distinguishing this notationally. The following result will be useful in order to locate the spectrum of L.

LEMMA 5.3. *$\mathscr{B}T \in Isom(h_0^{1+\alpha}(\mathscr{S}), h_0^{\alpha}(\mathscr{S}))$. Moreover,*

$$((\mathscr{B}T)^{-1}h \mid h) > 0, \qquad h \in h_0^{\alpha}(\mathscr{S}) \backslash \{0\}.$$

Proof. Let $N := \text{span}\{\mathbf{1}\}$ denote the kernel of $\mathscr{B}T$; see Remark 5.2(b). Moreover, let

$$P_1 g := \frac{1}{|\mathscr{S}|} \int_{\mathscr{S}} g \, d\sigma \, \mathbf{1}, \qquad g \in h^r(\mathscr{S}),$$

where $|\mathscr{S}|$ stands for the area of $\mathscr{S}$. Then P_1 is a continuous projection of $h^r(\mathscr{S})$ onto N, parallel to $N^c := \text{im}(I - P_1)$. It follows that the space $h^r(\mathscr{S})$ admits a direct topological decomposition $h^r(\mathscr{S}) = N \oplus N^c$, with $(I - P_1)$ being the projection onto N^c, parallel to N. Observe that $\text{im}(I - P_1) = \ker P_1$, which implies that $N^c = h_0^r(\mathscr{S})$. Next, observe that

$$\int_{\mathscr{S}} \mathscr{B}Tg \, d\sigma = \int_{\Omega^1 \cup \Omega^2} \Delta v \, dx = 0, \tag{5.4}$$

where $g \in h^{1+\alpha}(\mathscr{S})$, and where v is the harmonic extension of g on $\Omega^1 \cup \Omega^2$. We conclude that $P_1 \mathscr{B}T = \mathscr{B}TP_1 = 0$. Hence, the decomposition reduces $\mathscr{B}T$. Let $(\mathscr{B}T)^c$ be the part of $\mathscr{B}T$ in N^c. Since $h^{1+\alpha}(\mathscr{S})$ is compactly embedded in $h^{\alpha}(\mathscr{S})$ we conclude that $\mathscr{B}T$ has a compact resolvent. Consequently, the spectrum of $\mathscr{B}T$ consists only of eigenvalues. If follows that the same conclusion is also true for the spectrum of $(\mathscr{B}T)^c$. Since $(\mathscr{B}T)^c$ has trivial kernel, we conclude that $\mathscr{B}T$ is an isomorphism from $h_0^{1+\alpha}(\mathscr{S})$ into $h_0^{\alpha}(\mathscr{S})$.

Let $h \in h_0^{\alpha}(\mathscr{S}) \backslash \{0\}$ be given and set $g := (\mathscr{B}T)^{-1}h$. Then it follows as in (5.3) that

$$((\mathscr{B}T)^{-1} h \mid h) = (g \mid \mathscr{B}Tg) = \int_{\Omega^1 \cup \Omega^2} |\nabla v|^2 \, dx.$$

Since g has zero average, we see that $\int_{\Omega^1 \cup \Omega^2} |\nabla v|^2 \, dx > 0$. This concludes the proof of Lemma 5.3. ∎

We are now ready to characterize the spectrum of $-L$.

PROPOSITION 5.4. *The spectrum of* $-L$ *consists of real eigenvalues* $\{\mu_k; k \in \mathbb{N}\}$ *such that*

$$\cdots < \mu_{k+1} < \mu_k < \mu_{k-1} < \cdots < \mu_1 < \mu_0 = 0.$$

Moreover, 0 *is an eigenvalue of geometric multiplicity* $(n+1)$.

Proof. (a) It follows from Lemma 5.1 and the fact that $h^{3+\alpha}(\mathcal{S})$ is compactly embedded in $h^{\alpha}(\mathcal{S})$ that the spectrum of L consists entirely of eigenvalues.

(b) Let us first assume that $\mathcal{S} = S^n$ is the unit sphere centered at 0. We show that 0 is an eigenvalue of multiplicity $n+1$. Suppose $g \in h^{3+\alpha}(S^n)$ satisfies

$$\mathcal{B}T((n-1) + \Delta_{S^n})\, g = 0. \tag{5.5}$$

Then it follows from Remark 5.2(b) that

$$((n-1) + \Delta_{S^n})\, g = c \tag{5.6}$$

for some constant c. Note that $g_0 = (n-1)^{-1} c$ is a solution of (5.6). Any other solution of (5.5) in $h_0^{3+\alpha}(S^n)$ satisfies the homogeneous equation

$$((n-1) + \Delta_{S^n})\, g = 0. \tag{5.7}$$

A well-known result now implies that (5.7) has n linearly independent solutions, the spherical harmonics $\{Y_m;\ 1 \leqslant m \leqslant n\}$ of degree 1; see [35]. We conclude that $\{\mathbf{1},\ Y_m;\ 1 \leqslant m \leqslant n\}$ is a set of linearly independent solutions for the eigenvalue problem (5.5). Moreover, there exists a number $\gamma > 0$ such that

$$(D_{S^n} g \mid g) \geqslant \gamma (g \mid g), \qquad g \in h^{3+\alpha}(S^n), \quad g \perp \operatorname{span}\{\mathbf{1},\ Y_m;\ 1 \leqslant m \leqslant n\}. \tag{5.8}$$

Here the symbol $\perp$ indicates that g is orthogonal to the indicated subspace with respect to the scalar product $(\cdot \mid \cdot)$ in $L_2(S^n)$.

(c) Next we show that the remaining eigenvalues of $-\mathcal{B}TD_{S^n}$ are contained in $(-\infty, 0)$. Suppose that $z \in \mathbb{C} \backslash \{0\}$ and

$$(z + \mathcal{B}TD_{S^n})\, g = 0 \tag{5.9}$$

for some $g \in h^{3+\alpha}(S^n)$. It follows from (5.4) that $\mathcal{B}TD_{S^n} g$ has zero average, that is, $(\mathcal{B}TD_{S^n} g \mid \mathbf{1}) = 0$. We claim that g also has zero average. Indeed, since $(\mathcal{B}TD_{S^n} g \mid \mathbf{1}) = 0$ we conclude from (5.9) that $z(g \mid \mathbf{1}) = 0$. The assumption that $z \in \mathbb{C} \backslash \{0\}$ yields the claim. Consequently, we can apply

$(\mathscr{B}T)^{-1}$ to the identity in (5.9), see Lemma 5.3. We obtain, after multiplying the result with $\bar{g}$ and integrating over S^n,

$$z((\mathscr{B}T)^{-1} g \mid g) + (D_{S^n} g \mid g) = 0.$$

Step (b) shows that $g = g_1 + g_2$ with $g_2 \neq 0$, where $g_1 \in \text{span}\{\mathbf{1}, Y_m; 1 \leqslant m \leqslant n\}$ and $g_2 \perp \text{span}\{\mathbf{1}, Y_m; 1 \leqslant m \leqslant n\}$. We conclude that

$$z((\mathscr{B}T)^{-1} g \mid g) + (D_{S^n} g_2 \mid g_2) = 0,$$

using that D_{S^n} is symmetric on $L_2(S^n)$. Since g has zero average, Lemma 5.3 and (5.8) now imply that $z \in (-\infty, 0)$. Therefore, the spectrum of $-\mathscr{B}TD_{S^n}$ consists of a sequence of real numbers

$$\cdots < \mu_{k+1} < \mu_k < \mu_{k-1} < \cdots < \mu_1 < \mu_0 = 0$$

and μ_0 is an eigenvalue of multiplicity $(n+1)$.

(d) Suppose now that $\mathscr{S}$ is a sphere of radius R. Since the Mullins–Sekerka model is translation invariant we can assume without loss of generality that $\mathscr{S}$ has its center at the origin. Observe that

$$D_{\mathscr{S}} g = R^{-2} [\theta_R^*]^{-1} (D_{S^n} \theta_R^* g), \qquad g \in C^2(\mathscr{S}),$$

where θ_R stands for dilation with factor R and $\theta_R^* \in Isom(h^{3+\alpha}(\mathscr{S}), h^{3+\alpha}(S^n))$, $[\theta_R^* g](x) := g(Rx)$, denotes the corresponding pull-back operator. Then everything proved in steps (b)–(c) remains valid, where (5.8) is now replaced by

$$(D_{\mathscr{S}} g \mid g) \geqslant R^{-2} \gamma (g \mid g), \qquad g \in h^{3+\alpha}(\mathscr{S}), \quad g \perp \text{span}\{\mathbf{1}, Y_m^R; \ 1 \leqslant m \leqslant n\}.$$

and where Y_m^R are the spherical harmonics defined on the R-sphere $\mathscr{S}$. ∎

Remark 5.5. A more general result for multiple spheres will be proved in [4]. In addition, it will be shown that the operator A admits a self-adjoint realization on a properly chosen Hilbert space.

6. THE CENTER MANIFOLD

In this section we prove the existence of a locally invariant, $(n+1)$-dimensional center manifold for the evolution equation (4.5). We also show that this manifold attracts solutions at an exponential rate. Moreover, we prove that the center manifold is unique, consisting only of equilibria.

We first recall that any sphere in Ω is an equilibrium for the Mullins–Sekerka model. As in the previous sections, we fix a sphere $\mathscr{S}$ of radius R

which we assume to be centered at the origin of $\mathbb{R}^n$. Clearly, 0 is an equilibrium for the evolution equation (4.4). Note that (4.4) can be rewritten as

$$\partial_t \rho + L\rho = g(\rho), \qquad \rho(0) = \rho_0, \tag{6.1}$$

where

$$g(\rho) = L\rho - H(\rho),$$

and where L is the linearization of the mapping $[\rho \mapsto H(\rho)]$, see Lemma 5.1. It should be observed that (6.1) is now to be considered as a fully nonlinear evolution equation. In order to obtain a locally invariant center manifold for (6.1) we will resort to the theory of maximal regularity [13].

Next, recall that the kernel of L is the space $X_c := \operatorname{span}\{ Y_m;\ 0 \leqslant m \leqslant n\}$ spanned by $Y_0 = \mathbf{1}$ and by the n linearly independent spherical harmonics Y_m of degree 1, see the proof of Proposition 5.4. We can assume that

$$Y_m = R^{-1} p_m \mid \mathscr{S}, \qquad 1 \leqslant m \leqslant n, \tag{6.2}$$

where p_m is a harmonic polynomial of degree 1 given by $p_m(x) = x_m$ for $x \in \mathbb{R}^n$, and where $p_m \mid \mathscr{S}$ stands for the restriction of p_m to $\mathscr{S}$.

Since X_c is a finite-dimensional subspace of $h^{3+\alpha}(\mathscr{S})$, it is topologically complemented in $h^{3+\alpha}(\mathscr{S})$. Our next result shows that we can find a complementary subspace such that the corresponding direct topological sum reduces L.

Lemma 6.1. *There exists a closed subspace $h_s^{3+\alpha}(\mathscr{S})$ of $h^{3+\alpha}(\mathscr{S})$ such that $h^{3+\alpha}(\mathscr{S}) = X_c \oplus h_s^{3+\alpha}(\mathscr{S})$ is a direct topological sum which reduces L.*

Proof. (a) Note that $(Y_m \mid \mathbf{1}) = 0$ for $1 \leqslant m \leqslant n$, where $(\cdot \mid \cdot)$ denotes the scalar product in $L_2(\mathscr{S})$. Hence the spherical harmonics Y_m have zero average and $H := \operatorname{span}\{ Y_1, ..., Y_n\}$ is a subspace of $h_0^{3+\alpha}(\mathscr{S})$. Lemma 5.3 shows that

$$\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R}, \qquad \langle g, f \rangle := (g \mid (\mathscr{B}T)^{-1} f), \tag{6.3}$$

defines a scalar product on H. Let $\{\eta_1, \cdots, \eta_n\}$ be an orthogonal basis of H with respect to the scalar product (6.3). Then the mapping P_H defined by

$$P_H g := \sum_{m=1}^{n} \langle g, \eta_m \rangle \eta_m, \qquad g \in h^r(\mathscr{S}), \quad r > 0,$$

provides a continuous projection of $h^r(\mathcal{S})$ onto H. Next we show that $P_H Lg=0$ for each $g\in h^{3+\alpha}(\mathcal{S})$. Indeed, it follows from Lemma 5.1 and from the symmetry of $\mathcal{B}T$ and D that

$$P_H Lg = \sum_{m=1}^{n} (Lg \mid (\mathcal{B}T)^{-1}\eta_m)\eta_m = \sum_{m=1}^{n} (g \mid D\eta_m)\eta_m.$$

The statement follows now from $D\eta_m=0$ for $1\leqslant m\leqslant n$. Since $P_H g$ belongs to the kernel of L we see that $LP_H g=0$ for $g\in h^{3+\alpha}(\mathcal{S})$. Hence we have proved that $P_H L = LP_H$.

(b) Let $P_1 g := |\mathcal{S}|^{-1}(g \mid \mathbf{1})\mathbf{1}$ and set $Pg := P_1 g + P_H g$ for $g\in h^r(\mathcal{S})$. It follows that $P_1 P_H = P_H P_1 = 0$, where $P_H P_1 = 0$ is as consequence of Lemma 5.3. Therefore, we infer that P is a continuous projection of $h^r(\mathcal{S})$ onto X_c parallel to $h^r_s(\mathcal{S}) := \mathrm{im}(I_{h^r(\mathcal{S})} - P)$. Moreover, it follows from step (a) and from (5.4) that $PLg = LPg$ for $g\in h^{3+\alpha}(\mathcal{S})$. We conclude that the decomposition $X_c \oplus h^{3+\alpha}_s(\mathcal{S})$ reduces L. ∎

The invariant spaces X_c and $h^{3+\alpha}_s(\mathcal{S})$ are usually called the center subspace and the stable subspace of $-L$. For this reason we will from now on denote the projections onto X_c and $h^{3+\alpha}_s(\mathcal{S})$ by π^c and π^s, respectively.

Given any $\rho_0\in V$, let $\rho(\cdot,\rho_0)$ denote the solution of (4.5) and let $[0, t^+(\rho_0))$ be its maximal interval of existence. Let $k\in\mathbb{N}\backslash\{0\}$ be fixed and set $\eta := (2+\beta-\alpha)/3$.

PROPOSITION 6.2. (a) *There exists an open neighborhood $\mathcal{O}$ of 0 in X_c and a mapping*

$$\gamma \in C^k(\mathcal{O}, h^{3+\alpha}_s(\mathcal{S})) \qquad \textit{with} \quad \gamma(0)=0, \qquad \partial\gamma(0)=0, \tag{6.4}$$

such that the manifold $\mathcal{M}^c := \mathrm{graph}(\gamma) \subset h^{3+\alpha}(\mathcal{S})$ is locally invariant for the evolution equation (4.5) or (6.1). $\mathcal{M}^c$ contains all small global solutions of (6.1).

(b) *Let $\omega\in(0,-\mu_1)$ be given, where μ_1 is the first nonzero eigenvalue of $-L$, as in Proposition 5.4. Then there exists $c=c(\omega,\alpha,\beta)>0$ such that*

$$\|\pi^s\rho(t,\rho_0) - \gamma(\pi^c\rho(t,\rho_0))\|_{3+\alpha} \leqslant \frac{c}{t^{1-\eta}}\, e^{-\omega t}\, \|\pi^s\rho_0 - \gamma(\pi^c\rho_0)\|_{2+\beta} \tag{6.5}$$

for each ρ_0 in a sufficiently small neighborhood of 0 in $h^{2+\beta}(\mathcal{S})$. Estimate (6.5) is valid for all $t\in(0,t^+(\rho_0))$ satisfying $\pi^c\rho(t,\rho_0)\in\mathcal{O}$.

Proof. The construction of a locally invariant center manifold for the quasilinear evolution equation (4.5) or the fully nonlinear equation (6.1) relies on maximal regularity, see [7] for a short account of this theory.

Hereafter $(\cdot, \cdot)_\theta$ denotes the continuous interpolation method of Da Prato and Grisvard. It is known that the little Hölder spaces have the interpolation property

$$(h^{\sigma_0}(\mathscr{S}), h^{\sigma_1}(\mathscr{S}))_\theta = h^{(1-\theta)\sigma_0 + \theta\sigma_1}(\mathscr{S}) \qquad \text{if} \quad (1-\theta)\sigma_0 + \theta\sigma_1 \notin \mathbb{N}, \quad (6.6)$$

where $\theta \in (0, 1)$, $0 < \sigma_0 < \sigma_1$, see [6, Vol. II]. (Related results can also be found in [26, Sect. 1.2.4]). In the following, we verify that the assumptions in [34, Sect. 4] are satisfied. We begin by observing that the function spaces $h^\alpha(\mathscr{S})$ and $h^{3+\alpha}(\mathscr{S})$ correspond to X_0 and X_1 of [34, Sect. 4]. Next we note that $h^{2+\alpha}(\mathscr{S})$ and $h^{2+\beta}(\mathscr{S})$ can be realized as continuous interpolation spaces between X_1 and X_0. More precisely, it follows from (6.6) that $h^{2+\alpha}(\mathscr{S}) = (X_0, X_1)_{2/3}$ and that $h^{2+\beta}(\mathscr{S}) = (X_0, X_1)_{(2+\beta-\alpha)/3}$. The set $U := h^{2+\alpha}(\mathscr{S}) \cap \mathfrak{A}$ then corresponds to the set U_β of [34, Sect. 4], while $h^{2+\beta}(\mathscr{S}) \cap U$ corresponds to U_α. Let

$$A(\rho) := \mathscr{B}(\rho)\, T(\rho)\, P(\rho), \qquad F(\rho) := -\mathscr{B}(\rho)\, T(\rho)\, Q(\rho), \quad \rho \in U.$$

Lemmas 3.1 and 4.1 and (5.1) then yield (4.2) and (4.5), (4.6) in [34]. Next we fix $\sigma \in (0, \alpha)$ and set $E_1 := h^{3+\sigma}(\mathscr{S})$ and $E_0 := h^\sigma(\mathscr{S})$. It is clear that all assertions of Sections 4 and 5 of the present paper also hold true for the spaces $h^{3+\sigma}(\mathscr{S})$ and $h^\sigma(\mathscr{S})$. In order to keep the notation simple we do not distinguish between the realization of these mappings in different spaces. Theorem 4.2 then shows that $A(\rho) \in \mathscr{H}(E_1, E_0)$ for each $\rho \in U$ and (6.6) gives that $X_0 = (E_0, E_1)_\theta$ for $\theta = (\alpha - \sigma)/3$. Hence X_0 is a continuous interpolation space between E_1 and E_0. It is not difficult to see that the domain of the maximal X_0 realization of $A(\rho) \in L(E_1, E_0)$ coincides with X_1 for each $\rho \in U$. Theorem 4.2 and [34, Theorem 2.2] now imply the crucial maximal regularity result

$$A(\rho) \in \mathscr{M}_\mu(X_1, X_0), \qquad \rho \in U, \quad \mu \in (0, 1],$$

which renders assumption (4.3) of [34]. Finally, assumption (iv) holds true since $h^{2+\alpha}(\mathscr{S})$ can also be realized as a continuous interpolation space between E_1 and E_0. We conclude from Proposition 5.4 and Lemma 6.1 that the eigenvalue 0 of L also has algebraic multiplicity $(n+1)$. The existence of a locally invariant center manifold now follows from Proposition 5.4, Lemma 6.1, and from [34, Theorem 4.1], where we set $\mathscr{O} := \mathbb{B}_{X_c}(0, r)$ for a sufficiently small r, and where $\gamma := \sigma \mid \mathscr{O}$ with σ being the mapping in [34, Eq. (4.22)]. More precisely, it was shown in [34, Theorem 4.1] that there exists a globally invariant center manifold for a truncated version of the evolution equations (4.5) or (6.1), which constitutes a locally invariant center manifold for the unmodified equations in a sufficiently small neighborhood of 0. Since solutions of the modified evolution equation coincide with solutions $\rho(\cdot, \rho_0)$ of the original equation as long as $\pi^c \rho(t, \rho)$ is contained in $\mathcal{O}$, see [34, Eq. (4.19)], the estimate (6.5) is a consequence of [34, Theorem 5.8]. ∎

Remarks 6.3.

(a) The statement that $\mathcal{M}^c$ is locally invariant means that $\mathcal{M}^c$ is invariant as long as solutions stay in a small neighborhood of 0 in $h^{3+\alpha}(\mathcal{S})$.

(b) Note that $\mathcal{M}^c$ is a C^k-manifold of dimension $n+1$, since it is the graph of a C^k-function defined on an open subset of X_c. Moreover, it follows from (6.4) that the center space $X_c \equiv X_c \times \{0\}$ is tangential to $\mathcal{M}^c$ at 0.

(c) It is well known that in general local center manifolds are not unique.

(d) It is important to note that we get the exponential attractivity of the local center manifold $\mathcal{M}^c$ in the topology of $h^{3+\alpha}(\mathcal{S})$ for initial data ρ_0 in $h^{2+\beta}(\mathcal{S})$. This result is close to optimal and takes into account the smoothing property of the quasilinear evolution equation (4.5).

Let $\mathcal{C}$ denote the set of all spheres which are small perturbations of $\mathcal{S}$. Since spheres are equilibria for the Mullins–Sekerka model, Proposition 6.2 yields $\mathcal{C} \subset \mathcal{M}^c$. Observe that any $C \in \mathcal{C}$ is completely described by $n+1$ parameters, the radius and the coordinates of the center. We show that $\mathcal{C} = \mathcal{M}^c$.

PROPOSITION 6.4. *The local center manifold $\mathcal{M}^c$ consists of equilibria.*

Proof. Suppose C is a sphere that is sufficiently close to $\mathcal{S}$. Let $(z_1, ..., z_n)$ be the coordinates of its center and let z_0 be given such that $R + z_0$ corresponds to the radius. It follows from (6.2) that $(R+z_0)^2 = \sum_{m=1}^n ((R+\rho) Y_m - z_m)^2$, where ρ measures the distance from $\mathcal{S}$. Solving for ρ we obtain that C can be parameterized over $\mathcal{S}$ by the distance function

$$\rho(z) = \sum_{m=1}^{n} z_m Y_m - R + \sqrt{\left(\sum_{m=1}^{n} z_m Y_m\right)^2 + (R+z_0)^2 - \sum_{m=1}^{n} z_m^2}. \tag{6.7}$$

Assume that O is a small enough neighborhood of 0 in $\mathbb{R}^{n+1}$. It is then clear that any sphere C which is close to $\mathcal{S}$ can be characterized by (6.7) with $z \in O$. Note that $[z \mapsto \rho(z)]: O \to h^{3+\alpha}(\mathscr{S})$ depends smoothly on z and that the derivative at 0 is given by

$$\partial \rho(0)\, h = \sum_{m=0}^{n} h_m Y_m, \qquad h \in \mathbb{R}^{n+1}. \tag{6.8}$$

Next we write $\rho(z) = \rho^c(z) + \rho^s(z)$, where $\rho^c(z)$ is the part of $\rho(z)$ in X_c and where $\rho^s(z)$ is the corresponding part in $h_s^{3+\alpha}(\mathscr{S})$. It then follows that

$$\rho^c(z) = \sum_{m=1}^{n} z_m Y_m - R + \pi^c \sqrt{\left(\sum_{m=1}^{n} z_m Y_m\right)^2 + (R+z_0)^2 - \sum_{m=1}^{n} z_m^2}.$$

Let $(F_0(z), ..., F_n(z))$ be the coordinates of $\rho^c(z)$ with respect to the basis $\{Y_0, ..., Y_n\}$. Then (6.8) yields $\partial F(0) = I_{\mathbb{R}^{n+1}}$. If O is sufficiently small then the inverse function theorem implies that $\mathrm{im}(F)$ is an open neighborhood of 0 and that F is a smooth diffeomorphism of O into $\mathrm{im}(F)$. Let $\mathscr{C} := \{\rho(z); z \in O\}$. We have proved that $\pi^c \mathscr{C}$ is an open neighborhood of 0 in X_c which we can assume to coincide with the open set $\mathcal{O}$ of Proposition 6.2. Since we already know that $\mathscr{C} \subset \mathscr{M}^c$ we can now conclude that $\mathscr{C} = \mathscr{M}^c$. ▮

As before, we set $\eta := (2+\beta-\alpha)/3$. The next theorem shows that solutions which start out close enough to a sphere exist globally and converge to a sphere.

THEOREM 6.5.

(a) *There exists a neighborhood V of* 0 *in $h^{2+\beta}(\mathscr{S})$ such that solutions of* (4.5) *exist globally for every initial value $\rho_0 \in V$.*

(b) *Let $\omega \in (0, -\mu_1)$ be given and let $\rho_0 \in V$. Then there are $c = c(\omega, \alpha, \beta) > 0$ and a unique $z_0 = z_0(\rho_0) \in \mathcal{O}$ such that for all $t > 0$*

$$\|(\pi^c \rho(t, \rho_0), \pi^s \rho(t, \rho_0)) - (z_0, \gamma(z_0))\|_{3+\alpha} \leqslant \frac{c}{t^{1-\eta}}\, e^{-\omega t} \|\pi^s \rho_0 - \gamma(\pi^c \rho_0)\|_{2+\beta}.$$

Proof. It follows from Proposition 6.4 that solutions of the reduced ordinary differential equation in X_c,

$$\dot{z}(t) + \pi^c L z(t) = \pi^c g(z(t) + \gamma(z(t))), \qquad z(0) = z_0, \tag{6.9}$$

are given by $z(t) \equiv z_0$ for $z_0 \in \mathcal{O}$. Therefore, 0 is a stable equilibrium for the reduced equation (6.9). Now it follows from [33, Theorem 3.3] that $(0, \gamma(0)) = (0, 0) \in \mathscr{M}^c$ is also stable in $h^{2+\alpha}(\mathscr{S})$ for the evolution equation (4.5). This means, in particular, that solutions of (4.5) that start in a sufficiently small neighborhood V of $h^{2+\beta}(\mathscr{S})$ exist globally. Theorem 3.3 of

[33] can indeed be applied to our situation. Since we are dealing with a parameter independent equation, we can assume that $\lambda = 0$, that is, $F = \{0\}$, in [33]. In this case, the assumptions imposed in [33] coincide with the assumptions of [34, Sect. 4], which we have verified in the proof of Proposition 6.2. It remains to observe that we can choose $X = h^{2+\beta}(\mathscr{S})$ in [33, Theorem 3.3] since (4.5) defines a smooth semiflow on $h^{2+\beta}(\mathscr{S}) \cap U$. This proves the assertion (a).

The proof of part (b) relies on the fact that $\mathscr{M}$ is exponentially attracting with asymptotic phase. Here we adopt the proof of [26, Proposition 9.2.4] to our particular situation. The proof of part (a) shows that $\pi^c \rho(t, \rho_0)$ exists globally and that

$$x(t) := \pi^c \rho(t, \rho_0) \in (1/2)\mathscr{O}, \qquad t \in [0, \infty), \tag{6.10}$$

for all ρ_0 in a sufficiently small neighborhood V of 0 in $h^{2+\beta}(\mathscr{S})$. Let $z(\cdot, z_0, \gamma)$ denote the solution of the reduced ordinary differential equation (6.9) with $z_0 \in \mathscr{O}$ and let us recall that $z(t, z_0, \gamma) \equiv z_0$ for each $z_0 \in \mathscr{O}$ and for $t \in \mathbb{R}$. Therefore, we obtain that

$$w(\tau, t) := z(\tau - t, x(t), \gamma) = x(t) \in (1/2)\mathscr{O}, \qquad \tau, t \in \mathbb{R}. \tag{6.11}$$

We conclude from (6.10), (6.11) and from [34, Proposition 5.4.a)] that there exist constants $c > 0$ and $\mu \in (0, \omega)$ such that

$$\|x(\tau) - x(t)\|_{X_c} \leqslant c \int_\tau^t e^{\mu(\sigma - \tau)} \, \|\pi^s \rho(\sigma, \rho_0) - \gamma(\pi^c \rho(\sigma, \rho_0))\|_{3+\alpha} \, d\sigma \tag{6.12}$$

for $0 \leqslant \tau < t$. Let us point out that [34, Proposition 5.4.a)] was actually established for the truncated equations (4.20) and (5.2), (5.3) of that paper. But owing to (6.10), (6.11) above and to (4.19) in [34], we do not have to distinguish between the modified and the original equations. Note that

$$\pi^s \rho(\sigma, \rho_0) - \gamma(\pi^c \rho(\sigma, \rho_0)) = \pi^s \rho(\sigma - \tau, \rho(\tau, \rho_0)) - \gamma(\pi^c \rho(\sigma - \tau, \rho(\tau, \rho_0))),$$

for $0 \leqslant \tau \leqslant \sigma$. If follows from (6.10), (6.12), from Proposition 6.2.(b), and from the embedding of $h^{3+\alpha}(\mathscr{S})$ in $h^{2+\beta}(\mathscr{S})$ that

$$\begin{aligned} \|x(\tau) - x(t)\| &\leqslant c \int_\tau^t \frac{e^{(\mu - \omega)(\sigma - \tau)}}{(\sigma - \tau)^{1-\eta}} \| \, \pi^s \rho(\tau, \rho_0) - \gamma(\pi^c \rho(\tau, \rho_0))\|_{2+\beta} \, d\sigma \\ &\leqslant c\Gamma(\eta)(\omega - \mu)^{-\eta} \, \|\pi^s \rho(\tau, \rho_0) - \gamma(\pi^c \rho(\tau, \rho_0))\|_{2+\beta} \\ &\leqslant c\tau^{-(1-\eta)} e^{-\omega\tau} \, \|\pi^s \rho_0 - \gamma(\pi^c \rho_0)\|_{2+\beta} \end{aligned} \tag{6.13}$$

for $0<\tau<t$, where $\Gamma(\eta)$ denotes the Gamma function. Let (t_n) be a sequence in $\mathbb{R}_+$ that converges to infinity. Then (6.13) shows that $(x(t_n))$ is a Cauchy sequence in X_c. Let $z_0 \in X_c$ be its limit. It is not difficult to see that any other sequence (t'_n) will lead to the same limit. We conclude that there exists a unique $z_0 \in X_c$ such that $x(t) \to z_0$ as $t \to \infty$. Moreover, (6.10) implies that $z_0 \in (1/2)\,\overline{\mathcal{O}} \subset \mathcal{O}$. By letting t go to infinity in (6.13) we see that

$$\|x(\tau)-z_0\| \leqslant c\tau^{-(1-\eta)}e^{-\omega\tau}\,\|\pi^s\rho_0-\gamma(\pi^c\rho_0)\|_{2+\beta}, \qquad \tau>0. \tag{6.14}$$

It follows from (6.10) above and from equation (4.24) in [34] that there exists a positive number b such that

$$\|\pi^s\rho(\tau,\rho_0)-\gamma(z_0)\|_{3+\alpha} \leqslant \|\pi^s\rho(\tau,\rho_0)-\gamma(x(\tau))\|_{3+\alpha}+b\,\|x(\tau)-z_0\|_{X_c} \tag{6.15}$$

for all $\tau>0$. Now the assertion in (b) follows from Proposition 6.2.(b) and from (6.14), (6.15). ▮

It can be shown that $\mathscr{B}(\rho)\,T(\rho)\,P(\rho) \in \mathscr{H}(h^{l+3+\alpha}(\mathscr{S}), h^{l+\alpha}(\mathscr{S}))$ for each $\rho \in U_l$ and $l \in \mathbb{N}$, where $U_l := U \cap h^{l+2+\alpha}(\mathscr{S})$. It is then not difficult to see that Lemmas 5.1 and 5.3, Proposition 5.4, and Lemma 6.1 remain true if we replace α with $l+\alpha$. Clearly, $\mathscr{C}=\mathscr{M}^c$ is contained in $h^r(\mathscr{S})$ for each $r>0$. Due to Theorem 4.3 we already know that solutions to the Mullins–Sekerka model immediately regularize and become smooth for $t>0$. Hence we can also measure the distance of $\rho(t,\rho_0)$ to $(z_0,\gamma(z_0))$ in $h^r(\mathscr{S})$ for any $r>0$. By repeating the steps leading to Theorem 6.5 we obtain the following result:

Proposition 6.6. *Let $\omega \in (0, -\mu_1)$ be given and let $r>0$. Then there exists a neighborhood $V=V(\omega,r)$ of 0 in $h^{2+\beta}(\mathscr{S})$ and for each $\rho_0 \in V$ there is a unique $z_0=z_0(\rho_0) \in \mathcal{O}$ such that*

$$\|(\pi^c\rho(t,\rho_0),\pi^s(t,\rho_0))-(z_0,\gamma(z_0))\|_r \leqslant ce^{-\omega t}\,\|\pi^s\rho_0-\gamma(\pi^c\rho_0)\|_{2+\beta}, \qquad t \geqslant 1,$$

where $c=c(\omega,r)$. Thus the equilibrium $(z_0,\gamma(z_0))$ attracts the solution $\rho(t,\rho_0)$ at an exponential rate in the topology of $h^r(\mathscr{S})$.

Proof. Let $r>0$ be given. Then we find $l \in \mathbb{N}$ such that $l+\alpha>r$. Let $\tau := 1/(l+1)$. By choosing V small enough we infer from Theorem 6.5 that $\rho(\tau,\rho_0)$ is in a small neighborhood of 0 in $h^{3+\alpha}(\mathscr{S})$ for all initial values $\rho_0 \in V$. Let $\rho_0 \in V$ be given and set $\rho_1 := \rho(\tau,\rho_0)$. By repeating the proof of Theorem 6.5 we obtain that there exists $z_1=z_1(\rho_1) \in \mathcal{O}$ such that $(z_1,\gamma(z_1))$ attracts the solution $\rho(\cdot,\rho_1)$ at an exponential rate in the topology of $h^{4+\alpha_1}(\mathscr{S})$, where $\alpha_1 \in (0,\alpha)$ is chosen close to α. It is clear that $(z_1,\gamma(z_1))$ coincides with the equilibrium $(z_0,\gamma(z_0))$ obtained in Theorem 6.5. If necessary, we shrink the size of V to ensure that $\rho(\tau, \rho_1) = \rho(2\tau, \rho_0)$ is contained in a sufficiently small neighborhood of 0 in $h^{4+\alpha_1}(\mathscr{S})$. We can now repeat the arguments and we arrive, after l steps, at the conclusion of Proposition 6.6. ▌

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