Classical solutions for the quasi-stationary Stefan problem with surface tension

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We show that the quasi-stationary two-phase Stefan problem with surface tension has a unique smooth local solution. In addition we show that smooth solutions exist globally, provided that the initial interface is close to a sphere and no heat is supplied or withdrawn.

1 Introduction and main results

The classical Stefan problem is a model of phase transitions in solid-liquid systems accounting for heat diffusion and exchange of latent heat in a homogeneous medium. The strong formulation of this model corresponds to a moving boundary problem involving a parabolic diffusion equation for each phase and a transmission condition prescribed at the moving interface separating the phases. Molecular considerations attempting to explain dendritic growth of crystals suggest to also consider surface tension on the interface separating the solid from the liquid region.

In order to state our results, we introduce the following notations. We assume that Ω^1 is a bounded smooth domain in $\mathbb{R}^n$ such that its boundary $\partial\Omega^1$ consists of two disjoint components, the interior part J^1 and the exterior part Γ_0. In addition, let also Ω be a bounded smooth domain in $\mathbb{R}^n$ containing Ω^1 and possessing a boundary with two disjoint components. The interior part of $\partial\Omega$ is assumed to coincide with J^1 and the exterior part is called J^2. Finally, we let $\Omega^2 := \Omega\setminus\overline{\Omega}^1$. The domain Ω^1 is regarded as the region occupied by the fluid phase and Γ_0 is a sharp interface in contact with the solid phase occupying the region Ω^2. Heat is being supplied through the interior boundary J^1.

Given $t \geq 0$, let $\Gamma(t)$ be the position of Γ_0 at time t, and let $V(\cdot,t)$ and $\kappa(\cdot,t)$ be the normal velocity and the mean curvature of $\Gamma(t)$. Here we use the convention that the normal velocity is positive for expanding hypersurfaces and that the mean curvature is positive for spheres. Let $\Omega^1(t)$ and $\Omega^2(t)$ be the two regions in Ω separated by $\Gamma(t)$, with $\Omega^1(t)$ being the interior region. Moreover, let $\nu(\cdot, t)$ be the outer unit normal field of $\Gamma(t)$ with respect to $\Omega^1(t)$.

Then the strong formulation of the *two-phase Stefan problem with surface tension* consists of finding (u^1, u^2, Γ) satisfying

$$\begin{cases} c_i\partial_t u^i - k_i\Delta u^i = 0 & \text{in} \quad \Omega^i(t) \\ \mathcal{B}u^1 = g & \text{on} \quad J^1 \\ \partial_n u^2 = 0 & \text{on} \quad J^2 \\ u^i = \sigma\kappa & \text{on} \quad \Gamma(t) \\ lV = -[k\partial_\nu u] & \text{on} \quad \Gamma(t) \\ u^i(0) = u_0^i & \text{in} \quad \Omega^i(0) \\ \Gamma(0) = \Gamma_0\,. \end{cases} \tag{1.1}$$

The constants c_1, c_2 are the conductivity coefficients, k_1, k_2 are the diffusion coefficients, l is the latent heat, and σ is the surface tension. Moreover,

$$[k\partial_\nu u] := k_1\partial_\nu u^1 - k_2\partial_\nu u^2$$

denotes the jump of the normal derivatives of k_1u^1 and k_2u^2 across the boundary $\Gamma(t)$. Finally, $\mathcal{B}$ is either the Dirichlet or the Neumann boundary operator on the fixed boundary J^1 and g is a given function.

The condition $u = \sigma\kappa$ on the free interface is usually called the Gibbs-Thomson relation, see [**?**, **?**, **?**, **?**, **?**, **?**] and [**?**, **?**, **?**, **?**].

If u^2 on $\Omega^2(t)$ is replaced by a constant while all the other aspects of the problem are left unchanged, then the modified problem is called the *one-phase Stefan model with surface tension.*

If the interface moves slowly in comparison to the time scale for heat conduction, then the conductivity coefficients c_1 and c_2 are negligible and we obtain the *quasi-stationary Stefan problem with surface tension*

$$\begin{cases} \Delta u^i = 0 & \text{in} \quad \Omega^i(t) \\ \mathcal{B}u^1 = g & \text{on} \quad J^1 \\ \partial_n u^2 = 0 & \text{on} \quad J^2 \\ u^i = \sigma\kappa & \text{on} \quad \Gamma(t) \\ lV = -[k\partial_\nu u] & \text{on} \quad \Gamma(t) \\ \Gamma(0) = \Gamma_0\,. \end{cases} \tag{1.2}$$

This paper will address existence, uniqueness, and regularity of classical solutions for the quasi-stationary problem. A major difficulty in solving the set of equations in (1.2) comes from the fact that the problem has a nonlocal character, since the solution of an elliptic boundary value problem is needed in order to determine the normal velocity V of the moving interface $\Gamma(t)$. On the other side, the elliptic problem cannot be solved independently without having information on $\Gamma(t)$. Therefore (1.2) contains a coupled set of equations which have to be solved simultaneously. In the following we assume that $g \in C^\infty(J^1)$ and that $\sigma = 1$.

Theorem 1.1. *Assume* $\alpha \in (0,1)$ *and let* $\Gamma_0 \in C^{2+\alpha}$ *be given. Then*

a) *The quasi-stationary Stefan problem with surface tension has a unique local classical solution* (u^1, u^2, Γ) *on some interval* $(0,T)$. *Given* $t \in (0,T)$, *the interface* $\Gamma(t)$ *is smooth and* $\Gamma(\cdot)$ *depends smoothly on* $t \in (0,T)$. *Moreover,*
$$u^i(\cdot,t) \in C^\infty(\bar{\Omega}^i(t)), \quad t \in (0,T), \quad i = 1,2.$$

b) *Assume that* $\mathcal{B} = \partial_n$ *and* $g = 0$. *If* Γ_0 *is sufficiently close to a sphere in the* $C^{2+\alpha}$*-topology, then the solution exists globally and converges to a sphere exponentially fast in the* C^k*-topology, where* $k \in \mathbb{N}$ *is an arbitrary, fixed number.*

Remark 1.2. a) Although the system (1.2) is well established in applications, surprisingly few analytic results are available, and even weak solutions were not known to exist in the general setting presented here. In the case $k_1 = k_2$ local classical solutions are constructed in [**?**, **?**] and, independently, in [**?**]. Part b) of Theorem 1.1 is a generalization of results obtained in [**?**]. Finally, in two space dimensions and still in the case $k_1 = k_2$ the existence of weak solutions and the convergence of small perturbations of circles are shown in [**?**]. The methods employed in [**?**] rely on potential theory and the condition $k_1 = k_2$ is essential.

b) Suppose we start with a truly non-convex initial geometry so that the mean curvature κ_0 of Γ_0 takes positive and negative values. Since the function κ_0 enters in the elliptic problems contained in (1.2), it is impossible to use any elliptic comparison principle in this situation. This explains to some extent the fact that even weak solutions to problem (1.2) were not known to exist.

c) The quasi-stationary Stefan problem with surface tension can be viewed as a nonlocal generalization of the classical mean curvature flow $V = -\kappa$. More precisely, let $F(\kappa) := -l^{-1}[k\partial_\nu u_\kappa]$, where u_κ satisfies the first four equations in (1.2). Then the last two equations in (1.2) imply that the moving boundary

$\Gamma(\cdot)$ evolves according to the law

$$V = F(\kappa), \qquad \Gamma(0) = \Gamma_0. \tag{1.3}$$

F is a nonlocal, nonlinear operator of first order.

2 Proof

Let us first sketch the proof of part a) of Theorem 1.1. The basic idea is to transform problem (1.3) in a neighbourhood of Γ_0 into an evolution equation for the distance of the unknown interface to a fixed smooth hypersurface. More precisely, given $\Gamma_0 \in C^{2+\alpha}$, we find a smooth hypersurface Σ, a positive constant $r > 0$, and a function $\rho_0 \in C^{2+\alpha}(\Sigma)$ such that

$$X \,:\, \Sigma \times (-r, r) \to \mathbb{R}^n, \qquad X(s, \lambda) := s + \lambda\nu(s)$$

is a smooth diffeomorphism onto its image $Y := im(X)$ and such that $\theta_{\rho_0}(s) := X(s, \rho_0(s))$ is a $C^{2+\alpha}-$diffeomorphism mapping Σ onto Γ_0. Here, ν also denotes the outer normal at Σ with respect to the part of $\mathbb{R}^n$ being diffeomorphic to Ω^1. It is convenient to decompose the inverse of X into $X^{-1} = (S, \Lambda)$, where

$$S \in C^\infty(Y, \Sigma) \qquad \text{and} \qquad \Lambda \in C^\infty(Y, (-r, r)).$$

Note that $S(x)$ is the nearest point on Σ to x, and that $\Lambda(x)$ is the signed distance from x to Σ (that is, to $S(x)$). Moreover, the neighbourhood Y consists of those points with distance less than r to Σ.

Let $T > 0$ be a fixed number. We assume that $\Gamma(t)$ is a family of hypersurfaces given by

$$\Gamma(t) := \{x \in \mathbb{R}^n\,;\, x = X(s, \rho(s, t)),\ s \in \Sigma\}, \quad t \in [0, T],$$

for a function $\rho : \Sigma \times [0, T] \to (-r, r)$. Note that the hypersurfaces $\Gamma(t)$ are parameterized over Σ by the distance function ρ. In addition, $\Gamma(t)$ is the zero-level set of the function

$$\phi_\rho : Y \times [0, T] \to \mathbb{R}, \qquad \phi_\rho(x, t) := \Lambda(x) - \rho(S(x), t).$$

If ρ is differentiable with respect to the time variable then we can express the normal velocity V of $\Gamma(t)$ at the point $x = X(s, \rho(s, t))$ as

$$V(s, t) = -\left.\frac{\partial_t \phi_\rho(x, t)}{|\nabla_x \phi_\rho(x, t)|}\right|_{x = X(s, \rho(s, t))} = \left.\frac{\partial_t \rho(s, t)}{|\nabla_x \phi_\rho(x, t)|}\right|_{x = X(s, \rho(s, t))}.$$

Since the outer unit normal field on $\Gamma(t)$ is given by $\nu(t) = \nabla\phi_\rho(\cdot,t)/|\nabla\phi_\rho(\cdot,t)|$ we conclude that equation (1.3) which governs the motion of $\Gamma(t)$ takes the form

$$\partial_t\rho(s,t) = -(k_1\nabla u^1_\kappa - k_2\nabla u^2_\kappa|\nabla\phi_\rho)|_{X(s,\rho(s,t))}, \qquad \rho(s,0) = \rho_0(s). \tag{1.4}$$

Let $\Phi(\rho)$ be the transformed version of $(k_1\nabla u^1_\kappa - k_2\nabla u^2_\kappa|\nabla\phi_\rho)|_{X(s,\rho(s,t))}$. Then Φ is a quasilinear pseudo-differential operator of third order. Moreover, as in [**?**] Theorem 4.1, one shows by means of Fourier multiplier representations of Poisson operators and subtle perturbation arguments that the principal part of the linear part of Φ generates a strongly continuous analytic semigroup on an appropriate subspace of $C^\alpha(\Sigma)$. Once this generation property is verified, a unique classical solution $\rho \in C^\infty((0,T)\times\Sigma)$ of (1.4) is guaranteed by the general results of H. Amann, cf. [**?**], Section 12, and by a bootstrapping argument. The unique classical solution of (1.3) is then given by

$$\Gamma(t) := \{x \in \mathbb{R}^n\,;\, x = X(s,\rho(s,t)),\ s\in\Sigma\}, \quad t\in[0,T].$$

Since the interface $\Gamma(t)$ is now determined, the regularity of the temperature distributions can be obtained by standard elliptic theory.
Let us also mention that the unique solution $\rho(\cdot,\rho_0)$ of (1.4) governs an infinite-dimensional dynamical system φ on an appropriate phase space $V \subset C^{2+\alpha}(\Sigma)$ by letting $\varphi(t,\rho_0) := \rho(t,\rho_0)$.

Once the existence of classical solutions is established, it is easy to see that the homogeneous quasi-stationary Stefan problem preserves the volume of $\Omega^1(t)$ and minimizes the area of $\Gamma(t)$. In order to see this, assume that $g = 0$ and that $\mathcal{B} = \partial_n$. Moreover, let $\mathrm{Vol}(t)$ denote the volume of $\Omega^1(t)$ and let $A(t)$ be the area of the moving hypersurface $\Gamma(t)$. Then we can calculate

$$\begin{aligned}\frac{d}{dt}\mathrm{Vol}(t) &= \int\limits_{\Gamma(t)} V\,d\sigma = -\int\limits_{\Gamma(t)} [k\partial_\nu u_\kappa]\,d\sigma \\ &= -\int\limits_{\Omega^1(t)} k_1\Delta u^1_\kappa\,dx - \int\limits_{\Omega^2(t)} k_2\Delta u^2_\kappa\,dx = 0\end{aligned}$$

and

$$\begin{aligned}\frac{1}{n-1}\frac{d}{dt}A(t) &= \int\limits_{\Gamma(t)} \kappa V\,d\sigma = -\int\limits_{\Gamma(t)} u_\kappa[k\partial_\nu u_\kappa]\,d\sigma \\ &= -\int\limits_{\Omega^1(t)} k_1|\nabla u^1_\kappa|^2\,dx - \int\limits_{\Omega^2(t)} k_2|\nabla u^2_\kappa|^2\,dx \le 0,\end{aligned}$$

see [?] for more details. Notice that every Euclidean sphere is an equilibrium for problem (1.3). Hence the isoperimetric inequality suggests to analyze (1.3) near spheres. However, none of these equilibria are isolated. Thus the dynamic behaviour of the flow φ is quiet copious.

The proof of part b) of Theorem 1.1 consists of two steps. Suppose Σ is a fixed Euclidean sphere. We first show that the corresponding semiflow φ admits an asymptotically stable $(n+1)$-dimensional *local centre manifold* $\mathcal{M}^c$. In particular, this means that $\mathcal{M}^c$ is a locally invariant manifold, that $\mathcal{M}^c$ contains all small global solutions of φ, and that $\mathcal{M}^c$ attracts at an exponential rate all solutions which are $C^{2+\alpha}$-close to $\mathcal{M}^c$, see [?]. In a second step we then prove that $\mathcal{M}^c$ coincides with the set of all Euclidean spheres which are sufficiently close to Σ.

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