ANALYTICITY OF SOLUTIONS TO FULLY NONLINEAR PARABOLIC EVOLUTION EQUATIONS ON SYMMETRIC SPACES

JOACHIM ESCHER AND GIERI SIMONETT

In memoriam Philippe Bénilan

ABSTRACT. It is shown that solutions to fully nonlinear parabolic evolution equations on symmetric Riemannian manifolds are real analytic in space and time, provided the propagator is compatible with the underlying Lie structure. Applications to Bellman equations and to generalized mean curvature flows are also discussed.

1. INTRODUCTION

The smoothing property of solutions may be viewed as a characteristic feature of parabolic evolution equations. Roughly speaking, the smoothing property means that - under suitable assumptions - solutions to parabolic initial value problems enjoy more "spatial" regularity than the corresponding initial datum. This property is well-known for solutions to semilinear problems, see [18], and for solutions to classical quasilinear parabolic equations, see [10, 16, 21]. More recently, Amann [1] developed a theory for abstract quasi-linear problems in which the smoothing property of solutions appears as a cornerstone. Roughly speaking again, in these investigations smooth solutions are obtained by the property that parabolicity is preserved under (spatial) differentiation. This is obvious for linear problems. In the nonlinear situation considered in [1, 10, 16, 18, 21] this is due to the assumption that the nonlinearities are dominated by the parabolic linear part.

In this paper we are interested in fully nonlinear problems which we shall treat in the framework of maximal regularity. Here a completely different situation occurs. In fact, it can be viewed as a characteristic feature of this approach that it provides an existence and uniqueness theory without relying on a regularizing effect of solutions. This means that any form of smoothing of solutions to general fully nonlinear problems cannot be expected.

In order to be able to guarantee a smoothing action of solutions to fully nonlinear problems, one has to rely on additional structures of the problems under consideration. In this paper we consider the particular situation that the abstract equations come from nonlinear, possibly nonlocal operators acting on function spaces over a symmetric Riemannian manifold. Under the crucial assumption that these nonlinear operators are compatible with the underlying Lie structure we prove a strong regularizing property of solutions.

We illustrate the flexibility of our approach by discussing two different types of fully nonlinear parabolic evolution equations: First we treat a class of Bellman equations on $\mathbb{R}^m$, which arises in stochastic control theory. Secondly, we study a generalized mean curvature flow on spheres. A further example, occuring in the modelling of flows of incompressible fluids in rigid porous media, has been considered earlier in [11].

In the following we describe an important special case of our main result, Theorem 3.11. To make this more precise, let E_0 and E_1 be Banach spaces such that E_1 is continuously injected and dense in E_0. Assume further that $B \subset E_1$ is open and $P \in C^\omega(B, E_0)$. Given $u \in B$, let $\partial P(u) \in \mathcal{L}(E_1, E_0)$ denote the Fréchet derivative of P and assume that $\partial P(u)$ possess the property of maximal regularity in the sense of Da Prato-Grisvard, see Section 2 for a precise definition. Then, given $u_0 \in B$, the abstract evolution equation

$$\frac{d}{dt}u + P(u) = 0, \qquad u(0) = u_0, \tag{1.1}$$

possesses a unique solution

$$u := u(\cdot, u_0) \in C([0, t^+), B) \cap C^1([0, t^+), E_0), \tag{1.2}$$

where $t^+ := t^+(u_0) > 0$ stands for the maximal existence time of u, see again Section 2.

Assume now that M is a closed Riemannian manifold such that

$$E_1 \hookrightarrow buc^{1+\alpha}(M), \quad buc^\alpha(M) \hookrightarrow E_0 \hookrightarrow BUC(M), \tag{1.3}$$

for some $\alpha \in (0,1)$, where $buc^s(M)$ denotes the closure of the smooth functions in the usual Hölder spaces $BUC^s(M)$, cf. Section 2. Furthermore, we assume that M is a globally symmetric space. This means that there is a Lie group G which acts as a transformation group on M. Let $\cdot : G \times M \to M$ denote the action of G on M and set

$$g \cdot v : M \to \mathbb{R}, \quad p \mapsto v(g \cdot p) \quad \text{for} \quad (g, v) \in G \times E_j,$$

where $j = 0$, 1. We call G a **strongly continuous transformation group** on E_j if $[v \mapsto g \cdot v] \in \mathcal{L}(E_j)$ for all $g \in G$ and if

$$a_v : G \to E_j, \qquad g \mapsto g \cdot v$$

is for any $v \in E_j$ continuous at e, the unit element in G. We further need a structural condition which connects the underlying geometry of M with the operator P. To make this precise, we say that P is **equivariant** with respect to G if there is a neighborhood U ofo e in G such that $U \cdot B \subset B$ and

$$P(g \cdot v) = g \cdot P(v) \quad \text{for} \quad (g, v) \in U \times B.$$

Finally, letting $\hat{u}(t,p) := u(t)(p)$ for $(t,p) \in [0, t^+) \times M$, we have the following result:

Theorem 1.1. *Assume that G is a strongly continuous transformation group on E_j for $j = 0,\ 1$, and that P is equivariant with respect to G. Then the solution u to (1.1) is real analytic in space and time, i.e. $\hat{u} \in C^\omega((0,t^+) \times M)$.*

The above theorem is a special case of a more general result which is proved in the main body of this paper. We mention particularly that assumption (1.3), the assumption on M to be a symmetric space, as well as the equivariance of P can be weakened, see Theorem 3.11.

As mentioned above, the existence of solutions to (1.1) is obtained in the framework of continuous maximal regularity. We shall see in Section 3 that maximal regularity will also be instrumental in the proof of our main result, Theorem 3.11. In fact this property allows the application of the implicit function theorem in appropriate function spaces to show that, given $t \in (0,t^+)$, the mapping $(\lambda, X) \mapsto \exp(tX) \cdot u(\lambda t)$ is analytic on the Lie algebra $\mathbb{R} \times L(G)$ of $\mathbb{R} \times G$. This in turn, together with the analyticity of the exponential mapping, implies the analyticity of $\hat{u}$.

2. Continuous Maximal Regularity

In this section we briefly introduce the notion of maximal regularity in the sense of Da Prato-Grisvard. For this let E_0 and E_1 be Banach spaces such that E_1 is continuously injected and dense in E_0. Let $\mathcal{H}(E_1, E_0)$ denote the subset of all $A \in \mathcal{L}(E_1, E_0)$ such that $-A$, considered as a, in general, unbounded operator in E_0, generates a strongly continuous analytic semigroup on E_0. Let $B \subset E_1$ be open and assume that

$$P \in C^\omega(B, E_0) \quad \text{with} \quad \partial P(v) \in \mathcal{H}(E_1, E_0), \quad v \in B. \tag{2.1}$$

Given $T > 0$, set

$$\mathbb{E}_0 := C([0,T], E_0), \qquad \mathbb{E}_1 := C([0,T], E_1) \cap C^1([0,T], E_0),$$

and let $\gamma : \mathbb{E}_0 \to E_0,\ u \mapsto u(0)$ denote the trace operator in $\mathbb{E}_0$. We assume that $(\mathbb{E}_0, \mathbb{E}_1)$ is a pair of **maximal regularity** for $\partial P(v)$, this means we assume that

$$\Big(\frac{d}{dt} + \partial P(v), \gamma\Big) \in \mathcal{L}_{is}(\mathbb{E}_1, \mathbb{E}_0 \times E_1), \quad v \in B, \tag{2.2}$$

where $\mathcal{L}_{is}(X, Y)$ stands for the set of all bounded isomorphisms from the Banach space X into the Banach space Y. We are now ready to formulate the following existence and uniqueness result:

Theorem 2.1. *Assume that (2.1) and (2.2) hold true. Then, given any $u_0 \in B$ and $f \in C(\mathbb{R}_+, E_0)$, there exist $t^+ := t^+(u_0) > 0$ and a unique maximal solution*

$$u := u(\cdot, u_0) \in C([0,t^+), B) \cap C^1([0,t^+), E_0) \tag{2.3}$$

of the initial value problem

$$\frac{d}{dt}u + P(u) = f, \qquad u(0) = u_0. \tag{2.4}$$

Remarks 2.2. **(a)** Theorem 2.1 essentially goes back to Da Prato and Grisvard [9]. For some refinements and generalizations see also [5].

(b) Observe that assumption (2.2) and Theorem 2.1 coincide in the linear case, i.e., if $B = E_1$ and $P \in \mathcal{L}(E_1, E_0)$. Nevertheless, it is not at all clear whether or not property (2.2) can be verified if $E_1 \neq E_0$. In fact, it follows from a result of Baillon [7] that, in case $E_1 \neq E_0$, property (2.3) can only be expected if E_0 contains an isomorphic copy of the sequence space c_0. In particular, (2.3) will never be true in reflexive Banach spaces. However, in [9] the **continuous interpolation** functor $(\cdot,\cdot)^0_{\theta,\infty}$ was introduced, an interpolation method producing non-reflexive Banach spaces for which condition (2.2) can be verified.

(c) Let us briefly introduce an important scale of Banach spaces which may be realized as continuous interpolation spaces. Given $s \in \mathbb{R}$, define the **little Hölder spaces** to be

$$buc^s(\mathbb{R}^m) := \text{closure of } BUC^\infty(\mathbb{R}^m) \ \text{ in } \ B^s_{\infty,\infty}(\mathbb{R}^m),$$

where $B^s_{\infty,\infty}(\mathbb{R}^m)$ stands for the Besov spaces as defined in [26]. Note that the spaces $B^s_{\infty,\infty}(\mathbb{R}^m)$ coincides with the ususal Hölder spaces $BUC^s(\mathbb{R}^m)$, provided $s > 0$ is not an integer, see Theorem 2.5.7 and Remark 2.2.2.3 in [26]. Then it is shown in [22] Theorem 1.2.17 that

$$(BUC(\mathbb{R}^m), BUC^n(\mathbb{R}^m))^0_{\theta,\infty} = buc^{\theta n}(\mathbb{R}^m)$$

for all $n \in \mathbb{N}$ and $\theta \in (0,1)$ such that $\theta n \notin \mathbb{N}$.

(d) Assume that M is a smooth Riemannian manifold with bounded curvature and positive radius of injectivity. Then Lemma 2.26 in [6] ensures the existence of a uniformly locally finite covering of geodesic balls $M(p_j, \delta)$ with $p_j \in M$, $j \in \mathbb{N}$ and $\delta > 0$. As before, the spaces $buc^s(M)$ are defined to be the closure of $BUC^\infty(M)$ in $BUC^s(M)$. Again we have that

$$(BUC(M), BUC^n(M))^0_{\theta,\infty} = buc^{\theta n}(M)$$

for all $n \in \mathbb{N}$ and $\theta \in (0,1)$ such that $\theta n \notin \mathbb{N}$, cf. the proof of Corollary 1.2.19 in [22]. For simplicity we write $h^s(M) = buc^s(M)$ for $s \in \mathbb{R}$ if M is compact.

(e) Let M as above and fix s_0, $s_1 \in (0,\infty)$, $\theta \in (0,1)$. Setting $s_\theta := (1-\theta)s_0 + \theta s_1$, we have

$$(buc^{s_0}(M), buc^{s_1}(M))^0_{\theta,\infty} = buc^{s_\theta}(M),$$

provided s_0, s_1, and s_θ are not integers. This follows from (d), Theorem 7.4.4 in [27], and a density argument.

(f) A further scale of Banach spaces for which maximal regularity can be verified are the little Nikol'skii spaces. They can be realized as continuous interpolation spaces of Bessel potential spaces, cf. [9], Section 6 and [25], Section 6.

(g) Consider again the "linear" case $B = E_1$ and $P \in \mathcal{L}(E_1, E_0)$ and suppose in addition that $f \equiv 0$. Then problem (2.4) has for each $u_0 \in E_1$ a unique solution in the class $\mathbb{E}_1$ (for any $T > 0$, of course), provided $-P$ generates a strongly continuous semigroup, which does not need to be analytic. However, it is shown in [9] that the semigroup is automatically analytic if condition (2.2) is supposed to hold, see also Proposition III.3.1.1 in [2].

(h) A well-known characterization of generators of analytic semigroups yields that $A \in \mathcal{L}(E_1, E_0)$ belongs to $\mathcal{H}(E_1, E_0)$ if there are positive constants κ and ω such that $[\operatorname{Re}\lambda \geq \omega] \subset \rho(-A)$ and

$$|\lambda|\,\|(\lambda + A)^{-1}\|_{\mathcal{L}(E_0)} \leq \kappa, \qquad \operatorname{Re}\lambda \geq \omega.$$

(i) We mention that Theorem 2.1 remains true under a much weaker regularity assumption for P. Indeed, it suffices to assume that P is continuously Fréchet differentiable. Under this regularity assumption it can also be shown that the mapping

$$\bigcup_{x\in B} [0, t^+(x)) \times \{x\} \to B, \qquad (t, x) \mapsto u(t, x)$$

is a semiflow on B, provided f does not depend on t. However, since we are looking for possible smoothing properties of solutions, we presuppose analyticity of P from the very beginning.

(j) Let M be as in (d) and assume that $A \in \mathcal{H}(buc^{k+l+\beta}(M), buc^{k+\beta}(M))$ for some $k \in \mathbb{N}$, $l \in \mathbb{R}_+$, $\beta \in (0,1)$ with $\beta + l \notin \mathbb{N}$. Let further $\alpha \in (\beta, 1)$ with $\alpha + l \notin \mathbb{N}$ and suppose that $buc^{k+l+\alpha}(M)$ is the domain of the $buc^{k+\alpha}(M)$-realization of A. Setting $E_0 := buc^{k+\alpha}(M)$ and $E_1 := buc^{k+l+\alpha}(M)$, it follows from Théorème 3.1 in [9] and (e) that $(\mathbb{E}_0, \mathbb{E}_1)$ is a pair of maximal regularity for the operator A, see also [2, 5]. $\square$

3. The Smoothing Property

Let Σ be an analytic closed Riemannian manifold of dimension m and assume that E_0 and E_1 are Banach spaces of functions over Σ. More precisely, assume that E_1 is dense in E_0 and that

$$E_1 \hookrightarrow BUC(\Sigma), \qquad E_1 \hookrightarrow E_0 \hookrightarrow L_{1,loc}(\Sigma). \tag{A_1}$$

Throughout this section we presuppose (2.1) and (2.2) and we let u denote the solution of (2.4) on $[0, t^+)$, where $u_0 \in B$ is given and where we assume for simplicity that $f \equiv 0$. Moreover, we set $\hat{u}(t, q) := u(t)(q)$ for $(t, q) \in [0, t^+) \times \Sigma$. Our goal is to show that u enjoys a smoothing property. Hence, subdividing the interval of existence and using the semiflow property of u, see Remark 2.2(i), we may assume without loss of generality that $t^+ \leq 1$. Further, we fix $T \in (0, t^+)$ and set $I := [0, T]$.

From Theorem 2.1 we know that u belongs to $C(I, B) \cap C^1(I, E_0)$. Since we are dealing with nonlinear equations, including fully nonlinear partial differential equations involving nonlocal terms, there is no reason to expect u to have any further regularity, like

$$u \in C^\alpha(I \setminus \{0\}, E_1) \qquad \text{or} \qquad u \in C(I \setminus \{0\}, (E_1, E_2)_\alpha), \tag{3.1}$$

where E_2 stands for the domain of definition of $[\partial P(u_0)]^2$, equipped with the corresponding graph norm, and where $(\cdot, \cdot)_\alpha$ denotes a suitable interpolation method.

However, it turns out that there is actually a strong smoothing property for solutions of problem (2.4), provided we impose suitable symmetry properties for the manifold Σ as well as for the nonlinear operator P.

It should be remarked that if P carries a quasilinear structure in the sense that Theorem 12.1 in [1] is applicable, then it can be shown that the corresponding solutions do in fact possess a smoothing property in the sense of (3.1) without any geometrical condition on Σ or on P.

Concerning the manifold Σ we shall assume that it is analytically diffeomorphic to a globally Riemannian symmetric space. More precisely, we assume that

$$\begin{aligned}&\text{there exists a globally symmetric}\\ &\text{Riemannian space } M \text{ and a } \Phi \in \mathrm{Diff}^{\omega}(M,\Sigma).\end{aligned} \tag{A_2}$$

Recall that a Riemannian manifold M is called a globally symmetric space if it is connected and if for each $p \in M$ there is an involutive isometry $\sigma_p : M \to M$ such that p is an isolated fixed point of σ_p. Observe that σ_p reverses geodesics passing through the point p. This implies that M is complete and, by the Hopf-Rinow theorem, that the group $I(M)$ of all isometries acts transitively on M. Let g denote the metric on M and write $\tilde{g}$ for the metric on Σ induced by Φ and g. Then (A_2) implies that $(\Sigma, \tilde{g})$ is a globally symmetric Riemannian space. However, in view of applications, we prefer to keep the original metric on Σ.

Let now Φ^* and Φ_* denote the pull back and push forward operator induced by Φ. This means that, given $v \in L_{1,loc}(\Sigma)$ and $w \in L_{1,loc}(M)$, we have

$$\Phi^* v := v \circ \Phi \qquad \text{and} \qquad \Phi_* w := w \circ \Phi^{-1}.$$

For later purposes we need the following technical result.

Lemma 3.1. *$\Phi^* \in \mathcal{L}_{is}(L_{1,loc}(\Sigma), L_{1,loc}(M))$, $\Phi_* \in \mathcal{L}_{is}(L_{1,loc}(M), L_{1,loc}(\Sigma))$ and $[\Phi^*]^{-1} = \Phi_*$.*

Proof: It follows from Theorem 2.2.26 and Corollary 2.2.21 in [19] that M has bounded curvature and a positive radius of injectivity $\delta > 0$. Hence Lemma 2.26 in [6] ensures that there exists a uniformly locally finite covering of geodesic balls $M(p_j, \delta)$ on M and a smooth partition of unity $\{\pi_j\,;\, j \in \mathbb{N}\}$ subordinated to $\{M(p_j,\delta)\,;\, j \in \mathbb{N}\}$. Let

$$\|w\|_{j,M} := \|\pi_j w\|_{L_1(M)}, \qquad \|v\|_{j,\Sigma} := \|\Phi^* v\|_{j,M}, \quad j \in \mathbb{N},$$

for $v \in L_{1,loc}(\Sigma)$ and $w \in L_{1,loc}(M)$. Then, using the transformation theorem for the Lebesgue integral, it not difficult to see that

$$\{\|\cdot\|_{j,M}\,;\, j \in \mathbb{N}\} \quad \text{and} \quad \{\|\cdot\|_{j,\Sigma}\,;\, j \in \mathbb{N}\}$$

are separating families of seminorms, which induce the original topology on the spaces $L_{1,loc}(M)$ and $L_{1,loc}(\Sigma)$, respectively. The assertion follows now from the definition of $\|\cdot\|_{j,\Sigma}$. $\square$

For $j = 0,\ 1$, let

$$F_j := \{\Phi^* v\,;\, v \in E_j\}, \qquad \|w\|_{F_j} := \|\Phi_* w\|_{E_j}, \quad w \in F_j.$$

Then it is not difficult to verify that $F_j := (F_j\,;\,\|\cdot\|_{F_j})$ are well-defined Banach spaces such that F_1 is continuously injected and dense in F_0. Moreover, we have $F_0 \subset L_{1,\mathrm{loc}}(M)$ and $F_1 \subset BUC(M)$. We next introduce

$$Q(w,\Phi) := \Phi^* P(\Phi_* w), \quad v \in D,$$

where $D := \{\Phi^* v\,;\, v \in B\}$. Of course,

$$D \text{ is open in } F_1 \quad \text{and} \quad Q(\cdot,\Phi) \in C^\omega(D,F_0), \tag{3.2}$$

because of the fact that $\Phi^* : E_j \to F_j$ is an isometric isomorphism. We write $\partial_1 Q(w,\Phi) \in \mathcal{L}(F_1,F_0)$ for the Fréchet derivative of $Q(\cdot,\Phi)$. Further, we need the spaces

$$\mathbb{F}_0 := C(I,F_0), \qquad \mathbb{F}_1 := C(I,F_1) \cap C^1(I,F_0). \tag{3.3}$$

The pull back and push forward operator induced by Φ on $\mathbb{E}_0$ and $\mathbb{F}_0$ are defined pointwise with respect to $t \in I$, i.e., given $v \in \mathbb{E}_0$ and $w \in \mathbb{F}_0$, let

$$\Phi^* v : I \to F_0, \quad t \mapsto \Phi^* v(t), \qquad \Phi_* w : I \to E_0, \quad t \mapsto \Phi_* w(t).$$

Of course we do also not distinguish notationally between Φ^* and Φ_* and restrictions of these operators to linear subspaces of $\mathbb{E}_0$ and $\mathbb{F}_0$, respectively.

Lemma 3.2. *The following assertions hold true:*

(i) $\Phi^* \in \mathcal{L}_{is}(\mathbb{E}_j,\mathbb{F}_j)$, $\Phi_* \in \mathcal{L}_{is}(\mathbb{F}_j,\mathbb{E}_j)$ *and* $[\Phi^*]^{-1} = \Phi_*$ *for* $j = 0, 1$.

(ii) $(\mathbb{F}_1,\mathbb{F}_0)$ *is a pair of maximal regularity for* $\partial_1 Q(\cdot,\Phi)$, *i.e.,*

$$\Big(\frac{d}{dt} + \partial_1 Q(w,\Phi), \gamma\Big) \in \mathcal{L}_{is}(\mathbb{F}_1, \mathbb{F}_0 \times F_1), \quad w \in D. \tag{3.4}$$

Proof: The first assertion follows from the construction of the spaces $\mathbb{F}_j$, $j = 0, 1$. The second one is a consequence of (i) and the chain rule. □

Let $G := I_0(M)$ be the identity component of the group $I(M)$ of C^1-isometries on M. We already noticed that G acts transitively on M, see the remark after (A_2). Furthermore, it follows from (A_2) and Theorem I.4.6 in [20] that G acts analytically as a Lie transformation group on M, so that M is a homogeneous Riemannian space with respect to G. Fix $p_0 \in M$ and let $H := \{g \in G\,;\, g \cdot p_0 = p_0\}$ denote the isotropy group of p_0, where $\cdot : G \times M \to M$ denotes the action of G on M. Then G/H admits a real analytic structure and

$$j : G/H \to M, \qquad g \cdot H \mapsto g \cdot p_0$$

is a real analytic diffeomorphism, cf. Proposition I.4.2 in [20]. By means of this diffeomorphism we always identify M with the coset manifold G/H. Finally, recall that

- $\mathbb{R}^m$,
- the unit sphere $\mathbb{S}^m = SO(m+1)/SO(m)$,
- products of Riemannian globally symmetric spaces

are Riemannian globally symmetric spaces.

Let Y be a nonempty set. Given $f \in Y^M$ and $g \in G$, we define $g \cdot f \in Y^M$ by $g \cdot f(p) := f(g \cdot p)$. The next result contains the transformation rule for the operator Q with respect to G which will be needed in the following.

Lemma 3.3. *Let $g \in G$ be given. Then*

(i) $g \cdot \Phi \in Diff^\omega(M, \Sigma)$*;*

(ii) $g \cdot Q(w, \Phi) = Q(g \cdot w, g \cdot \Phi), \quad w \in D.$

Proof: (i) This follows from the analyticity of the group action of G on M.

(ii) Given $g \in G$ we have

$$(g \cdot \Phi)^* v = g \cdot (\Phi^* v), \quad v \in E_0, \qquad (g \cdot \Phi)_*(g \cdot w) = \Phi_* w, \quad w \in F_0. \tag{3.5}$$

Thus we find

$$\begin{aligned} Q(g \cdot w, g \cdot \Phi) &= (g \cdot \Phi)^* P((g \cdot \Phi)_*(g \cdot w)) = (g \cdot \Phi)^* P(\Phi_* w) \\ &= g \cdot (\Phi^* P(\Phi_* w)) = g \cdot Q(w, \Phi), \end{aligned}$$

for any $w \in D$. $\square$

We next assume that

$$G \text{ is a strongly continuous transformation group on } F_j \text{ for } j = 0, 1. \tag{A_3}$$

Writing $L(G)$ for the Lie algebra of G, we define

$$T_X(t)w := \exp(tX) \cdot w \quad \text{for} \quad (X, t, w) \in L(G) \times \mathbb{R} \times F_j.$$

It follows from (A_3) that $\{T_X(t)\,;\, t \in \mathbb{R}\}$ is for any $X \in L(G)$ a strongly continuous group on F_0. We write A_X for the infinitesimal generator of $\{T_X(t)\,;\, t \in \mathbb{R}\}$ and we assume that

$$F_1 \subset \mathrm{dom}(A_X) \quad \text{for any} \quad X \in L(G), \tag{A_4}$$

where $\mathrm{dom}(A_X)$ is given the graph norm of A_X. Observe that (A_4) and the closed graph theorem imply that

$$A_X \in \mathcal{L}(F_1, F_0) \quad \text{for any} \quad X \in L(G). \tag{3.6}$$

Recall that, given $w \in F_j$, we have set

$$a_w : G \to F_j, \qquad g \mapsto g \cdot w.$$

We write $d_g a_w$ for the differential of a_w at $g \in G$, provided a_w is differentiable, of course.

Lemma 3.4. *(i) If $w \in F_j$ then $a_w \in C(G, F_j)$ for $j = 0, 1$.*

(ii) If $w \in F_1$ then $a_w \in C^1(G, F_0)$ and, given $(g, X) \in G \times L(G)$, we have

$$d_g a_w X = A_X(g \cdot w). \tag{3.7}$$

Proof: (i) Let $g \in G$ and choose a sequence (g_k) in G such that $g_k \to g$. Since G is a strongly continuous transformation group on F_j we find

$$a_w(g_k) = a_{g \cdot w}(g^{-1} g_k) \to g \cdot w = a_w(g) \quad \text{in} \quad F_j.$$

(ii) Due to (i) and (3.6) it suffices to prove (3.7). Identifying $T_e G \cong L(G)$, we have that $\{g \cdot \exp(tX) \,;\, t \in \mathbb{R}\}$ is an integral curve on G through g with tangent vector X. Thus (A_3) yields

$$\lim_{t \to 0} \frac{a_w(g \cdot \exp(tX)) - a_w(g)}{t} = \lim_{t \to 0} \frac{\exp(tX) \cdot (g \cdot w) - (g \cdot w)}{t} = A_X(g \cdot w)$$

in F_0, since w belongs to $F_1 \subset \operatorname{dom}(A_X)$. $\square$

Remark 3.5. Let $M = \Sigma = \mathbb{R}$ and $E_j = BUC^j(\mathbb{R})$ for $j = 0, 1$. Then $F_j = E_j$ for $j = 0, 1$ and, given $w \in F_j$ and $\lambda \in \mathbb{R}$, we have that $a_w(\lambda) = \tau_\lambda w$, where $\tau_\lambda w$ denotes the left translation by $\lambda \in \mathbb{R}$ of w. This shows that the regularity of a_w with respect to G as stated in Lemma 3.4 is optimal. $\square$

We note that $L(G)$ is finite dimensional, as Theorem VI 3.3 and Theorem VI 3.4 (4) in [20] imply. Let $\mathcal{B} = \{X_1, \dots, X_N\}$ be a fixed basis of $L(G)$. Given $(\mu_1, \dots, \mu_N) \in \mathbb{R}^N$, we write

$$\mu\mathcal{B} := \sum_{k=1}^{N} \mu_k X_k, \quad T_\mu(t) := T_{\mu\mathcal{B}}(t), \quad A_\mu := A_{\mu\mathcal{B}}.$$

Let E, F, and G be Banach spaces. Writing $\mathcal{L}^2(E \times F, G)$ for the Banach space of all bilinear continuous mappings from $E \times F$ to G, we have

Corollary 3.6. $[(\mu, w) \mapsto A_\mu w] \in \mathcal{L}^2(\mathbb{R}^N \times \mathbb{F}_1, \mathbb{F}_0)$.

Proof: It follows readily from Lemma 3.4 that $[(\mu, w) \mapsto A_\mu w]$ is bilinear. Let $C := \sum_{k=1}^{N} \|A_{X_k}\|_{\mathcal{L}(F_1, F_0)}$ and observe that (3.6) implies that $C < \infty$. Now, given $w \in \mathbb{F}_1$, $\mu \in \mathbb{R}^N$, and $t \in I$, we have

$$\|A_\mu w(t)\|_{F_0} = \|d_e a_{w(t)} \sum_{k=1}^{N} \mu_k X_k\|_{F_0} \le |\mu|_{\mathbb{R}^N} \sum_{k=1}^{N} \|A_{X_k} w(t)\|_{F_0} \le C|\mu| \|w\|_{\mathbb{F}_1}.$$

Taking the maximum over $t \in I$, we get the assertion. $\square$

We shall now formulate the compatibility condition for the operator P with respect to G. For this we first have to introduce the following assumption:

$$D \text{ is invariant under } T_\mu(t) \text{ for all } (\mu, t) \in (-\varepsilon_0, \varepsilon_0)^N \times I. \qquad (A_5)$$

Further, let $\mathbb{D}_1 := C(I, D) \cap C^1(I, F_0)$, pick $(\mu, w) \in (-\varepsilon_0, \varepsilon_0)^N \times \mathbb{D}_1$, and define

$$\mathbb{Q}(\mu, w)(t) := Q(w(t), T_\mu(t)\Phi), \qquad t \in I,$$

where $T_\mu(t)\Phi := \exp(t\mu\mathcal{B}) \cdot \Phi$ for $(t, \mu) \in \mathbb{R} \times \mathbb{R}^N$. Observe that (A_5) ensures that this definition is meaningful. Moreover, we have

$$\mathbb{Q}(\mu, w)(t) = (T_\mu(t)\Phi)^* P((T_\mu(t)\Phi)_* w(t)), \quad t \in I.$$

Hence $\mathbb{Q}(\mu, w) \in F_0^I$. We need that this function belongs to $\mathbb{F}_0$ and that the operator $\mathbb{Q}(\mu, w)$ depends analytically on (μ, w). For this we first observe that

$$\mathbb{D}_1 \text{ is an open subset of } \mathbb{F}_1. \tag{3.8}$$

Indeed, this follows from the compactness of I and the fact that D is open in F_1. We now assume that there is a $r \in (0, \varepsilon_0)$ such that

$$[(\mu, w) \mapsto \mathbb{Q}(\mu, w)] \in C^\omega(\mathbb{B}_{\mathbb{R}^N}(0, r) \times \mathbb{D}_1, \mathbb{F}_0). \tag{A_6}$$

Remarks 3.7. **(a)** Assume that $D \to F_0$, $w \mapsto \Phi^* P(\Phi_* w)$ is equivariant with respect to G. Then assumption (A_6) is satisfied. Indeed, recalling (3.5) we get

$$\begin{aligned} \mathbb{Q}(\mu, w)(t) &= (T_\mu(t)\Phi)^* P((T_\mu(t)\Phi)_* w(t)) \\ &= T_\mu(t)\Phi^* P(\Phi_* T_\mu(-t) w(t)) \\ &= \Phi^* P(\Phi_* w(t)) = Q(w(t), \Phi) \end{aligned}$$

for $w \in \mathbb{D}_1$ and $t \in I$. Now the assertion follows from (3.2).

(b) Assume that $\Sigma = M = \mathbb{R}^m$ and that $\Phi = id$. Furthermore, set

$$E_0 := buc^\alpha(\mathbb{R}^m), \quad B := E_1 := buc^{2+\alpha}(\mathbb{R}^m)$$

for some $\alpha \in (0, 1)$. We fix $a \in buc^\alpha(\mathbb{R}^m)$ and define

$$P(w) := a\Delta w \quad \text{for} \quad w \in E_1. \tag{3.9}$$

Obviously, P is only equivariant under translations if a is constant. In order to satisfy (A_6) for nonconstant coefficient functions a, observe that

$$\mathbb{Q}(\mu, w)(t) = (\tau_{t\mu} a)\Delta w(t) \quad \text{for} \quad w \in \mathbb{E}_1,$$

where $\tau_{t\mu} a$ stands for the translation of a by the vector $t\mu \in \mathbb{R}^m$. Assume now

$$\begin{aligned} &a \in C^\infty(\mathbb{R}^m) \text{ and there is a } M_0 > 0 \text{ such that} \\ &\|\partial^\beta a\|_{BUC^\alpha(\mathbb{R}^m)} \le M_0 \beta! \text{ for all } \beta \in \mathbb{N}^m. \end{aligned} \tag{3.10}$$

Clearly, (3.10) implies that $a \in C^\omega(\mathbb{R}^m)$, but observe that (3.10) is in general stronger than pointwise analyticity. If (3.10) holds then P satisfies (A_6). Indeed, we first observe that the mapping $[(b, w) \mapsto b\Delta w] : \mathbb{E}_0 \times \mathbb{E}_1 \to \mathbb{E}_0$ is bilinear and bounded. Thus it remains to show that $[\mu \mapsto \tau_{t\mu} a] \in C^\omega(\mathbb{B}_{\mathbb{R}^m}(0, r), \mathbb{E}_0)$ for an appropriate number $r > 0$. An easy computation shows that

$$\|\partial_\mu^\beta \tau_{t\mu} a\|_{BUC^\alpha(\mathbb{R}^m)} = t^{|\beta|} \|\tau_{t\mu} \partial^\beta a\|_{BUC^\alpha(\mathbb{R}^m)} = t^{|\beta|} \|\partial^\beta a\|_{BUC^\alpha(\mathbb{R}^m)} \le M_0 T^{|\beta|} \beta!$$

for all $(\beta, \mu) \in \mathbb{R}^m \times \mathbb{R}^m$ and $t \in I$. Recall that we have $T < 1$ by assumption. Hence we conclude that (A_6) is satisfied if $r = 1$, for instance.

(c) It has been shown in [15] that the strategy of using maximal regularity in conjunction with the implicit function theorem does also guarantee that the solutions to $\partial_t u - a\Delta u = 0$ are analytic in space and time, provided a satisfies the weaker assumption $a \in C^\omega(\mathbb{R}^m) \cap buc^\alpha(\mathbb{R}^m)$ instead of (3.10). $\square$

We next fix $\varepsilon_0 > 0$ such that $\lambda t \in [0, t^+)$ for all $\lambda \in (1-\varepsilon_0, 1+\varepsilon_0)$ and all $t \in [0,T]$. Given now any $(\lambda,\mu) \in (1-\varepsilon_0, 1+\varepsilon_0) \times \mathbb{R}^N$, we define $w_{\lambda,\mu} \in F_0^I$ by $w_{\lambda,\mu}(t) := T_\mu(t)\Phi^* u(\lambda t)$ for $t \in I$, where we recall that u is the solution to (2.4).

Our next result shows that the function $w_{\lambda,\mu}$ solves a parameter dependent evolution equation involving the operators $\mathbb{Q}$ and A_μ. In order to economize our notation, we set $\Pi := \Pi(\varepsilon_0) := (1-\varepsilon_0, 1+\varepsilon_0) \times (-\varepsilon_0, \varepsilon_0)^N$.

Lemma 3.8. *Given $(\lambda,\mu) \in \Pi$, we have*

(i) $w_{\lambda,\mu} \in \mathbb{D}_1$.

(ii) $w_{\lambda,\mu}$ solves the evolution equation

$$\frac{d}{dt} w + \lambda \mathbb{Q}(\mu, w) = A_\mu w, \qquad w(0) = w_0, \tag{3.11}$$

where $w_0 := \Phi^* u_0$.

Proof: (i) This follows from (2.3), the definition of D, see (3.2), assumption (A_5), Lemma 3.4, and the analyticity of the exponential mapping. Furthermore, Lemma 3.4 implies that

$$\frac{d}{dt} w_{\lambda,\mu}(t) = T_\mu(t) A_\mu(\Phi^* u(\lambda t)) + \lambda (T_\mu(t)\Phi)^* \frac{d}{dt} u(\lambda t), \quad t \in I.$$

Observing (3.5) and the fact that $T_\mu(t)$ and A_μ commute on F_1 we get

$$\frac{d}{dt} w_{\lambda,\mu}(t) = A_\mu w_{\lambda,\mu}(t) + \lambda (T_\mu(t)\Phi)^* \frac{d}{dt} u(\lambda t), \quad t \in I. \tag{3.12}$$

(ii) Using (3.12) and (2.4) we now find

$$\frac{d}{dt} w_{\lambda,\mu}(t) = A_\mu w_{\lambda,\mu}(t) - \lambda (T_\mu(t)\Phi)^* P(u(\lambda t)), \quad t \in I.$$

From Lemma 3.3(ii) and the definitions of the operators Q and $\mathbb{Q}$ we further conclude

$$\begin{aligned}(T_\mu(t)\Phi)^* P(\Phi_* \Phi^* u(\lambda t)) &= T_\mu(t) Q(\Phi^* u(\lambda t), \Phi) \\ &= Q(T_\mu(t)\Phi^* u(\lambda t), T_\mu(t)\Phi) = \mathbb{Q}(\mu, w_{\lambda,\mu})(t)\end{aligned}$$

for $t \in I$. This completes the proof. □

Our next lemma contains the key result to show via the implicit function theorem that the mapping $(\lambda,\mu) \mapsto w_{\lambda,\mu}$ is analytic.

Lemma 3.9. *Given $((\lambda,\mu), w) \in \Pi \times \mathbb{D}_1$, let*

$$F((\lambda,\mu), w) := \Big(\frac{d}{dt} w + \lambda \mathbb{Q}(\mu, w) - A_\mu w, w(0) - w_0\Big).$$

Then

$$F \in C^\omega(\Pi \times \mathbb{D}_1, \mathbb{F}_0 \times F_1) \tag{3.13}$$

and

$$\partial_2 F((1,0), w) \in \mathcal{L}_{is}(\mathbb{F}_1, \mathbb{F}_0 \times F_1), \qquad w \in \mathbb{D}_1, \tag{3.14}$$

where $\partial_2 F$ is the derivative of F with respect to $w \in \mathbb{D}_1$.

Proof: (i) Clearly, we have that

$$\Big(\frac{d}{dt},\gamma\Big)\in\mathcal{L}(\mathbb{F}_1,\mathbb{F}_0\times F_1).$$

Hence it follows from Corollary 3.6 that

$$\Big[((\lambda,\mu),w)\mapsto\Big(\frac{d}{dt}w-A_\mu w,w(0)-w_0\Big)\Big]\in C^\omega(\Pi\times\mathbb{F}_1,\mathbb{F}_0\times F_1).$$

Thus we obtain (3.13) from assumption (A_6).

(ii) Let $w\in\mathbb{D}_1$ and $h\in\mathbb{F}_1$ be given. Then we have

$$\partial_2F((1,0),w)h=\frac{d}{d\varepsilon}F((1,0),w+\varepsilon h)\big|_{\varepsilon=0}=\Big(\frac{d}{dt}h+\partial_1Q(w,\Phi)h,h(0)\Big).$$

Combining Lemma 3.2(ii) with Remark III 3.4.2(c) in [2] it follows that, given $(f,\varphi)\in\mathbb{F}_0\times F_1$, there is a unique solution $h\in\mathbb{F}_1$ to the inhomogeneous evolution equation

$$\frac{d}{dt}h+\partial_1Q(w(t),\Phi)h=f(t),\qquad h(0)=\varphi.$$

(3.14) is now a consequence of the open mapping theorem. □

We are now prepared to show that $w_{\lambda,\mu}$ depends analytically on the parameter (λ,μ).

Proposition 3.10. *There is an $\varepsilon_0>0$ such that $[(\lambda,\mu)\mapsto w_{\lambda,\mu}]\in C^\omega(\Pi(\varepsilon_0),\mathbb{D}_1)$.*

Proof: Let F be given as in Lemma 3.9 and observe that $F((\lambda,\mu),w)=0$ if and only if $w\in\mathbb{D}_1$ is a solution to

$$\frac{d}{dt}w+\lambda\mathbb{Q}(\mu,w)=A_\mu w,\qquad w(0)=w_0.$$

Now the assertion follows from Lemma 3.8, Lemma 3.9, and the implicit function theorem in Banach spaces. □

It remains to translate the above Proposition into the desired analyticity of $\hat{u}$, see the beginning of this section.

Theorem 3.11. *Assume that (A_1)-(A_6) hold true. Then $\hat{u}\in C^\omega((0,t^+)\times\Sigma)$.*

Proof: (i) Let $\hat{w}:=\Phi^*\hat{u}$, i.e. $\hat{w}(t,p):=\hat{u}(t,\Phi(p))$ for $(t,p)\in(0,t^+)\times M$. It suffices to show that $\hat{w}\in C^\omega((0,t^+)\times M)$. For this we fix $(t_0,p_0)\in(0,t^+)\times M$. Moreover there exists a subset $\{j_1,\ldots,j_m\}$ of $\{1,\ldots,N\}$ such that $\{X_{j_1},\ldots,X_{j_m}\}\subset L(G)$ induces via the integral curves $[t\mapsto\exp(tX_{j_k})\cdot p_0]$ a basis of $T_{p_0}M$ (recall that we identify M with the coset manifold G/H, see also Theorem IV.3.3(iii) in [17]). Without loss of generality we may assume that $j_1=1,\ldots,j_m=m$. Moreover, in the following we write $\hat{\mu}=(\mu_1,\ldots,\mu_m,0,\ldots,0)\in\mathbb{R}^N$ for $(\mu_1,\ldots,\mu_m)\in\mathbb{R}^m$ and we identify

$$\Pi_m:=\Pi_m(\varepsilon_0):=(1-\varepsilon_0,1+\varepsilon_0)\times(-\varepsilon_0,\varepsilon_0)^m\quad\text{with}\quad\Pi\cap(\mathbb{R}^{m+1}\times\{0\}).$$

Shrinking $\varepsilon_0>0$ if necessary, we have that

$$\varphi:\Pi_m\to(0,t^+)\times M,\qquad(\lambda,\hat{\mu})\mapsto(\lambda t_0,T_{\hat{\mu}}(t_0)\cdot p_0)$$

is an analytic parametrization of an open neighborhood O of (t_0, p_0) in $(0, t^+) \times M$.

(ii) Observe that by assumption (A_1) we know that $\mathbb{D}_1 \subset C(I, BUC(M))$. Thus the evaluation mapping

$$\mathbb{D}_1 \to \mathbb{R}, \qquad w \mapsto w(t_0)(p_0)$$

is well-defined and clearly analytic. Combining this with Proposition 3.10 we find

$$[(\lambda, \hat{\mu}) \mapsto w_{\lambda,\hat{\mu}}(t_0)(p_0)] \in C^\omega(\Pi_m, \mathbb{R}).$$

But $\varphi^* \hat{w}(\lambda, \hat{\mu}) = w_{\lambda,\hat{\mu}}(t_0)(p_0)$ for $(\lambda, \hat{\mu}) \in \Pi_m$. This shows that $\hat{w} \in C^\omega(O, \mathbb{R})$ and completes the proof. $\square$

Proof of Theorem 1.1: Setting $\Sigma = M$ and $\Phi = id$, it is clear that the hypotheses of Theorem 1.1 on M and G imply that (A_1)-(A_3) are fulfilled. Moreover, (A_5) and, by Remark 3.7(a) also (A_6) are satisfied by the assumed equivariance of P with respect to G. Finally, let $X \in L(G)$ be given. Then, using local coordinates, it is not difficult to verify that

$$\lim_{t \to 0} \frac{\exp(tX) \cdot w - w}{t} = A_X w \quad \text{in} \quad buc^\alpha(M)$$

for all $w \in buc^{1+\alpha}(M)$. Since $buc^\alpha(M) \hookrightarrow E_0$ it follows from Lemma 3.4 that $buc^{1+\alpha}(M)$ is contained in $\mathrm{dom}(A_X)$. Hence (1.3) implies that (A_4) is true as well. Now the assertion follows from Theorem 3.11. $\square$

4. Applications

This section is devoted to two applications of Theorem 3.11. First we consider the Bellman equation on $\mathbb{R}^m$, an equation occuring in stochastic control otheory. Secondly, we discuss a generalized mean curvature flow on spheres. For a further example of a fully nonlinear parabolic evolution equation on $\mathbb{R}^m$ which is equivariant with respect to translations we refer to [11].

4.1. Bellman Equations.

In certain cases the Bellman equation occuring in stochastic control theory leads to the following fully nonlinear partial differential equation, cf. Section 8.5.5 in [22],

$$\partial_t u = \frac{1}{2} \sum_{j,k=1}^m a_{jk} \partial_j \partial_k u - \frac{1}{2} \langle (I + \partial^2 u)^{-1} \partial u | \partial u \rangle + \sum_{j=1}^m b_j \partial_j u, \tag{4.1}$$

to be satisfied on $(0, T) \times \mathbb{R}^m$, where $T > 0$ is given. In (4.1), the Euclidean inner product on $\mathbb{R}^m$ is denoted by $\langle \cdot | \cdot \rangle$ and $I := id_{\mathbb{R}^m}$. Moreover, ∂u and $\partial^2 u$ stand for the gradient and the Hessian of u, respectively. Concerning the coefficients we assume that

$$a_{jk},\ b_j \in buc^\alpha(\mathbb{R}^m) \quad \text{with} \quad a_{jk} = a_{kj} \tag{4.2}$$

for $j,\, k = 1,\dots,m$. Here, $\alpha \in (0,1)$ is fixed. Given $x \in \mathbb{R}^m$, we set $A(x) := [a_{jk}(x)] \in \mathbb{R}^{m\times m}$ and assume the following ellipticity condition to hold: There is a constant $c > 0$ such that

$$\langle A(x)\xi|\xi\rangle \geq c|\xi|^2, \qquad (x,\xi) \in \mathbb{R}^m \times \mathbb{R}^m. \tag{4.3}$$

The equation (4.1) is complemented by the initial condition

$$u(0,x) = u_0(x), \qquad x \in \mathbb{R}^m. \tag{4.4}$$

Here we suppose that $u_0 \in buc^{2+\alpha}(\mathbb{R}^m)$ and that there is a $\delta > 0$ such that

$$\det(I + \partial^2 u_0(x)) \geq \delta, \qquad x \in \mathbb{R}^m. \tag{4.5}$$

By continuity, there is an $\varepsilon > 0$ such that, given $v \in B := \mathbb{B}_{buc^{2+\alpha}(\mathbb{R}^m)}(u_0,\varepsilon)$, we have

$$\det(I + \partial^2 v(x)) \geq \delta/2, \qquad x \in \mathbb{R}^m. \tag{4.6}$$

In order to economize our notation, we set $j(v) := (I+\partial^2 v)^{-1}$ for $v \in B$, $b = (b_1,\dots,b_m)$, and

$$P(v) := \frac{1}{2}\mathrm{trace}(A\partial^2 v) - \frac{1}{2}\langle j(v)\partial v|\partial v\rangle + \langle b|\partial v\rangle, \quad v \in B.$$

It follows from (4.6) and the fact that inversion maps $\mathcal{L}_{is}(\mathbb{R}^m)$ analytically into $\mathcal{L}(\mathbb{R}^m)$, that

$$P \in C^\omega(B, buc^\alpha(\mathbb{R}^m)). \tag{4.7}$$

Moreover, given $v \in B$ and $h \in buc^{2+\alpha}(\mathbb{R}^m)$, we have $\partial j(v)h = -j(v)\partial^2 h j(v)$, and therefore

$$\partial P(v)h = \frac{1}{2}\mathrm{trace}(A\partial^2 h) + \frac{1}{2}\langle \partial^2 h j(v)\partial v|j(v)\partial v\rangle - \langle j(v)\partial h|\partial v\rangle + \langle b|\partial h\rangle.$$

Let now

$$p(v) := \frac{1}{2}\left(A + (j(v)\partial v \otimes j(v)\partial v)\right), \qquad v \in B,$$

denote the principal symbol of the second order operator $\partial P(v)$. Then we have

$$\langle p(v)(x)\xi|\xi\rangle = \frac{1}{2}\left(\langle A(x)\xi|\xi\rangle + \langle \xi|j(v)(x)\partial v(x)\rangle^2\right), \quad (x,\xi) \in \mathbb{R}^m \times \mathbb{R}^m.$$

Hence (4.3) implies that $\partial P(v)$ is uniformly elliptic and we conclude from Theorem 4.2 and Remark 4.6 in [3] that

$$\partial P(v) \in \mathcal{H}(buc^{2+\beta}(\mathbb{R}^m), buc^\beta(\mathbb{R}^m)), \qquad \beta \in (0,\alpha],\ v \in B. \tag{4.8}$$

Setting $E_0 := buc^\alpha(\mathbb{R}^m)$ and $E_1 := buc^{2+\alpha}(\mathbb{R}^m)$, it follows from (4.7), (4.8), and Remark 2.2(j) that (2.1) and (2.2) hold true. Hence we obtain the following existence and uniqueness result for (4.1), (4.4).

Proposition 4.1. *Let $u_0 \in buc^{2+\alpha}(\mathbb{R}^m)$ satisfy (4.5) and assume that (4.2), (4.3) hold true. Then there is a $t^+ := t^+(u_0) > 0$ such that the Bellman equation*

$$\begin{cases} \partial_t u = \frac{1}{2}\mathit{trace}(A\partial^2 u) - \frac{1}{2}\langle j(u)\partial u|\partial u\rangle + \langle b|\partial u\rangle, & (t,x) \in (0,T)\times\mathbb{R}^m, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^m \end{cases} \tag{4.9}$$

has a unique solution $u \in C([0,t^+), buc^{2+\alpha}(\mathbb{R}^m)) \cap C^1([0,t^+), buc^{\alpha}(\mathbb{R}^m))$.

Using classical Hölder spaces of parabolic type, a similar result is derived in [22]. In order to apply Theorem 3.11, we have to increase the regularity assumptions upon the coefficients in the following way. We suppose that

$$\begin{aligned} & a_{jk},\ b_j \in C^\infty(\mathbb{R}^m), \quad j,\, k = 1, \dots, m, \\ & \text{and that there is a } M_0 > 0 \text{ such that for all } \beta \in \mathbb{N}^m : \\ & \|\partial^\beta a_{jk}\|_{BUC^\alpha(\mathbb{R}^m)},\ \|\partial^\beta b_j\|_{BUC^\alpha(\mathbb{R}^m)} \le M_0 \beta! \,. \end{aligned} \tag{4.10}$$

We are now prepared to show that the solutions to the Bellman equation (4.9) constructed in Proposition 4.1 are in fact real analytic in space and time.

Theorem 4.2. *Let* $u_0 \in buc^{2+\alpha}(\mathbb{R}^m)$ *satisfy (4.5). Morevoer, assume that (4.3) and (4.10) hold true and let* $u \in C([0,t^+), buc^{2+\alpha}(\mathbb{R}^m)) \cap C^1([0,t^+), buc^{\alpha}(\mathbb{R}^m))$ *be the solution to (4.9) as constructed in Proposition 4.1. Then*

$$[(t,x) \mapsto u(t)(x)] \in C^\omega((0,t^+) \times \mathbb{R}^m, \mathbb{R}).$$

Proof: Let $\Sigma = \mathbb{R}^m$ and $\Phi = id_{\mathbb{R}^m}$. Then it is not difficult to verify that the assumptions (A_1)-(A_5) of Section 3 are satisfied. Moreover, given $\mu \in \mathbb{R}^m$, $w \in \mathbb{E}_1$, and $t \in I$, we have

$$\begin{aligned} \mathbb{Q}(\mu, w)(t) \quad &= \quad \frac{1}{2} \sum_{j,k=1}^{m} (\tau_{t\mu} a_{jk}) \partial_j \partial_k w(t) - \frac{1}{2} \langle (I + \partial^2 w(t))^{-1} \partial w(t) | \partial w(t) \rangle \\ &+ \quad \sum_{j=1}^{m} (\tau_{t\mu} b_j) \partial_j w(t). \end{aligned}$$

Arguing as in Remark 3.7(b), one shows that also assumption (A_6) is satisfied and the assertion follows from Theorem 3.11. □

4.2. **A Generalized Motion by Mean Curvature.** Consider the following nonlocal geometric evolution equation:

$$V(t) = -(-\Delta_{M(t)})^\beta \left(1 - \Delta_{M(t)}\right)^\gamma H(t) \qquad \text{on } M(t) \text{ for } t > 0. \tag{4.11}$$

Here $M(t)$ is an unknown, with respect to time $t > 0$ evolving closed compact oriented hypersurface in $\mathbb{R}^{m+1}$. We write $\Delta_{M(t)}$ for the Laplace-Beltrami operator on $M(t)$ with respect to the Euclidean metric. The mean curvature of $M(t)$ is denoted by $H(t)$ and V stands for the normal velocity of the family $\{M(t)\,;\, t > 0\}$. Finally, β and γ are given real numbers. The evolution equation (4.11) does not depend on the local choice of orientation. However, if $M(t)$ encloses a domain $\Omega(t)$, we always choose the orientation such that $V(t)$ is positive if $\Omega(t)$ grows and such that $H(t)$ is positive if $M(t)$ is convex with respect to $\Omega(t)$. We mention that

in the plane sometimes the opposite orientation is used.
For the values $\beta = 1$ and $\gamma = -1$ the evolution equation (4.11) is a special case of the so-called intermediate surface diffusion flow, introduced by Cahn and Taylor, see [8]. Assume next that $\beta = 0$. Then equation (4.11) reduces to the (negative) gradient flow of the area functional $\int_{M(t)} d\sigma(t)$ in the Sobolev space $H^{-\gamma}$. Also in the case $\gamma = 0$ the equation (4.11) is the (negative) gradient flow of the area functional in $H^{-\beta}$, but with the constrain that the volume of the domain $\Omega(t)$ enclosed by $M(t)$ is preserved during the evolution. Indeed, consider first the situation when $\beta = 0$. We assume that we are given a smooth solution $\{M(t)\,;\, t \in (0,T]\}$ to

$$V(t) = -\left(1 - \Delta_{M(t)}\right)^{\gamma} H(t) \quad \text{on} \quad M(t) \quad \text{for} \quad t \in (0,T]. \tag{4.12}$$

We fix $t \in (0,T]$ and set $M := M(t)$. The normal field on M is denoted by ν_M. Given $h \in C^\infty(M)$, let

$$X_h \,:\, M \times (-\varepsilon, \varepsilon) \to \mathbb{R}^{m+1}, \qquad X_h(s,\tau) := s + \tau h \nu_M(s)$$

be the normal variation of M. Then $X_h(\cdot,\tau)$ is a smooth diffeomorphism from M onto its range $M_{\tau,h} := X_h(M,\tau)$, provided $\varepsilon > 0$ is chosen sufficiently small. Let

$$A_h(\tau) := m \int_{M_{\tau,h}} d\sigma(\tau), \quad \tau \in (-\varepsilon, \varepsilon),$$

denote the m-fold area functional of $M_{\tau,h}$. Here, the factor m in front of the integral is introduced just to simplify some of the calculations below. Using this notation, we say that the geometric evolution law

$$V(t) = F(M(t)), \quad t \in (0,T] \tag{4.13}$$

is the (negative) gradient flow of the area functional in $H^{-\gamma}$ if

$$A_h'(0) = \frac{d}{d\tau} \int_{M_{\tau,h}} d\sigma(\tau)|_{\tau=0} = (-F|h)_{H^{-\gamma}(M)}, \qquad h \in C^\infty(M).$$

Recall that $A_h'(0) = \int_M H_M h \, d\sigma$, where H_M stands for the mean curvature of M. Besides, the inner product in $H^{-\gamma}(M)$ is given by

$$(f|g)_{H^{-\gamma}(M)} = ((1-\Delta_M)^{-\gamma/2} f | (1-\Delta_M)^{-\gamma/2} g)_{L_2(M)},$$

see Section 4.2 in [23]. Consequently, we get

$$(H_M|h)_{L_2(M)} = -((1-\Delta_M)^{-\gamma/2} F | (1-\Delta_M)^{-\gamma/2} h)_{L_2(M)}.$$

Using the fact that $(1-\Delta_M)^{-\gamma/2}$ is self-adjoint in $L_2(M)$ and the density of $C^\infty(M)$ in $L_2(M)$, we find that

$$F = -(1-\Delta_M)^{\gamma} H_M.$$

We consider next (4.11) in the case when $\gamma = 0$. As before, let $\{M(t)\,;\, t \in (0,T]\}$ be a smooth solution to (4.13) and assume in addition that each $M(t)$ encloses a well-defined domain $\Omega(t)$. Writing $vol(t) := \int_{\Omega(t)}$ for the $m+1$-dimensional volume of $\Omega(t)$, we have

$$vol'(t) = \int_{M(t)} V(t)\, d\sigma(t) = \int_{M(t)} F(M(t))\, d\sigma(t), \quad t \in (0,T]. \tag{4.14}$$

Hence the flow (4.13) preserves the volume if $\int_{M(t)} F(M(t))\, d\sigma(t) = 0$. We fix again $t \in (0,T]$ and set $M := M(t)$. Moreover, we let

$$\mathbf{1}^\perp := \left\{ f \in L_2(M)\,;\, \int_M f\, d\sigma = 0 \right\}$$

and write

$$P : L_2(M) \to \mathbf{1}^\perp, \quad f \mapsto f - \overline{f}$$

for the projection onto $\mathbf{1}^\perp$. Here, we used the notation $\overline{f} := \int_M f\, d\sigma / \int_M d\sigma$. Note that $f \mapsto \|(-\Delta_M)^{\beta/2} f\|_{L_2(M)}$ is an equivalent norm on $H^\beta(M) \cap \mathbf{1}^\perp$ and that the normal variation X_h is volume preserving, provided h belongs to $C^\infty(M) \cap \mathbf{1}^\perp$. Indeed, the normal velocity of the variation X_h is given by

$$V = (\partial_\tau X_h | \nu_M)|_{\tau=0} = h,$$

and the assertion follows from (4.14). According to these observations we say that (4.13) is the (negative) volume preserving gradient flow of the area functional in $H^{-\beta}$ if $\overline{F} = 0$ and if

$$A_h'(0) = -((-\Delta_M)^{-\beta/2} F | (-\Delta_M)^{-\beta/2} h)_{L_2(M)}, \qquad h \in C^\infty(M) \cap \mathbf{1}^\perp.$$

This implies that

$$\int_M HPg\, d\sigma = (PH|g)_{L_2(M)} = -((-\Delta_M)^{-\beta} F | Pg)_{L_2(M)} = -((-\Delta_M)^{-\beta} F | g)_{L_2(M)}$$

for all $g \in C^\infty(M)$, since $P = P^*$, $P(-\Delta_M)^{-\beta} P = (-\Delta_M)^{-\beta} P$, and $PF = F$. Using the density of $C^\infty(M)$ in $L_2(M)$, we therefore obtain

$$F = -(-\Delta_M)^\beta PH = -(-\Delta_M)^\beta (H - \overline{H}).$$

Finally, observe that $(-\Delta_M)^\beta \mathbf{1} = 0$ if $\beta > 0$. Hence we find

$$F = \begin{cases} -(-\Delta_M)^\beta H & \text{if } \beta > 0, \\ \overline{H} - H & \text{if } \beta = 0. \end{cases}$$

We remark that in the case $\beta = 0$ the evolution law $V = \overline{H} - H$ is known as the volume preserving mean curvature. In case $\beta = 1$, one calls $V = \Delta_M H$ the surface diffusion flow.

For simplicity, we restrict ourselves in the following to the case $\beta = 0$ and $\gamma \in [0, 1/2]$, i.e. we consider

$$V(t) = -(1 - \Delta_{M(t)})^{-\gamma} H(t), \quad t > 0, \quad M(0) = M_0. \tag{4.15}$$

The change of sign in the exponent has been made for simplicity only. We shall see that, given $\alpha \in (1/2, 1)$, the flow (4.15) is well-posed in $h^{\alpha-1+2\gamma}$ for initial data M_0 belonging to $h^{1+\alpha}$. For values of β, γ with $\beta \geq 0$ and $\beta - \gamma > 0$, the general equation (4.11) has to be treated in spaces with more regularity. If $\beta < 0$ one has to work in suitable subspaces of $\mathbf{1}^{\perp}$.

We parametrize (4.15) in a neighborhood of an analytic compact closed immersed oriented hypersurface Σ in $\mathbb{R}^{m+1}$. To make this precise, let ν denote the unit outer normal field on Σ. Moreover, given $a > 0$, choose a localization system $\{(U_l, \varphi_l)\,;\, l = 1, \ldots, n\}$ for Σ such that $\Sigma = \cup_{l=1}^{n} U_l$ and

$$\varphi_l \,:\, (-a, a)^m \to U_l, \quad l \in \{1, \ldots, n\},$$

is an analytic parametrization of U_l. Shrinking $a > 0$ if necessary, we may assume that

$$X_l \,:\, U_l \times (-a, a) \to \mathbb{R}^{m+1}, \qquad X_l(s, r) := s + r\nu(s).$$

is a smooth diffeomorphism onto its image $\mathcal{R}_l := \mathrm{im}(X_l)$, i.e.

$$X_l \in \mathrm{Diff}^{\omega}(U_l \times (-a, a), \mathcal{R}_l).$$

The inverse of X_l can be decomposed in the following way. Writing $S_l \in C^{\omega}(\mathcal{R}_l, U_l)$ and $\Lambda_l \in C^{\omega}(\mathcal{R}_l, (-a, a))$ for the metric projection of $\mathcal{R}_l$ onto U_l and for the signed distance function with respect to U_l, respectively, we have $X_l^{-1} = (S_l, \Lambda_l)$. In particular, observe that $\mathcal{R} := \cup_{l=1}^{n} \mathcal{R}_l$ consists of those points in $\mathbb{R}^{m+1}$ with distance less than a to Σ.

We now fix $\alpha > 1/2$, let

$$W(\Sigma) := W_a(\Sigma) := \{\rho \in h^{1+\alpha}(\Sigma)\,;\, \|\rho\|_{C^1(\Sigma)} \leq a/2\},$$

and define

$$M_\rho := \bigcup_{l=1}^{n} \{X_l(s, \rho(s))\,;\, s \in U_l\}$$

for $\rho \in W(\Sigma)$. Then M_ρ is a compact closed oriented immersed hypersurface in $\mathbb{R}^{m+1}$ of class $C^{1+\alpha}$, which can be seen as a graph in normal direction over Σ. Of course, ρ measures the signed distance of Σ to M_ρ. For convenience let us also introduce the mapping

$$\theta_\rho \,:\, \Sigma \to M_\rho, \quad s \mapsto X_l(s, \rho(s)) \quad \text{for} \quad s \in U_l.$$

Then θ_ρ is a well-defined global diffeomorphism of class $C^{2+\alpha}$ from Σ onto M_ρ. By means of this diffeomorphism we can pull back the Euclidean metric on M_ρ to Σ, producing in that way a Riemannian manifold which we denote in the following by $\Sigma(\rho)$. We now consider a family of hypersurfaces in $\mathcal{R}$. More precisely, let $T > 0$ be given, and define $I := [0, T]$, as well as

$$W(\Sigma_T) := W_a(\Sigma_T) := \{\rho \in C(I, h^{1+\alpha}(\Sigma))\,;\, \|\rho\|_{C(I, C^1(\Sigma))} \leq a/2\}.$$

Then, given $\rho \in W(\Sigma_T)$, we transform the evolution equation (4.15) for the family $\{M_{\rho(t)}\,;\, t \in [0, T]\}$ into an evolution equation on Σ. For this we first calculate the normal velocity of $[t \mapsto M_{\rho(t)}]$. We have, cf. [12],

$$V(t, s) = \partial_t \rho(t, s) \Big/ |\nabla_x \Phi_\rho(x, t)|_{x = \theta_{\rho(t)}(s)} \qquad \text{for} \quad (t, s) \in I \times \Sigma,$$

where we used the function

$$\Phi_\rho : \mathcal{R} \times [0,T] \to \mathbb{R}, \quad (x,t) \mapsto \Lambda(x) - \rho(t, S(x))$$

to represent $M_{\rho(t)}$ as the 0−level set of $\Phi_\rho(\cdot,t)$, i.e. $M_{\rho(t)} = \Phi^{-1}(\cdot,t)(0)$. To shorten our notation, let

$$L(\rho)(t,s) := |\nabla_x \Phi_\rho(x,t)|_{x=\theta_{\rho(t)}(s)}, \quad (t,s) \in I \times \Sigma.$$

Moreover, we write $K(\rho) := \theta_\rho^* H$ and Δ_ρ for the mean curvature and Laplace-Beltrami operator of $\Sigma(\rho)$, respectively. Here, θ_ρ^* denotes the pull-back operator induced by the diffeomorphism θ_ρ, i.e. $\theta_\rho^* f = f \circ \theta_\rho$ for $f \in C(M_\rho)$. This means in particular that we have

$$\theta_\rho^* \Delta_{M_\rho} = \Delta_\rho \theta_\rho^*.$$

Given $\rho_0 \in W_a(\Sigma)$, consider now the following nonlinear nonlocal partial differential equation

$$\frac{d\rho}{dt} = -L(\rho)(1-\Delta_\rho)^{-\gamma} K(\rho) \quad \text{in} \quad I \times \Sigma, \qquad \rho(0) = \rho_0 \quad \text{on} \quad \Sigma. \tag{4.16}$$

In order to treat (4.16) in the framework of Theorem 3.11, we set

$$P(\rho) := L(\rho)(1-\Delta_\rho)^{-\gamma} K(\rho) \quad \text{for} \quad \rho \in W(\Sigma) \cap h^{2+\alpha}(\Sigma).$$

Our first result in this section shows that P can be extended to an analytic mapping with values in $h^{\alpha-1+2\gamma}(\Sigma)$.

Lemma 4.3. *There exists an extension of P, again denoted by P, such that $P \in C^\omega(W(\Sigma), h^{\alpha-1+2\gamma}(\Sigma))$.*

Proof: **(i)** We first express the terms $L(\rho)$, $K(\rho)$, and Δ_ρ in local coordinates. To make this precise, let

$$\hat\rho_l(s) := \rho(\varphi_l(s)), \quad \hat X_l(s,r) := X_l(\varphi_l(s), r), \quad (s,r) \in (-a,a)^{m+1},$$

be the local representations of ρ_l and X_l with respect to U_l. In the following we do not always distinguish between ρ_l, X_l and their local representations $\hat\rho_l$, $\hat X_l$, as well as between local coordinates $s \in (-a,a)^m$ and the corresponding points $\varphi_l(s)$ on U_l. Moreover, we suppress the index $l \in \{1,\dots,n\}$ if no confusion seems likely. Given $\rho \in W(\Sigma)$, define

$$w_{jk}(\rho)(s) := (\partial_j X | \partial_k X)|_{(s,\rho(s))}, \quad s \in (-a,a)^m,$$

for j, $k \in \{1,\dots,m\}$, where $(\cdot|\cdot)$ stands for the Euclidean metric in $\mathbb{R}^{m+1}$ and ∂_j denotes the partial derivative with respect to the j-th variable of s. Since ρ belongs to $W(\Sigma)$, the matrix $[w_{jk}(\rho)]$ is invertible and we write $w^{jk}(\rho)$ for the entries of its inverse. Then we have

$$L(\rho) = \sqrt{1 + w^{jk}(\rho)\partial_j\rho\partial_k\rho} \tag{4.17}$$

cf. (2.3) in [14]. In (4.17) and in what follows we use summation convention over repeated indices. Moreover, we write

$$\Gamma^i_{jk}(\rho) := \frac{1}{2} w^{il}(\rho)\Big(\partial_k(\partial_l X|\partial_j X) - \partial_l(\partial_j X|\partial_k X) + \partial_j(\partial_k X|\partial_l X)\Big)\Big|_{(\cdot,\rho)},$$

for the corresponding Christoffel symbols. Then Lemma 2.1 in [14] shows that $K(\rho)$ carries a quasi-linear structure, i.e. given $\rho \in W(\Sigma)$, there are

$$K_1(\rho) \in \mathcal{L}(h^{2+\alpha}(\Sigma), h^\alpha(\Sigma)) \quad \text{and} \quad K_2(\rho) \in h^\alpha(\Sigma) \tag{4.18}$$

such that

$$K(\rho) = K_1(\rho)\rho + K_2(\rho) \quad \text{for} \quad \rho \in W(\Sigma) \cap h^{2+\alpha}(\Sigma). \tag{4.19}$$

In the chosen local coordinates these mappings are represented as:

$$\begin{aligned} K_1(\rho) =& \frac{1}{mL(\rho)^3}\Big[\Big\{ - L(\rho)^2 w^{jk}(\rho) + w^{jl}(\rho)w^{kn}(\rho)\partial_l\rho\partial_n\rho\Big\}\partial_j\partial_k \\ &+ \Big\{L(\rho)^2 w^{jk}(\rho)\Gamma^i_{jk}(\rho) + w^{jl}(\rho)w^{ki}(\rho)\Gamma^{m+1}_{jk}(\rho)\partial_l\rho \\ &\quad + 2w^{kn}(\rho)\Gamma^i_{(m+1)k}(\rho)\partial_n\rho - w^{jl}(\rho)w^{kn}(\rho)\Gamma^i_{jk}(\rho)\partial_l\rho\partial_n\rho\Big\}\partial_i\Big] \end{aligned} \tag{4.20}$$

and

$$K_2(\rho) = -\frac{1}{mL(\rho)} w^{jk}(\rho)\Gamma^{m+1}_{jk}(\rho). \tag{4.21}$$

In order to express Δ_ρ in local coordinates, let η be the Euclidean metric and write $\sigma(\rho) := \theta^*_\rho\eta$ for the Riemannian metric on Σ induced by the diffeomorphism θ_ρ. This means that, using the above introduced notation, we have $\Sigma(\rho) = (\Sigma, \sigma(\rho))$. Let further $\sigma_{jk}(\rho)$ denote the components of $\sigma(\rho)$ in local coordinates and write $\sigma^{jk}(\rho)$ for the components of the inverse of $[\sigma_{jk}(\rho)]$. Using again summation convention over repeated indices, the Christoffel symbols of $\sigma(\rho)$ in the chosen coordinates are given by

$$\gamma^l_{jk}(\rho) = \frac{\sigma^{ln}(\rho)}{2}\left[\frac{\partial\sigma_{kn}(\rho)}{\partial s^j} + \frac{\partial\sigma_{jn}(\rho)}{\partial s^k} - \frac{\partial\sigma_{jk}(\rho)}{\partial s^n}\right],$$

and we have

$$\Delta_\rho = \sigma^{jk}(\rho)\left[\frac{\partial^2}{\partial s^j\partial s^k} - \gamma^l_{jk}(\rho)\frac{\partial}{\partial s^l}\right], \quad \rho \in W(\Sigma) \cap h^{2+\alpha}(\Sigma), \tag{4.22}$$

cf. the proof of Lemma 2.1 in [14].

(ii) It follows from [12, p. 1037] that $w_{jk}(\rho)$ is a quadratic polynomial in ρ. Moreover, we have that

$$[\rho \mapsto \theta_\rho - id_\Sigma] \in \mathcal{L}(h^{1+\alpha}(\Sigma), h^{1+\alpha}(\Sigma, \mathbb{R}^{m+1})).$$

This obviously implies that

$$w_{jk} \in C^\omega(W(\Sigma), h^{1+\alpha}(\Sigma)), \qquad \sigma_{jk} \in C^\omega(W(\Sigma), h^\alpha(\Sigma)), \tag{4.23}$$

and consequently:

$$w^{jk},\ \Gamma^i_{jk} \in C^\omega(W(\Sigma), h^{1+\alpha}(\Sigma)), \quad \sigma^{jk} \in C^\omega(W(\Sigma), h^\alpha(\Sigma)). \tag{4.24}$$

(iii) Combining (4.24) and (4.17), we see that

$$L \in C^\omega(W(\Sigma), h^\alpha(\Sigma)). \tag{4.25}$$

Recall that we have assumed that $\alpha > 1/2$. Hence $\alpha > \max\{1-\alpha, \alpha-1\}$, and we conlcude from [26, Theorem 2.8.2] and a localization argument that

$$h^\alpha(\Sigma) \times h^{\alpha-1}(\Sigma) \to h^{\alpha-1}(\Sigma), \quad (f,g) \mapsto fg \tag{4.26}$$

is continuous and bilinear. Using this and Theorem 2.3.8 in [26], it follows from (4.20), (4.21), and (4.24) that

$$[\rho \mapsto K(\rho)] \in C^\omega(W(\Sigma), h^{\alpha-1}(\Sigma)). \tag{4.27}$$

(iv) Let $\rho_0 \in W(\Sigma)$ be given and choose $\varepsilon > 0$ such that

$$B_0 := \mathbb{B}_{h^{1+\alpha}(\Sigma)}(\rho_0, \varepsilon_0) \subset W(\Sigma).$$

For simplicity we set $X_0 := h^{\alpha-1}(\Sigma)$ and $X_1 := h^{1+\alpha}(\Sigma)$, and denote the X_0-realization of $1 - \Delta_\rho$ by $A(\rho)$. Shrinking $\varepsilon_0 > 0$, it follows from Theorem 4.2 in [3] and a localization argument that

$$A(\rho) \in \mathcal{H}(X_1, X_0), \qquad \rho \in B_0. \tag{4.28}$$

Moreover, we conclude from Theorem 7.4.3 and Remark 7.2.5.1 in [27] that $\mathbb{R}_+$ belongs to the resolvent set $\mathrm{res}(A(\rho))$ of $A(\rho)$ for all $\rho \in B_0$. Thus, given $\rho \in B_0$ and $s \in \mathbb{R}$ we conclude that $A(\rho)^s$ is a well-defined linear operator in X_0. Observing (4.22), we infer from (4.28) that

$$[\rho \mapsto A(\rho)A(0)^{-1}] \in C^\omega(B_0, \mathcal{L}(X_0)).$$

Hence we find

$$[\rho \mapsto A(0)^\gamma A(\rho)^{-\gamma}] \in C^\omega(B_0, \mathcal{L}(X_0)), \tag{4.29}$$

since $[B \mapsto B^{-\gamma}] \in C^\omega(\mathcal{L}(X_0))$, see Theorem VIII.7 in [28]. Recall that $A(0) = 1 - \Delta_0 = 1 - \Delta_\Sigma$ is a uniformly elliptic operator with smooth coefficients on the compact smooth manifold Σ. Thus Theorem III.4.7.5 in [2] implies that $1 - \Delta_0$ possesses bounded imaginary powers, i.e., there exists a $K > 0$ such that

$$\|(1-\Delta_0)^{it}\|_{\mathcal{L}(X_0)} \le K, \quad |t| \le 1.$$

Hence it follows from Theorem 3 in [24] and the fact that the spaces $h^s(\Sigma)$ are stable under complex interpolation, cf. Theorem 7.4.4 in [27], that

$$A(0)^{-\gamma} \in \mathcal{L}(h^{\alpha-1}(\Sigma), h^{\alpha-1+2\gamma}(\Sigma)). \tag{4.30}$$

Combining this with (4.29) we get

$$[\rho \mapsto A(\rho)^{-\gamma}] \in C^\omega(B_0, \mathcal{L}(h^{\alpha-1}(\Sigma), h^{\alpha-1+2\gamma}(\Sigma))). \tag{4.31}$$

Recall that $\gamma \leq 1/2$. Thus pointwise multiplication maps $h^{\alpha-1+2\gamma}(\Sigma) \times h^\alpha(\Sigma)$ bilinearly and continuously into $h^{\alpha-1+2\gamma}(\Sigma)$ and it remains to combine (4.25), (4.27), and (4.31) to complete the argumentation. □

In the following example we choose $\gamma = 1/2$. Given $\rho \in W(\Sigma)$, set

$$A_0(\rho) := L(\rho)(1-\Delta_\rho)^{-1/2}K_1(\rho) \quad \text{and} \quad f(\rho) := -L(\rho)(1-\Delta_\rho)^{-1/2}K_2(\rho).$$

Obviously, we have $P(\rho) = A_0(\rho)\rho - f(\rho)$ for $\rho \in W(\Sigma)$. Since also

$$A_0(\rho) \in \mathcal{L}(h^{1+\alpha}(\Sigma), h^\alpha(\Sigma)), \quad f \in C^\omega(W(\Sigma), h^\alpha(\Sigma)),$$

it follows that P consists of the quasilinear operator A_0 and the perturbation f. Nevertheless, the abstract equation

$$\frac{d\rho}{dt} + A_0(\rho)\rho = f(\rho)$$

cannot be treated in the framework of [1], since neither $W(\Sigma)$ nor the range of f is contained in any interpolation space of $h^\alpha(\Sigma)$ and $h^{1+\alpha}(\Sigma)$ obtained by an admissible interpolation functor in the sense of [1]. □

We are now going to verify that the operator P satisfies the assumptions (2.1) and (2.2). To avoid too many technicalities, we consider here the particular situation that $M = \Sigma = \mathbb{S}^m$, where $\mathbb{S}^m$ denotes the standard unit sphere in $\mathbb{R}^{m+1}$. It is possible to study (4.16) in a more general situation, which will be done elsewhere. We fix $\alpha > 1/2$ and set

$$E_0 := h^{\alpha-1+2\gamma}(\mathbb{S}^m), \qquad E_1 := h^{1+\alpha}(\mathbb{S}^m).$$

Moreover, P denotes the operator constructed in Lemma 4.1 and $\Delta := \Delta_0$ stands for the Laplace-Beltrami operator on $\mathbb{S}^m$. Then we have

Lemma 4.4. $\partial P(0) = \frac{1}{m}(1-\Delta)^{1-\gamma} - \frac{m+1}{m}(1-\Delta)^{-\gamma}$

Proof: Let $h \in E_1$ be given. Observe that $L(0) = 1$ and $K(0) = 1$, and that

$$\partial L(0)h = \frac{d}{d\varepsilon}\sqrt{1 + \varepsilon^2 w^{jk}\partial_j h \partial_k h}\Big|_{\varepsilon=0} = 0.$$

Further, we infer from (4.22) that $(s+1-\Delta_{\varepsilon h})(1/(s+1)) = 1$ for all $s \geq 0$ and all $\varepsilon \in \mathbb{R}$ which are in modulus sufficientlty small . Using the representation formula

$$(1-\Delta_{\varepsilon h})^{-\gamma} = \frac{\sin(\pi\gamma)}{\pi}\int_0^\infty s^{-\gamma}(s+1-\Delta_{\varepsilon h})^{-1}\, ds, \qquad \gamma \in (0, 1/2],$$

and $\int_0^\infty s^{-\gamma}(1+s)^{-1}ds = \pi/\sin(\pi\gamma)$, see (III.4.6.9) and (III.4.6.10) in [2], this implies that $(1-\Delta_{\varepsilon h})^{-\gamma}1 = 1$, and consequently

$$\partial(1-\Delta_\rho)^{-\gamma}K(0)|_{\rho=0}h = \frac{d}{d\varepsilon}(1-\Delta_{\varepsilon h})^{-\gamma}|_{\varepsilon=0}1 = 0.$$

Therefore we obtain

$$\partial P(0) = (1-\Delta)^{-\gamma}\partial K(0). \tag{4.32}$$

But $\partial K(0) = -\frac{1}{m}(m+\Delta)$, see Lemma 3.1 in [13]. Combining this with (4.32) we easily get the assertion. □

Corollary 4.5. *There exists an open neighborhood B of 0 in E_1 such that P satisfies (2.1) and (2.2).*

Proof: **(i)** Since $-(1-\Delta)$ generates a strongly continuous analytic semigroup on E_0, we know from Remark III.4.6.12 in [2] that the very same is true for $-(1-\Delta)^{1-\gamma}$. Repeating the arguments which lead to (4.30), we conclude that the domain of $-(1-\Delta)^{1-\gamma}$ is given by $h^{1+\alpha}(\mathbb{S}^m)$. This shows that $-(1-\Delta)^{1-\gamma} \in \mathcal{H}(E_1, E_0)$.
Since $(1-\Delta)^{-\gamma} \in \mathcal{L}(E_0, E_0)$, it follows from Lemma 4.2 and a well-known perturbation result for generators of analytic semigroups that $\partial P(0) \in \mathcal{H}(E_1, E_0)$. But $\mathcal{H}(E_1, E_0)$ is open in $\mathcal{L}(E_1, E_0)$, see Theorem I.1.3.1 in [2]. Thus there is a open neighborhood V of $\partial P(0)$ such that $V \subset \mathcal{H}(E_1, E_0)$. From Lemma 4.1 we know that $\partial P \in C^\omega(W(\mathbb{S}^m), \mathcal{L}(E_1, E_0))$. Hence there is a open neighborhood B of 0 in $W(\mathbb{S}^m)$ such that $\partial P(B) \subset V$. In particular, we see that (2.1) is satisfied.
(ii) Let $\beta \in (1/2, \alpha)$ be fixed. Then the very same arguments as in (i) ensure that, given $\rho \in B$, we have that $\partial P(\rho) \in \mathcal{H}(h^{1+\beta}(\mathbb{S}^m), h^{\beta-1+2\gamma}(\mathbb{S}^m))$. Observing Remark 2.2(j), we see that assumption (2.2) is satisfied as well. □

Theorem 4.6. *Let $\alpha \in (1/2, 1)$ and $\gamma \in [0, 1/2]$. Let further $\rho_0 \in h^{1+\alpha}(\mathbb{S}^m)$ and assume that M_{ρ_0} is the graph of ρ_0 over $\mathbb{S}^m$ in normal direction. Then there exists a $t^+ > 0$ such that the generalized mean curvature flow*

$$V = -(1-\Delta_{M_{\rho(t)}})^{-\gamma} H(t), \quad M_{\rho(0)} = M_{\rho_0}$$

possesses a unique solution $\{M_{\rho(t)}\,;\, t \in [0, t^+)\}$ with

$$\rho \in C([0, t^+), h^{1+\alpha}(\mathbb{S}^m)) \cap C^1([0, t^+), h^{\alpha-1+2\gamma}(\mathbb{S}^m)),$$

provided $\|\rho_0\|_{C^{1+\alpha}(\mathbb{S}^m)}$ is small enough. In addition,

$$[(t,p) \mapsto \rho(t)(p)] \in C^\omega((0, t^+) \times \mathbb{S}^m), \mathbb{R}).$$

Proof: (i) It follows from Remark 2.2 d) that (1.3) holds true.

(ii) Clearly, $SO(m+1) \cdot BUC^\infty(\mathbb{S}^m) \subset BUC^\infty(\mathbb{S}^m)$. Moreover, it is not difficult to verify that $SO(m+1)$ is a transformation group of isometries on $BUC^j(\mathbb{S}^m)$ for $j = 0, 1, 2$. By Remark 2.2(c) and a density argument we therefore conclude that $SO(m+1)$ is a strongly continuous transformation group on $h^{j+\alpha}(\mathbb{S}^m)$ for $j = 0, 1$.

(iii) Observe that the metric on $\mathbb{S}^m$ is invariant under $SO(m+1)$. This implies that the transformation group on $h^{1+\alpha}(\mathbb{S}^m)$ induced by $SO(m+1)$ consists of isometries, i.e. given $R \in SO(m+1)$, the mapping

$$h^{1+\alpha}(\mathbb{S}^m) \to h^{1+\alpha}(\mathbb{S}^m), \quad v \mapsto R \cdot v$$

is an isometry. Thus, replacing B by a sufficiently small ball in $h^{1+\alpha}(\mathbb{S}^m)$ around 0, we have that $SO(m+1)\cdot B\subset B$.

(iv) Fix $\rho\in B$ and $R\in SO(m+1)$. Obviously, we have

$$R\circ\theta_{R\cdot\rho}=R\cdot\theta_\rho. \tag{4.33}$$

Thus, given $p\in\mathbb{S}^m$, we get

$$K(R\cdot\rho)(p)=H(\theta_{R\cdot\rho}(p))=H(R^{-1}\theta_\rho(Rp))=H(\theta_\rho(Rp))=[R\cdot K(\rho)](p), \tag{4.34}$$

since R is an isometry of the Euclidean space $(\mathbb{R}^{m+1},\eta)$. Moreover, (4.33) implies that

$$R:(\mathbb{S}^m,(R\circ\theta_{R\cdot\rho})^*\eta)\to(\mathbb{S}^m,\theta_\rho^*\eta)$$

is an isometry as well. Hence we find that

$$R\cdot(\Delta_\rho v)=\Delta_{R\cdot\rho}(R\cdot v),\quad v\in h^{1+\alpha}(\mathbb{S}^m), \tag{4.35}$$

see e.g. Remark XI.6.9(c) in [4]. Consequently, we obtain

$$R\cdot((1-\Delta_\rho)^{-\gamma}w)=(1-\Delta_{R\cdot\rho})^{-\gamma}(R\cdot w)\quad w\in h^{\alpha-1+2\gamma}(\mathbb{S}^m). \tag{4.36}$$

Finally, it follows from the chain rule and (4.33) that $L(R\cdot\rho)=R\cdot L(\rho)$. Combining (4.34), (4.35), (4.36), and (iii) we see that

$$P\,:\,B\to h^{\alpha-1+2\gamma}(\mathbb{S}^m),\quad \rho\mapsto L(\rho)(1-\Delta_\rho)^{-\gamma}K(\rho)$$

is equivariant with respect to $SO(m+1)$. Now the assertion follows from Theorem o1.1. □

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Institute for Applied Mathematics, University of Hannover, D-30167 Hannover, Germany

E-mail address: escher@ifam.uni-hannover.de

Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA

E-mail address: simonett@math.vanderbilt.edu