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006

THE SURFACE DIFFUSION FLOW FOR IMMERSED HYPERSURFACES*

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Abstract. We show existence and uniqueness of classical solutions for the motion of immersed hypersurfaces driven by surface diffusion. If the initial surface is embedded and close to a sphere, we prove that the solution exists globally and converges exponentially fast to a sphere. Furthermore, we provide numerical simulations showing the creation of singularities for immersed curves.

Key words. surface diffusion, mean curvature, free boundary problem, immersed hypersurfaces, center manifolds, maximal regularity, numerical simulations

AMS subject classifications. 35R35, 35K55, 35S30, 65C20, 80A22

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1. Introduction.

In this paper we study the motion of a family of immersed hypersurfaces whose normal velocity is equal to its surface diffusion. More precisely, let Γ_0 be a compact closed immersed orientable hypersurface in $\mathbb{R}^n$ of class $C^{2+\beta}$. We are looking for a family $\Gamma = \{\Gamma(t); t \geq 0\}$ of smooth immersed orientable hypersurfaces satisfying the following evolution equation:

$$V(t) = \Delta_{\Gamma(t)} H_{\Gamma(t)}\,, \qquad \Gamma(0) = \Gamma_0\,. \tag{1.1}$$

Here $V(t)$ denotes the velocity in the normal direction of Γ at time t, while $\Delta_{\Gamma(t)}$ and $H_{\Gamma(t)}$ stand for the Laplace–Beltrami operator and the mean curvature of $\Gamma(t)$, respectively. Both the normal velocity and the curvature depend on the local choice of the orientation; however, (1.1) does not, and so we are free to choose whichever one we like. In particular, if $\Gamma(t)$ is embedded and encloses a region $\Omega(t)$, we always choose the *outer* normal, so that V is positive if $\Omega(t)$ grows and so that $H_{\Gamma(t)}$ is positive if $\Gamma(t)$ is convex with respect to $\Omega(t)$. Due to the local nature of the evolution, we may assume the hypersurface Γ_0 to be connected.

In order to give precise results, let us introduce the following notation. Given an open set $U \subset \mathbb{R}^n$, let $h^s(U)$ denote the little Hölder spaces of order $s > 0$, that is, the closure of $BUC^\infty(U)$ in $BUC^s(U)$, the latter space being the Banach space of all bounded and uniformly Hölder continuous functions of order s. If Σ is a (sufficiently) smooth submanifold of $\mathbb{R}^n$ then the spaces $h^s(\Sigma)$ are defined by means of a smooth atlas for Σ.

THEOREM 1.1. *Assume that $0 < \beta < 1$, and let Γ_0 be a compact closed immersed orientable hypersurface in $\mathbb{R}^n$ belonging to the class $h^{2+\beta}$.*

(a) *The surface diffusion flow (1.1) has a unique local classical solution $\Gamma = \{\Gamma(t); t \in [0,T)\}$ for some $T > 0$. Each hypersurface $\Gamma(t)$ is of class C^∞ for $t \in (0,T)$. Moreover, the mapping $[t \mapsto \Gamma(t)]$ is continuous on $[0,T)$ with respect to the $h^{2+\beta}$-topology and smooth on $(0,T)$ with respect to the C^∞-topology.*

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(b) *Suppose that the initial hypersurface* Γ_0 *is a* $h^{2+\beta}$*-graph in normal direction over some smooth immersed hypersurface* Σ*. Then the mapping* $\varphi := [(t, \Gamma_0) \mapsto \Gamma(t)]$ *induces a smooth local semiflow on an open subset of* $h^{2+\beta}(\Sigma)$.

Remark. The assumption that the initial surface be orientable is not necessary for Theorem 1.1(a) to hold true.

This follows by evolving the double cover in case the initial hypersurface is not orientable. The double cover remains a double cover by uniqueness of smooth solutions, and (1.1) is invariant with respect to the local orientation, as noted before. Hence, one can go back to the quotient space which therefore also evolves according to the surface diffusion flow. □

The motion given by (1.1) has some interesting geometrical features. Assume that Γ is a smooth orientable solution to (1.1) and let $\mathcal{A}(t)$ denote the area of $\Gamma(t)$. Then the function $\mathcal{A}$ is smooth and we find for its derivative (e.g., see [22, Theorem 4] or [15, p. 70])

$$\begin{aligned}\frac{1}{n-1}\frac{d}{dt}\mathcal{A}(t) &= \int_{\Gamma(t)} V(t) H_{\Gamma(t)}\, d\sigma = \int_{\Gamma(t)} [\Delta_{\Gamma(t)} H_{\Gamma(t)}] H_{\Gamma(t)}\, d\sigma \\ &= -\int_{\Gamma(t)} |\mathrm{grad}_{\Gamma(t)} H_{\Gamma(t)}|^2_{\Gamma(t)} d\sigma \le 0\,. \end{aligned} \tag{1.2}$$

Hence the motion driven by surface diffusion is area decreasing.

It is also possible to identify the surface diffusion flow as an H^{-1}-gradient flow for the area functional; see [10, 25]. The notion of such gradient flows was proposed by Fife [19, 20].

Assume that Γ is a smooth solution to (1.1) consisting of embedded hypersurfaces which enclose a region $\Omega(t)$, and let $\mathrm{Vol}(t)$ denote the volume of $\Omega(t)$. The derivative of the smooth function Vol is then given by

$$\frac{d}{dt}\mathrm{Vol}(t) = \int_{\Gamma(t)} V(t)\, d\sigma = \int_{\Gamma(t)} \Delta_{\Gamma(t)} H_{\Gamma(t)}\, d\sigma = 0\,,$$

thus the motion driven by surface diffusion is also volume preserving in the embedded case. Every Euclidean sphere is an equilibrium for (1.1), and it follows from Alexandrov's characterization [1] of embedded constant mean curvature surfaces that spheres are the only equilibria. However, none of these equilibria is isolated, since in every neighborhood of a fixed sphere there is a continuum of further spheres. Thus the dynamics of the flow generated by (1.1) is even locally quite copious.

THEOREM 1.2. *Let* S *be a fixed Euclidean sphere and let* $\mathcal{M}$ *denote the set of all spheres which are sufficiently close to* S*. Then* $\mathcal{M}$ *attracts all embedded solutions which are* $h^{2+\beta}(S)$*-close to* $\mathcal{M}$ *at an exponential rate. In particular, if* Γ_0 *is sufficiently close to* S *in* $h^{2+\beta}(S)$*, then* Γ *exists globally and converges exponentially fast to some sphere in* $\mathcal{M}$ *enclosing the same volume as* Γ_0*. The convergence is in the* C^k*-topology for every initial hypersurface* Γ_0 *which is in a sufficiently small* $h^{2+\beta}(S)$*-neighborhood* $W = W(k)$ *of* S*, where* $k \in \mathbb{N}$ *is a fixed number.*

The surface diffusion flow (1.1) was first proposed by Mullins [26] to model surface dynamics for phase interfaces when the evolution is only governed by mass diffusion in the interface. It has also been examined in a more general mathematical and physical context by Davì and Gurtin [13], and by Cahn and Taylor [9]. More recently, Cahn, Elliott, and Novick-Cohen [8] showed by formal asymptotics that the surface diffusion flow is the singular limit of the zero level set of the solution to the Cahn–Hilliard equation with a concentration-dependent mobility. In the case of constant mobility in the Cahn–Hilliard equation, Alikakos, Bates, and Chen [2] proved that the motion of the singular limit is governed by the Mullins–Sekerka model (also called the Hele–Shaw model with surface tension), rigorously establishing a result that was formally derived by Pego [27]. The Mullins–Sekerka model shares many properties with the surface diffusion flow (1.1). They both preserve the enclosed volume and decrease the area of the interface, and for both the invariant manifold $\mathcal{M}$ of spheres is exponentially attracting; see [16, 17, 18].

In two dimensions and for strip-like domains, the surface diffusion flow was investigated by Baras, Duchon, and Robert [7]. They prove global existence of weak solutions. Also in two dimensions, the surface diffusion flow for closed embedded curves was analytically investigated by Elliott and Garcke [14]. They show local existence and regularization for C^4-initial curves, and global existence for small perturbations of circles. Furthermore, assuming global existence, they show that any closed curve will become circular under this evolution. They do not obtain uniqueness of solutions. Recently, Giga and Ito [21] established the existence of unique (local) solutions for immersed H^4-initial curves. Moreover, they prove that the surface diffusion flow can drive an initially embedded curve to a self intersection. The techniques in [14, 21] seem to be restricted to two dimensions.

Our methods work in any dimension, and we obtain existence and uniqueness for immersed hypersurfaces. This is of particular interest since embedded hypersurfaces can become immersed under the surface diffusion flow, which is in clear contrast to the mean curvature flow where smooth solutions remain embedded if their initial surface is embedded. Our numerical simulations show that an immersed curve can develop singularities under the surface diffusion flow. Our example consists of a curve with a loop within a loop where the inner loop tightens and then contracts to a point. This situation has been analyzed in great detail by Angenent [6] for the mean curvature flow. In case of surface diffusion we do not have an analytical proof for the occurrence of singularities. We also give an example showing that an immersed curve evolves towards a stable limiting configuration which is not an embedded circle, but a multiply covered immersed circle. Finally, we provide evidence that the surface diffusion flow shrinks a figure eight to a point in finite time. Our approach for proving existence and uniqueness of solutions can be used to set up the numerical scheme for our simulations.

In case the initial hypersurface has several components, it is clear that some components may collide under the surface diffusion flow. This is most easily seen by choosing any nonstationary initial hypersurface, and then placing a stationary sphere in its path.

Theorem 1.1 constitutes a precise local existence and uniqueness result for classical solutions to (1.1) starting out as immersed hypersurfaces. In particular, the results disclose a parabolic regularization of the flow φ since we are allowed to choose initial surfaces Γ_0 of class $h^{2+\beta}$, although $\Delta_{\Gamma_0} H_{\Gamma_0}$ is for such Γ_0 in general not a classical function. This parabolic structure also provides the foundation for the study of the qualitative behavior of the semiflow φ. Our approach for proving existence, uniqueness, and regularity of solutions is based on the general theory of Amann [3, 4] for quasi-linear parabolic evolution equations.

The proof of Theorem 1.2 consists of two steps. We first show that the semiflow φ admits a stable $(n+1)$-dimensional local center manifold $\mathcal{M}^c$. This means, in particular, that $\mathcal{M}^c$ is a locally invariant manifold and that $\mathcal{M}^c$ contains all small global solutions of φ. In a second step we then prove that $\mathcal{M}^c$ coincides with the manifold $\mathcal{M}$ of the theorem. It is well-known that local center manifolds are generally not unique. However, since each local center manifold of the surface diffusion flow consists only of equilibria, this forces uniqueness. Under suitable spectral assumptions for the linearization, the existence of center manifolds is well known for finite-dimensional dynamical systems. The corresponding construction for quasi-linear infinite-dimensional semiflows (e.g., for φ) is considerably more involved. The basic technical tool here is the theory of maximal regularity, due to Da Prato and Grisvard [11]; see also [4, 5, 23]. In particular, these results allow to treat (1.1) as a fully nonlinear perturbed linear evolution equation; see [12, 23, 29].

2. Existence and uniqueness. In this section we introduce the mathematical setting in order to reformulate (1.1) as a quasi-linear parabolic evolution equation. Let Σ be a smooth compact closed immersed oriented hypersurface in $\mathbb{R}^n$, and assume that Γ_0 is close to this fixed reference manifold Σ. Let ν be the unit normal field on Σ commensurable with the chosen orientation. Choose $a > 0$ and an open covering $\{U_l\,;\, l = 1, \dots, m\}$ of Σ such that

$$X_l : U_l \times (-a, a) \to \mathbb{R}^n\,, \qquad X_l(s, r) := s + r\nu(s)$$

is a smooth diffeomorphism onto its image $\mathcal{R}_l := \mathrm{im}(X_l)$, that is,

$$X_l \in \mathit{Diff}^{\infty}(U_l \times (-a, a), \mathcal{R}_l)\,, \qquad 1 \le l \le m\,.$$

This can be done by choosing the open sets $U_l \subset \Sigma$ in such a way that they are embedded in $\mathbb{R}^n$ instead of only immersed, and then taking $a > 0$ sufficiently small so that each of the U_l has a tubular neighborhood of radius a. It is convenient to decompose the inverse of X_l into $X_l^{-1} = (S_l, \Lambda_l)$, where

$$S_l \in C^{\infty}(\mathcal{R}_l, U_l) \qquad \text{and} \qquad \Lambda_l \in C^{\infty}(\mathcal{R}_l, (-a, a))\,.$$

Note that $S_l(x)$ is the nearest point on U_l to $x \in \mathcal{R}_l$, and that $\Lambda_l(x)$ is the signed distance from x to U_l (that is, to $S_l(x)$). Moreover, the union of the sets $\mathcal{R}_l$, $1 \le l \le m$, consists exactly of those points in $\mathbb{R}^n$ with distance less than a to Σ.

Let $T > 0$ be a fixed number. We assume that $\Gamma := \{\Gamma(t),\, t \in [0, T)\}$ is a family of immersed graphs in normal direction over Σ. To be precise, we ask that there is a globally defined function

$$\rho : \Sigma \times [0, T) \to (-a, a)$$

such that for fixed $t \in [0, T)$, a manifold $\Gamma(t)$ is locally given by the images of the maps $[s \mapsto X_l(s, \rho(s, t))]$, $1 \le l \le m$.

Conversely, given any (sufficiently) smooth function $\rho : \Sigma \times [0, T) \to (-a, a)$, let

$$\Phi_{l,\rho} : \mathcal{R}_l \times [0, T) \to \mathbb{R}\,, \qquad \Phi_{l,\rho}(x, t) := \Lambda_l(x) - \rho(S_l(x), t)\,, \quad 1 \le l \le m\,. \tag{2.1}$$

Then for each $t \in [0, T)$, the zero-level set $\Phi_{l,\rho}^{-1}(0, t) \subset \mathcal{R}_l$ defines a smooth hypersurface, and the hypersurfaces $\Phi_{l,\rho}^{-1}(0, t)$ can be glued together to constitute a compact closed immersed orientable hypersurface $\Gamma_{\rho(t)}$. It is then easy to see that

$$\Gamma_{\rho(t)} = \Gamma(t) = \bigcup_{l=1}^{m} \mathrm{Im}\left(X_l : U_l \to \mathbb{R}^n\,,\ [s \mapsto X_l(s, \rho(s, t))]\right).$$

In addition, the normal velocity V of $\Gamma := \{\Gamma_{\rho(t)}\ ;\ t \in [0,T)\}$ at time t and at the point $x = X_l(s, \rho(s,t))$, expressed as a function over U_l, is given by

$$V(s,t) = -\left.\frac{\partial_t \Phi_{l,\rho}(x,t)}{|\nabla_x \Phi_{l,\rho}(x,t)|}\right|_{x=X_l(s,\rho(s,t))} = \left.\frac{\partial_t \rho(s,t)}{|\nabla_x \Phi_{l,\rho}(x,t)|}\right|_{x=X_l(s,\rho(s,t))}$$

for $(s,t) \in U_l \times (0,T)$. In the following, we fix $t \in [0,T)$ and drop it in our notation. Moreover, we fix $0 < \alpha < \beta < 1$ and define

$$\mathfrak{A} := \{\rho \in h^{2+\alpha}(\Sigma)\ ;\ \|\rho\|_{C(\Sigma)} < a\}\,.$$

Then for any $\rho \in \mathfrak{A}$,

$$\theta_\rho\,:\,\Sigma \to \Gamma_\rho, \qquad \theta_\rho(s) := X_l(s,\rho(s)) \text{ for } s \in U_l\,,$$

is a well-defined global $(2+\alpha)$-diffeomorphism. We write Δ_{Γ_ρ} for the Laplace–Beltrami operator of Γ_ρ and H_{Γ_ρ} for the mean curvature of Γ_ρ. Finally, let

$$G(\rho) := -L_\rho \theta_\rho^*(\Delta_{\Gamma_\rho} H_{\Gamma_\rho}) \quad \text{for} \quad \rho \in h^{4+\alpha}(\Sigma) \cap \mathfrak{A}\,,$$

where $L_\rho(s) := \theta_\rho^* |\nabla_x \Phi_{l,\rho}|(s)$ for $s \in U_l$, $1 \le l \le m$. On the phase space

$$\mathcal{V} := h^{2+\beta}(\Sigma) \cap \mathfrak{A},$$

we are now considering the following evolution equation for the distance function ρ:

$$\partial_t \rho + G(\rho) = 0\,, \qquad \rho(0) = \rho_0\,, \tag{2.2}$$

where ρ_0 is a function on Σ determined by Γ_0. More precisely, given $\rho \in \mathcal{V}$, we call a family $\rho\,:\,[0,T) \to \mathcal{V}$ a classical solution of (2.2) if

$$\rho \in C([0,T),\mathcal{V}) \cap C^\infty((0,T), C^\infty(\Gamma))$$

and if ρ satisfies (2.2) pointwise for $t \in (0,T)$. It is not difficult to see that the surface diffusion flow (1.1) and the evolution equation (2.2) are equivalent on $\mathcal{R} := \cup_{l=1}^m \mathcal{R}_l$. That is, if $\Gamma := \{\Gamma(t)\ ;\ t \in [0,T)\}$ is a classical solution of (1.1) such that $\Gamma(t) \subset \mathcal{R}$ for $t \in [0,T)$, then the above construction yields a classical solution of (2.2) and vice-versa; if $\rho\,:\,[0,T) \to \mathcal{V}$ is a classical solution of (2.2), then $\Gamma := \{\Gamma_{\rho(t)}\ ;\ t \in [0,T)\}$ is a classical solution of (1.1).

In order to state our next result, let E_1 and E_0 be Banach spaces with $E_1 \hookrightarrow E_0$, and let $\mathcal{H}(E_1,E_0)$ be the set of all $A \in \mathcal{L}(E_1,E_0)$ such that $-A$, considered as an unbounded operator in E_0, generates a strongly continuous analytic semigroup on E_0. It can be shown that $\mathcal{H}(E_1,E_0)$ is open in $\mathcal{L}(E_1,E_0)$; cf. [4, Theorem 1.3.1]. We always assume that $\mathcal{H}(E_1,E_0)$ carries the corresponding relative topology. Recall that we already fixed $0 < \alpha < \beta < 1$. Now, in addition, pick $\beta_0 \in (\alpha,\beta)$ and let

$$\mathcal{U} := h^{2+\beta_0}(\Sigma) \cap \mathfrak{A}\,.$$

Lemma 2.1. *There exist*

$$P \in C^\infty(\mathcal{U}, \mathcal{H}(h^{4+\alpha}(\Sigma), h^\alpha(\Sigma)))\,, \quad F \in C^\infty(\mathcal{U}, h^{\beta_0}(\Sigma))\,,$$

such that

$$G(\rho) = P(\rho)\rho + F(\rho), \qquad \rho \in h^{4+\alpha}(\Sigma) \cap \mathfrak{A}\,.$$

Proof. (a) The first step is to define a metric on Σ that lends itself well for computations in local coordinates. We choose a system of local coordinates on Σ, where we can assume that the sets U_l, $1 \le l \le m$, are exactly the domains of the charts. We fix some index $l \in \{1, \dots, m\}$ and we let η be the restriction of the Euclidean metric on $\mathcal{R}_l \subset \mathbb{R}^n$. Now define the pull-back metric

$$g_l := X_l^* \eta \quad \text{on} \quad T(U_l \times (-a, a)) .$$

The mapping X_l is exactly translation into the normal direction of U_l, and hence it is easily seen that the metric g_l splits along the fibers of $U_l \times (-a, a)$, i.e., $g_l = w_l(r) + dr \otimes dr$. Here r denotes the coordinate in the normal direction of U_l, and $w_l(r)$ is a metric on the tangent space to $U_l \times \{r\} \equiv U_l$. Given $\rho \in \mathcal{U}$, we set

$$g(\rho) := w(\rho) + dr \otimes dr := g_l|_{(s,\rho(s))} \quad \text{on} \quad T_{(s,\rho(s))}(U_l \times (-a, a)) .$$

In particular $w(\rho)$ constitutes a metric on $T(U_l)$ with components $w_{jk}(\rho)$. Furthermore, let $w^*(\rho)$ be the induced metric on the cotangent bundle $T^*(U_l)$, that is, $w^*(\rho)(\xi, \zeta) := w^{jk}(\rho)\xi_j\zeta_k$ for $\xi, \zeta \in T^*(U_l)$, where $w^{jk}(\rho)$ are the entries of the inverse matrix of $[w_{jk}(\rho)]$. Note that the metric $w(\rho)$ is not the same as $\theta_\rho^* \eta$. In particular, $w(\rho)$ does not involve any derivatives of ρ, whereas $\theta_\rho^* \eta$ does. We define $U_{l,\rho} := (U_l, w(\rho))$ and $\Xi_l := (U_l \times (-a, a), g_l)$. As a consequence of the special form (2.1) of $\Phi_{l,\rho}$, we have $\widehat{\Phi}_{l,\rho}(s, r) := \Phi_{l,\rho}(X_l(s, r)) = r - \rho(s)$ on $\mathcal{R}_l$, and hence

$$\nabla_{\Xi_l} \widehat{\Phi}_{l,\rho}(s, r) = \tfrac{\partial}{\partial r} - \nabla_{U_{l,\rho}} \rho(s) , \qquad (s, r) \in U_l \times (-a, a) .$$

Therefore we get for $s \in U_l$

$$\begin{aligned} L_\rho^2(s) = {}& |\nabla_{\mathbb{R}^n} \Phi_{l,\rho}|^2 \Big|_{x = X_l(s,\rho(s))} = g_l(\nabla_{\Xi_l} \widehat{\Phi}_{l,\rho}, \nabla_{\Xi_l} \widehat{\Phi}_{l,\rho}) \Big|_{(s,\rho(s))} \\ = {}& 1 + w(\rho)(\nabla_{U_{l,\rho}} \rho, \nabla_{U_{l,\rho}} \rho) \Big|_s = 1 + w^*(\rho)(d\rho, d\rho) \Big|_s , \end{aligned} \tag{2.3}$$

where $d\rho := \partial_j \rho dx^j \in T^*(\Sigma)$ denotes the exterior differential of any $\rho \in C^1(\Sigma)$. We did not label the metrics $g(\rho)$ and $w(\rho)$ with an index l as they can be defined globally on Σ.

To simplify the notation we set $H_\rho := \theta_\rho^* H_{\Gamma_\rho}$. It is known that the mean curvature operator H_ρ is a second order quasi-linear elliptic operator acting on functions defined on Σ; see, for instance, [18, Lemma 3.1]. Moreover, it follows from the proof of that lemma presented in [18] that

$$H_\rho = P_1(\rho)\rho + F_1(\rho), \qquad \rho \in \mathcal{U} .$$

$P_1(\rho)$ and $F_1(\rho)$ are represented in local coordinates by

$$\begin{aligned} P_1(\rho) = \tfrac{1}{(n-1)L_\rho^3} \Big[& \big(- L_\rho^2 w^{jk}(\rho) + w^{jl}(\rho) w^{km}(\rho) \partial_l \rho \partial_m \rho \big) \partial_j \partial_k \\ & + \big(L_\rho^2 w^{jk}(\rho) \Gamma_{jk}^i(\rho) + w^{jl}(\rho) w^{ki}(\rho) \Gamma_{jk}^n(\rho) \partial_l \rho \\ & \quad + 2 w^{km}(\rho) \Gamma_{nk}^i(\rho) \partial_m \rho - w^{jl}(\rho) w^{km}(\rho) \Gamma_{jk}^i(\rho) \partial_l \rho \partial_m \rho \big) \partial_i \Big], \\ F_1(\rho) = {}& -\frac{1}{(n-1)L_\rho} w^{jk}(\rho) \Gamma_{jk}^n(\rho). \end{aligned}$$

Here the summation runs from 1 to $(n-1)$ for all repeated indices. Moreover, Γ_{jk}^i are the Christoffel symbols of the metric g_l and

$$\Gamma_{jk}^i(\rho) := \Gamma_{jk}^i \Big|_{(s,\rho(s))} \quad \text{on} \quad T_{(s,\rho(s))}(\Xi_l) .$$

An important observation here is that $w^{jk}(\rho)$ and $\Gamma^i_{jk}(\rho)$ are all independent of the derivatives of ρ, and hence the above equations together with (2.3) give complete information on how derivatives of ρ go into the operators $P_1(\rho)$ and $F_1(\rho)$.

Given $\xi \in T^*(\Sigma)$, let $p_1^\pi(\rho)(\xi)$ denote the symbol of the principal part of $P_1(\rho)$. Then (2.3) and the Cauchy–Schwarz inequality yield

$$\begin{aligned} p_1^\pi(\rho)(\xi) &= \frac{1}{(n-1)L_\rho^3}\big[w^*(\rho)(\xi,\xi) + w^*(\rho)(d\rho,d\rho)w^*(\rho)(\xi,\xi) - (w^*(\rho)(d\rho,\xi))^2\big] \\ &\geq \frac{w^*(\rho)(\xi,\xi)}{(n-1)L_\rho^3} \end{aligned}$$

for any $\xi \in T^*(\Sigma)$.

(b) Let us now turn to the operator $\theta_\rho^*\Delta_{\Gamma_\rho}$. Since θ_ρ is a diffeomorphism between Σ and Γ_ρ, we obtain that

$$\theta_\rho^*\Delta_{\Gamma_\rho} = \Delta_\rho\theta_\rho^*\,, \quad \rho \in \mathcal{U}\,,$$

where Δ_ρ is the Laplace–Beltrami operator on $(\Sigma, \theta_\rho^*\eta)$. Here η is the Euclidean metric on the immersed manifold Γ_ρ and $\theta_\rho^*\eta$ denotes the Riemannian metric that is induced by θ_ρ on the manifold Σ. To simplify the notation we set $\sigma(\rho) := \theta_\rho^*\eta$. Let $\sigma_{jk}(\rho)$ be the components of $\sigma(\rho)$ in local coordinates and let $\sigma^*(\rho)$ be the induced metric on $T^*(\Sigma)$, that is, $\sigma^*(\rho)(\xi,\zeta) := \sigma^{jk}(\rho)\xi_j\zeta_k$ for $\xi,\zeta \in T^*(\Sigma)$. As usual, $\sigma^{jk}(\rho)$ are the entries of the inverse matrix of $[\sigma_{jk}(\rho)]$. Finally, $\gamma^i_{jk}(\rho)$ denote the Christoffel symbols of $\sigma(\rho)$. Using local coordinates, we find

$$\Delta_\rho = \sigma^{jk}(\rho)\big(\partial_j\partial_k - \gamma^i_{jk}(\rho)\partial_i\big)\,, \qquad \rho \in \mathcal{U}\,.$$

(c) Let $\rho \in \mathcal{U}$ be given. Then we define $P^\pi(\rho) \in \mathcal{L}(h^{4+\alpha}(\Sigma), h^\alpha(\Sigma))$ by

$$P^\pi(\rho) := -\frac{1}{(n-1)L_\rho^2}\,\sigma^{rs}(\rho)\big[-L_\rho^2 w^{jk}(\rho) + w^{jl}(\rho)w^{km}(\rho)\partial_l\rho\partial_m\rho\big]\partial_r\partial_s\partial_j\partial_k\,.$$

We show that there exists a mapping $Q(\rho) \in \mathcal{L}(h^{3+\alpha}(\Sigma), h^\alpha(\Sigma))$ such that

$$-L_\rho\Delta_\rho P_1(\rho)\rho - P^\pi(\rho)\rho = Q(\rho)\rho, \quad \rho \in \mathcal{U} \cap h^{4+\alpha}(\Sigma)\,.$$

Using the representations of Δ_ρ and $P_1(\rho)\rho$ in local coordinates we see that fourth order derivatives of ρ can only occur when $\partial_r\partial_s$ falls on $\partial_j\partial_k\rho$, and these terms are collected exactly in the operator $P^\pi(\rho)$. Third order derivatives of ρ can only enter in a linear way. For this recall that $w^{jk}(\rho)$ and $\Gamma^i_{jk}(\rho)$ do depend on ρ, but not on its derivatives. So $\partial_r\partial_s$ applied to these functions will only generate second order derivatives of ρ. Hence

$$\partial_r\partial_s(w^{jl}(\rho)w^{km}(\rho)\partial_l\rho\partial_m\rho)$$

will, for instance, produce a third order derivative of ρ exactly when $\partial_r\partial_s$ falls on $\partial_l\rho$ or on $\partial_m\rho$. The result is clearly linear in the third order derivatives. Next observe that L_ρ is represented in local coordinates by

$$L_\rho = \sqrt{1 + w^{jk}(\rho)\partial_j\rho\partial_k\rho}\,,$$

see (2.3). It is then easily seen that $\partial_r\partial_s L_\rho^{-1}$ generates, once again, third order derivatives which enter linearly. Similar arguments apply to all of the remaining terms. Finally, in case that an expression does not contain third order derivatives we can always split off a linear term $\partial_j\partial_k\rho$ or $\partial_i\rho$. By similar arguments we also conclude that there exists a mapping $R(\rho)\in\mathcal{L}(h^{3+\alpha}(\Sigma),h^{\alpha}(\Sigma))$ such that

$$-L_\rho\left(\Delta_\rho\frac{1}{L_\rho}\right)L_\rho F_1(\rho)=R(\rho)\rho\,,\quad \rho\in\mathcal{U}\cap h^{3+\alpha}(\Sigma)\,.$$

We set

$$\begin{aligned}P(\rho)&:=\ P^\pi(\rho)+Q(\rho)+R(\rho),\quad \rho\in\mathcal{U}\,,\\ F(\rho)&:=\ -L_\rho\Delta_\rho F_1(\rho)-R(\rho)\rho,\quad \rho\in\mathcal{U}\cap h^{3+\alpha}(\Sigma)\,.\end{aligned}$$

It follows from the above considerations and from the representation of $F_1(\rho)$ in local coordinates that

$$P\in C^\infty(\mathcal{U},\mathcal{L}(h^{4+\alpha}(\Sigma),h^{\alpha}(\Sigma)))\,,\quad F\in C^\infty(\mathcal{U},h^{\beta_0}(\Sigma))$$

and that $G(\rho)=P(\rho)\rho+F(\rho)$ for $\rho\in h^{4+\alpha}(\Sigma)\cap\mathfrak{A}$.

(d) It remains to show that $P(\rho)\in\mathcal{H}(h^{4+\alpha}(\Sigma),h^{\alpha}(\Sigma))$ for $\rho\in\mathcal{U}$. Given $\rho\in\mathcal{U}$, let $p^\pi(\rho)$ denote the symbol of $P^\pi(\rho)$. Then the results in steps (a)–(c) yield

$$p^\pi(\rho)(\xi)=L_\rho\,\sigma^*(\rho)(\xi,\xi)\,p_1^\pi(\rho)(\xi)\geq\frac{1}{(n-1)L_\rho^2}\sigma^*(\rho)(\xi,\xi)\,w^*(\rho)(\xi,\xi)$$

for all $\xi\in T^*(\Sigma)$. Hence, for any fixed $\rho\in\mathcal{U}$, the operator $P^\pi(\rho)$ is a uniformly elliptic fourth order operator acting on functions over the compact manifold Σ. Consequently, $-P^\pi(\rho)$ generates a strongly continuous analytic semigroup on $h^\alpha(\Sigma)$, that is, we have that

$$P^\pi(\rho)\in\mathcal{H}(h^{4+\alpha}(\Sigma),h^{\alpha}(\Sigma)),\quad \rho\in\mathcal{U}\,.$$

Since $Q(\rho)$ and $R(\rho)$ are lower order perturbations, we can now conclude that $-P(\rho)$ generates an analytic semigroup on $h^\alpha(\Sigma)$ as well. ☐

Now we are in a position to apply the general theory of quasi-linear evolution equations developed by H. Amann providing a unique classical solution of problem (2.2). More precisely, we have the following theorem.

THEOREM 2.2. *Given any $\rho\in\mathcal{V}$, there exists a unique classical solution*

$$\rho\in C([0,t^+),\mathcal{V})\cap C^\infty((0,t^+),C^\infty(\Sigma))$$

of problem (2.2). *Here, $t^+:=t^+(\rho_0)>0$ stands for the maximal time of existence. The map $[(t,\rho_0)\mapsto\rho(t,\rho_0)]$ defines a smooth local semiflow on $\mathcal{V}$.*

Proof. Set $E_0:=h^\alpha(\Sigma)$ and $E_1:=h^{4+\alpha}(\Sigma)$ and let

$$E_\theta:=(E_0,E_1)^0_{\theta,\infty},\qquad \theta\in(0,1)\,,$$

denote the continuous interpolation spaces between E_1 and E_0; see [23] or [4]. Next we fix

$$\theta_1:=\frac{2+\beta-\alpha}{4}\,,\quad \theta_0:=\frac{2+\beta_0-\alpha}{4}\,,\quad \theta:=\frac{\beta_0-\alpha}{4}\,.$$

Since the little Hölder spaces are stable under continuous interpolation, we get the following identities

$$E_{\theta_1}=h^{2+\beta}(\Sigma)\,,\quad E_{\theta_0}=h^{2+\beta_0}(\Sigma)\,,\quad E_\theta=h^{\beta_0}(\Sigma)\,.$$

Hence Lemma 2.1 and [3, Theorem 12.1] imply that there exists a unique solution in the class

$$C([0,t^+),\mathcal{V})\cap C((0,t^+),h^{4+\alpha}(\Sigma))\cap C^1((0,t^+),h^\alpha(\Sigma))\,.$$

The additional regularity in the assertion follows from a bootstrapping argument in the scale of Banach spaces $h^s(\Sigma)$, cf. the proof of [17, Theorem 1]. Moreover, the results in [3, section 12] also show that the map $[(t,\rho_0)\mapsto\rho(t,\rho_0)]$ defines a smooth local semiflow on $\mathcal{V}$. ∎

3. Global existence. To prove Theorem 1.2, we fix a Euclidean sphere S and set $\Sigma=S$ in the construction of section 2. Without loss of generality we may assume that S is the unit sphere centered at 0. Observe that Lemma 2.1 implies that

$$G\;:\;\mathcal{U}\cap h^{4+\alpha}(S)\to h^\alpha(S)\,,\qquad \rho\mapsto G(\rho)$$

is smooth. Let $A:=\partial G(0)$ be the Fréchet derivative of G at 0. Then we have the following representation of A.

LEMMA 3.1.

$$A=\frac{1}{n-1}\Delta_S^2+\Delta_S\,,$$

where Δ_S denotes the Laplace–Beltrami operator on S.

Proof. Recall that $G(\rho)=-L_\rho\Delta_\rho H_\rho$ for $\rho\in\mathcal{U}\cap h^{4+\alpha}(S)$. Thus we get

$$Ah=\partial G(0)h=-\partial\big(L_\rho\Delta_\rho\big)\big|_{\rho=0}[h,H_0]-L_0\Delta_0\partial H_\rho\big|_{\rho=0}h \tag{3.1}$$

for $h\in h^{4+\alpha}(S)$. Furthermore, observe that

$$L_0\equiv 1,\qquad \Delta_0=\Delta_S,\qquad H_0\equiv 1\,. \tag{3.2}$$

In particular, given $\rho\in\mathcal{U}$, we have that $L_\rho\Delta_\rho H_0=0$. Hence

$$\partial\big(L_\rho\Delta_\rho\big)\big|_{\rho=0}[h,H_0]=\frac{d}{d\varepsilon}\big(L_{\varepsilon h}\Delta_{\varepsilon h}\big)\big|_{\varepsilon=0}H_0=0\,. \tag{3.3}$$

Finally, it was shown in [18, Lemma 3.1] that

$$\partial H_\rho\big|_{\rho=0}h=-\frac{1}{n-1}\big(n-1+\Delta_S\big)h, \tag{3.4}$$

and the assertion follows from (3.1)–(3.4). ∎

LEMMA 3.2. *The spectrum of $-A$ consists of a sequence of real eigenvalues*

$$\cdots<\mu_{k+1}<\mu_k<\mu_{k-1}<\cdots<\mu_1<\mu_0=0\,.$$

In addition, μ_0 is an eigenvalue of geometric multiplicity $(n+1)$.

Proof. (a) Due to the compact embedding of $h^{4+\alpha}(S)$ in $h^\alpha(S)$ it is clear that the spectrum of $-A$ consists only of eigenvalues.

(b) Assume that

$$Ah = \frac{1}{n-1}\Delta_S(n-1+\Delta_S)h = 0$$

for some $h \in h^{4+\alpha}(S)$. Then

$$(n-1+\Delta_S)h = c \tag{3.5}$$

for some constant c. Observe that $g_0 = c/(n-1)$ is a solution of (3.5). Consequently, we find that

$$(n-1+\Delta_S)(h-g_0) = 0\,.$$

On the other hand it is well known that $(n-1)$ is an eigenvalue of $-\Delta_S$ of multiplicity n and that the spherical harmonics $\{Y_k; 1 \le k \le n\}$ of degree 1 span the corresponding eigenspace. Let $Y_0 = 1$ and set $N := \text{span}\{Y_k\,;\, 0 \le k \le n\}$. We have shown that 0 is an eigenvalue of A of geometric multiplicity $(n+1)$ with eigenspace N.

(c) Suppose that $\lambda \in \mathbb{C} \setminus \{0\}$ and $h \in h^{4+\alpha}(S)$ satisfy the equation $(\lambda + A)h = 0$. Then h belongs to $N^\perp$, where the orthogonal complement has to be taken in $L_2(S)$. Indeed, given $k \in \{0, \ldots, n\}$, we have that

$$0 = \big((\lambda + A)h|Y_k\big) = \lambda\,(h|Y_k)\,,$$

showing that $h \in N^\perp$. Next observe that there are positive constants c_1 and c_2 such that

$$\big((\Delta_S)^{-1}g|g\big) \le -c_1\big(g|g\big), \qquad \big((n-1+\Delta_S)g|g\big) \le -c_2\big(g|g\big)$$

for all $g \in h^{2+\alpha}(S) \cap N^\perp$, where $(\,\cdot\,|\,\cdot\,)$ denotes the inner product in $L_2(S)$. Now, multiplying the equation $(\lambda + A)h = 0$ in $L_2(S)$ with $(\Delta_S)^{-1}h$, we get

$$\lambda\,(h|(\Delta_S)^{-1}h) + \frac{1}{n-1}\big((n-1+\Delta_S)h|h\big) = 0\,.$$

It follows that $\lambda < 0$ and this completes the proof. ◻

We are now ready to prove our Theorem 1.2. Here we follow [18].

(i) In a first step we sketch the construction of a center manifold $\mathcal{M}^c$ over N. For $g \in h^r(S)$, $r > 0$, let $Pg := \sum_{k=0}^n (g|Y_k)Y_k$. Then it is easily verified that P is a continuous projection of $h^r(S)$ onto $N = \{Y_k\,;\, 0 \le k \le n\}$, the kernel of the operator A. Moreover, P commutes with A, that is, $PAg = APg = 0$ for all $g \in h^{4+\alpha}(S)$. Therefore, N and $h_s^{4+\alpha}(S) := \ker(P)$ provide topologically complementary subspaces of $h^{4+\alpha}(S)$, which reduce the operator A. We conclude that $\sigma(-\pi^c A) = \{0\}$ and $\sigma(-\pi^s A) \subset (-\infty, \mu_1]$ with $\mu_1 < 0$, where $\pi^c = P$ and $\pi^s = id - P$ denote the projections onto N and $h_s^{4+\alpha}(S)$, respectively, the center subspace and the stable subspace of $-A$. It is now clear that the eigenvalue 0 also has algebraic multiplicity $(n+1)$. We can now apply [29, Theorem 4.1]; see also [23, Theorem 9.2.2]. These results imply that, given $m \in \mathbb{N}^*$, there exists an open neighborhood U of 0 in N and a mapping

$$\gamma \in C^m(U, h_s^{4+\alpha}(S)) \quad \text{with} \quad \gamma(0) = 0,\;\; \partial\gamma(0) = 0$$

such that $\mathcal{M}^c := \text{graph}(\gamma)$ is a locally invariant manifold for the semiflow generated by the solutions of (2.2). $\mathcal{M}^c$ is an $(n+1)$-dimensional submanifold of $h^{4+\alpha}(S)$

with $T_0(\mathcal{M}^c) = N$. In addition, the manifold $\mathcal{M}^c$ is exponentially attractive. More precisely, it follows from [29, Theorem 5.8] that given $\omega \in (0, -\mu_1)$, there exist a positive constant c and a neighborhood W of 0 in $h^{2+\beta}(S)$ such that

$$\|\pi^s\rho(t,\rho_0) - \gamma(\pi^c\rho(t,\rho_0))\|_{h^{4+\alpha}(S)} \leq \frac{c}{t^{1-\theta}} e^{-\omega t} \|\pi^s\rho_0 - \gamma(\pi^c\rho_0)\|_{h^{2+\beta}(S)} \tag{3.6}$$

for each ρ_0 in W. Estimate (3.6) is valid for all $t \in (0, t^+(\rho_0))$ with $\pi^c\rho(t,\rho_0) \in U$. Moreover, $\theta := (2+\beta-\alpha)/4$.

(ii) Step (i) implies that $\mathcal{M}^c$ contains all small equilibria of (2.2). We show that $\mathcal{M}^c$ and $\mathcal{M}$ coincide near 0. Suppose that S' is a sphere which is sufficiently close to S. Let $(z_1, \dots, z_n)$ be the coordinates of its center and r be its radius. Recall that S is the unit sphere in $\mathbb{R}^n$ and let $z_0 := 1 - r$. If ρ measures the distance from S to S' in normal direction with respect to S, we get the identity

$$(1+z_0)^2 = \sum_{k=1}^{n} \big((1+\rho)Y_k - z_k\big)^2. \tag{3.7}$$

Here we used that the spherical harmonics Y_k, $k = 1, \dots, n$, are just the restrictions of the harmonic polynomials $[x \mapsto x_k]$. Solving (3.7) for ρ we obtain that S' can be parameterized over S by the distance function

$$\rho(z) = \sum_{k=1}^{n} z_k Y_k - 1 + \sqrt{\left(\sum_{k=1}^{n} z_k Y_k\right)^2 + (1+z_0)^2 - \sum_{k=1}^{n} z_k^2}\,, \tag{3.8}$$

where $z := (z_0, \dots, z_n) \in \mathbb{R}^{n+1}$. If O is a sufficiently small neighborhood of 0 in $\mathbb{R}^{n+1}$, then it is clear that any sphere S' which is close to S can be characterized by (3.8) with $z \in O$. Furthermore, the mapping $[z \mapsto \rho(z)] : O \to h^{4+\alpha}(S)$ is smooth and its derivative at 0 is given by

$$\partial\rho(0)h = \sum_{k=0}^{n} h_k Y_k, \qquad h \in \mathbb{R}^{n+1}\,. \tag{3.9}$$

Now, let $\{F_0(z), \dots, F_n(z)\}$ be the coordinates of $\pi^c\rho(z)$ with respect to the basis $\{Y_0, \dots, Y_n\}$ of N. Then (3.9) yields that $\partial F(0) = \mathrm{id}_{\mathbb{R}^{n+1}}$. Consequently, the inverse function theorem implies that F is a smooth diffeomorphism from O onto its image $V := \mathrm{im}(F)$, provided O is small enough. Let $\mathcal{M} := \{\rho(z)\,;\, z \in O\}$. Then it follows that $\pi^c\mathcal{M}$ is an open neighborhood of 0 in N which can be assumed to coincide with the open neighborhood U of 0 in N obtained in step (i). Hence we conclude that $\mathcal{M} = \mathcal{M}^c$.

(iii) It follows from step (ii) that the reduced flow of (2.2) on $\mathcal{M}^c$ consists of equilibria. Therefore, 0 is a stable equilibrium for the reduced flow and we conclude that 0 is also stable for the evolution equation (2.2); see [28, Theorem 3.3]. In particular, there exists a neighborhood W of 0 in $h^{2+\beta}(S)$ such that solutions of (2.2) exist globally for every initial value $\rho_0 \in W$ and such that estimate (3.6) is satisfied for all $t > 0$.

(iv) As in [18, Theorems 6.5 and 6.6] one can show the following result. Given $k \in \mathbb{N}$ and $\omega \in (0, -\mu_1)$, there exists a neighborhood $W = W(k, \omega)$ of 0 in $h^{2+\beta}(S)$

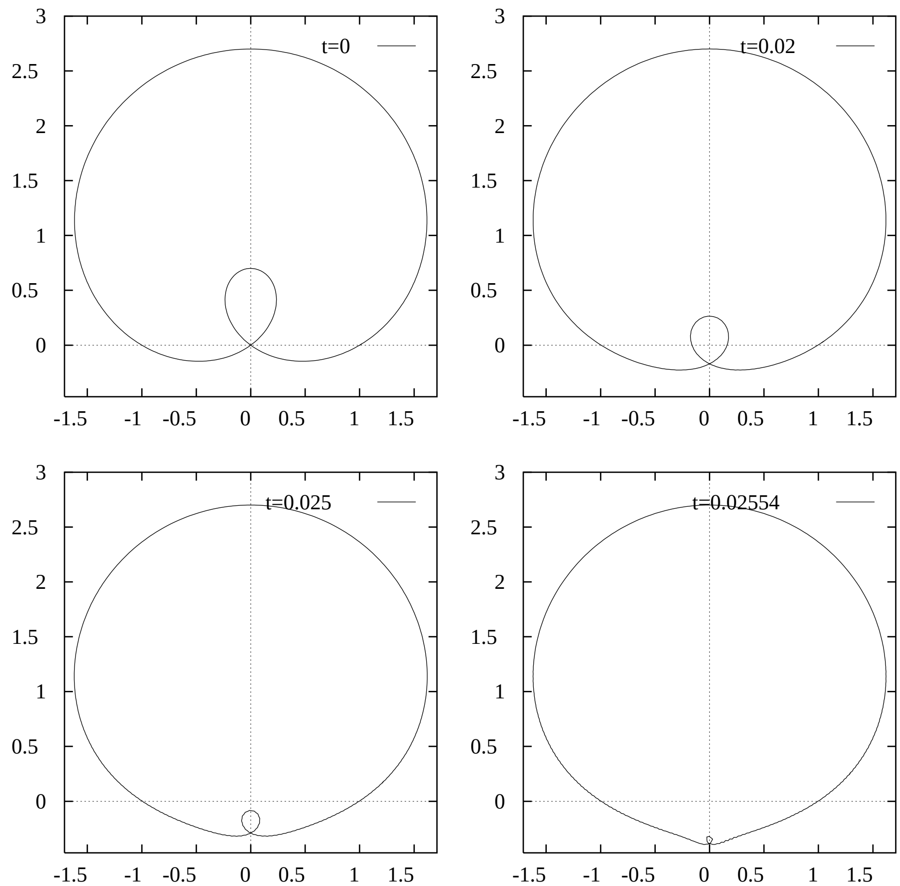

FIG. 1. *The limaçon* $r(\theta) = 1 + 1.7\sin(\theta)$.

with the following property. Given $\rho_0 \in W$, the solution $\rho(\cdot, \rho_0)$ of (2.2) exists globally and there exist $c = c(k, \omega) > 0$ and a unique $z_0 = z_0(\rho_0) \in U$ such that

$$\|\big(\pi^c\rho(t,\rho_0), \pi^s\rho(t,\rho_0)\big) - \big(z_0, \gamma(z_0)\big)\|_{C^k(S)} \le ce^{-\omega t}\|\pi^s\rho_0 - \gamma(\pi^c\rho_0)\|_{h^{2+\beta}(S)}$$

for $t \ge 1$. According to step (ii), $(z_0, \gamma(z_0)) \in \mathcal{M}^c$ is a sphere. Hence we have proved that given $\rho_0 \in W$ the solution $\rho(t, \rho_0)$ of (2.2) exists globally and converges to the sphere $(z_0, \gamma(z_0))$ exponentially fast in the C^k-topology as $t \to \infty$. And so, the proof of Theorem 1.2 is now completed.

4. Numerical simulations.

The general theory from the previous sections can be used to set up a numerical scheme as well. The idea is to discretize in time and to use an implicit scheme for stability reasons. Linearization of the dependence on the next time step leads to a semi-implicit scheme. Discretization of the interface leads then to a front-tracking method. We implement this here for two space dimensions; for three space dimensions, and for further details see [24, 25].

4.1. A limaçon.

The example of a limaçon shows that the surface diffusion flow can produce singularities. This is not unlike the mean curvature flow, for which Angenent [6] has investigated the singularities arising from the evolution of this shape.

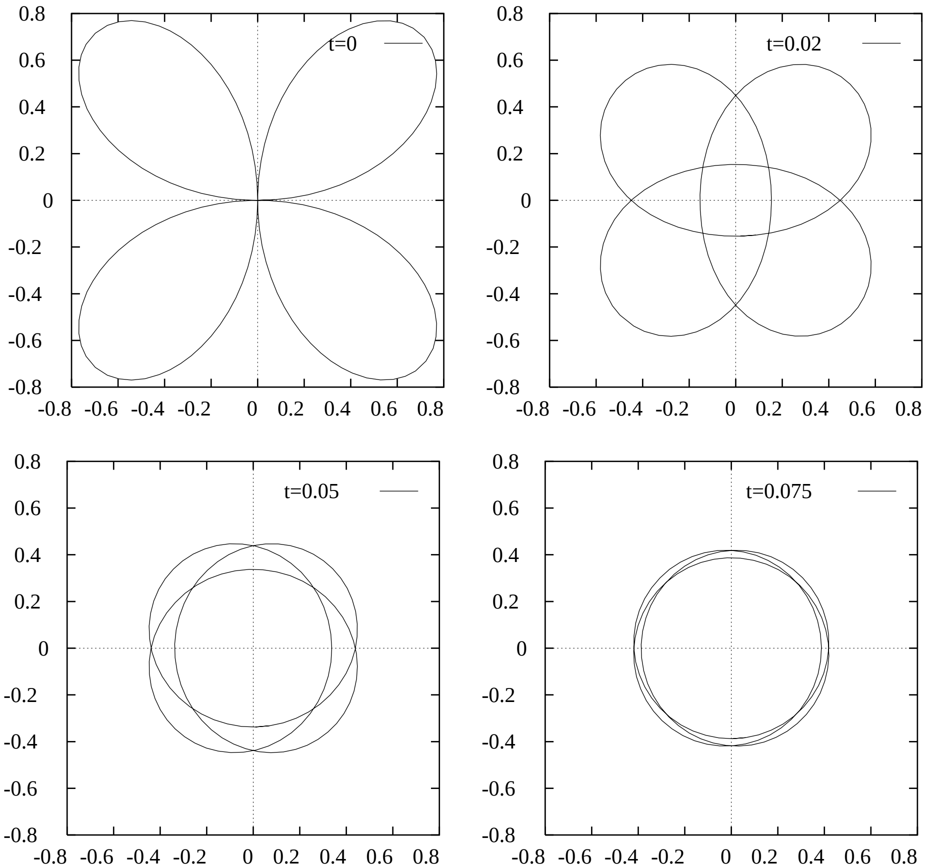

FIG. 2. *The rose* $r(\theta) = \sin(2\theta)$.

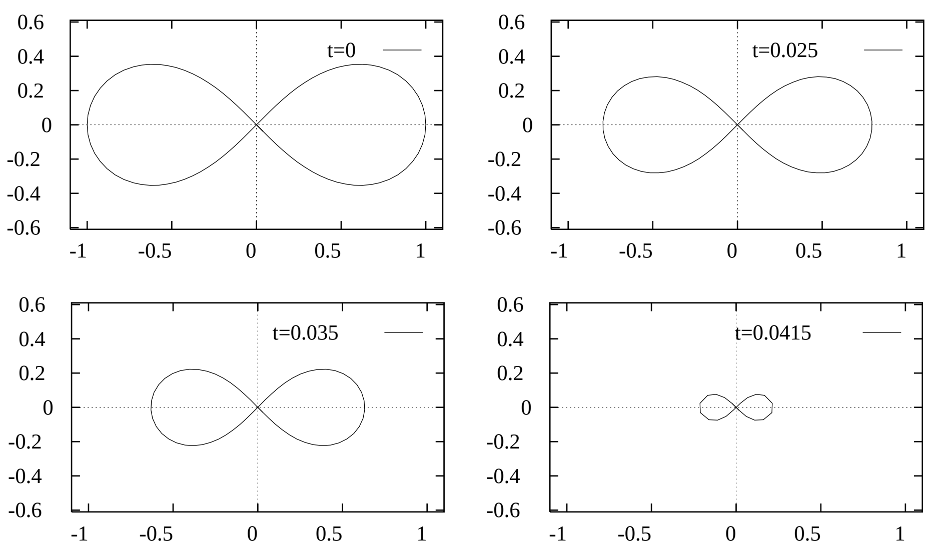

FIG. 3. *The figure eight* $r(\theta) = \sqrt{\cos(\theta)}$.

The smaller loop tightens, having a maximum for the curvature. Therefore the curvature increases for the smaller loop. This leads to a blow-up of the curvature in finite time (Figure 1.)

4.2. A four-leafed rose. The rose exhibits the phenomenon that the stable limiting configuration need not necessarily be an embedded circle, it can also be a multiply covered immersed circle. For positive time the winding number of the curve with respect to the origin does not change, and hence the limiting curve is a triply covered circle (Figure 2.)

4.3. A figure eight. One can make perfect sense of the enclosed signed area of a figure eight, which is for a symmetric figure eight equal to zero. As the evolution decreases the length of the curve and preserves the enclosed area, it can be expected that the limiting figure has zero area and zero length. This is exactly what happens, the figure eight shrinks in finite time to a point. As the curve shortens it is necessary to remove vertices from the numerical simulation to maintain the ratio of temporal versus spatial resolution. In other words, as the length of the curve decreases one needs to remove vertices to maintain a given lower bound on the distance between any two consecutive points of the discretized curve. This is somewhat visible in the last picture where the curve has shrunk so much that because of the increased curvature one can discern faint corners (Figure 3.)

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