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Qualitative Behavior of Solutions for Thermodynamically Consistent Stefan Problems with Surface Tension

Jan Prüss, Gieri Simonett & Rico Zacher

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Abstract

We study the qualitative behavior of a thermodynamically consistent two-phase Stefan problem with surface tension and with or without kinetic undercooling. It is shown that these problems generate local semiflows in well-defined state manifolds. If a solution does not exhibit singularities in a sense made precise herein, it is proved that it exists globally in time and its orbit is relatively compact. In addition, stability and instability of equilibria are studied. In particular, it is shown that multiple spheres of the same radius are unstable, reminiscent of the onset of Ostwald ripening.

1. Introduction

The Stefan problem is a model for phase transitions in liquid–solid systems that accounts for heat diffusion and exchange of latent heat in a homogeneous medium. The strong formulation of this model corresponds to a free boundary problem involving a parabolic diffusion equation for each phase and transmission conditions prescribed at the interface separating the phases.

(i) In order to describe the physical situation, let us consider a domain Ω that is occupied by a liquid and a solid phase, say water and ice, that are separated by an interface Γ. Due to melting or freezing, the corresponding regions occupied by water and ice will change and, consequently, the interface Γ will also change its position and shape. This leads to a free boundary problem.

The basic physical law governing this process is conservation of energy. The unknowns are the temperatures v_i, $i = 1, 2$, of the two phases, and the position of

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the interface Γ separating the phases. The conservation laws can then be expressed by a diffusion equation for v_i in the respective regions Ω_i occupied by the different phases, and by the so-called Stefan condition which accounts for the exchange of latent heat due to melting or solidifying. In the *classical* Stefan problem one assumes, in addition, that the temperatures v_i coincide with the melting temperature at the interface Γ (where the two phases are in contact), that is, one requires

$$v_1 = v_2 = 0 \quad \text{on} \quad \Gamma, \tag{1.1}$$

where 0 is the (properly scaled) melting temperature. Molecular considerations suggest that the condition (1.1) on the free boundary Γ be replaced by the *Gibbs–Thomson correction*

$$v_1 = v_2 = -\sigma\mathcal{H} \quad \text{on} \quad \Gamma, \tag{1.2}$$

where σ is a positive constant, called surface tension, and where $\mathcal{H}$ denotes the mean curvature of Γ. We will occasionally refer to the Stefan problem with condition (1.2) as the *classical Stefan problem with surface tension*.

It should be emphasized that the Stefan problem with Gibbs–Thomson correction (1.2) differs from the classical Stefan problem in a much more fundamental way than just in the modification of an interface condition. This becomes evident, for instance, by the fact that the classical Stefan problem allows for a comparison principle, a property that is no longer shared by the Stefan problem with surface tension. A striking difference is also provided by the fact that in the classical Stefan model, the temperature completely determines the phases, that is, the liquid region can be characterized by the condition $v > 0$, whereas $v < 0$ characterizes the solid region. The inclusion of surface tension will no longer allow us to determine the phases merely by the sign of v.

The main reason for introducing the Gibbs–Thomson correction (1.2) stems from the need to account for so-called *supercooling*, in which a fluid supports temperatures below its freezing point, or *superheating*, the analogous phenomena for solids; or dendrite formation, in which simple shapes evolve into complicated tree-like structures. The effect of supercooling can be on the order of hundreds of degrees for certain materials, see [15, Chapter 1] and [95]. We also refer to [14,15,32,36–38,42,52,63,64,94] for additional information.

The Stefan problem has been studied in the mathematical literature for over a century, see [58,87] and [95, pp. 117–120] for a historic account. The classical Stefan problem is known to admit unique long time weak solutions, see for instance [29,30,45] and [51, pp. 496–503]. It is important to point out that the existence of weak solutions is closely tied to the maximum principle.

Important results concerning the regularity of weak solutions for the multidimensional classical one-phase Stefan problem were established in [10,11,13,31, 46,47,62], and regularity results for the classical two-phase Stefan problem are contained in [7,8,12,22,23,73,88,96], to list only a few references. We remark that classical solutions for the Stefan problem with condition (1.1) were first established in [41,57]. We also refer to [75] for a more detailed account of the literature concerning the classical Stefan problem. Although the Stefan problem with the

Gibbs–Thomson correction (1.2) has been around for many decades, only few analytical results concerning existence, regularity and qualitative properties of solutions are known. The authors in [32] consider the case with small surface tension $0 < \sigma \ll 1$ and linearize the problem about $\sigma = 0$. Assuming the existence of smooth solutions for the case $\sigma = 0$, that is, for the classical Stefan problem, the authors prove existence and uniqueness of a weak solution for the *linearized* problem and then investigate the effect of small surface tension on the shape of $\Gamma(t)$. Existence of long time weak solutions is established in [5,54,85]. A more detailed discussion will be given below. A proof for existence—without uniqueness—of local time classical solutions is obtained in [83,84]. In [59], the way in which a spherical ball of ice in a supercooled fluid melts down is investigated. The case of a strip-like geometry, where the free surface Γ is given as the graph of a function, is considered in [28], and existence as well as uniqueness of local time classical solutions is established. Moreover, it is shown that solutions instantaneously regularize to become analytic in space and time. The approach is based on the theory of maximal regularity, which also forms the basis for the local existence theory in the current paper. In [79] linearized stability and instability of equilibria are studied. Finally, the authors in [39] consider a strip-like geometry over a torus and establish asymptotic stability of flat surfaces.

If the diffusion equation $\partial_t v_i - \Delta v_i = 0$ in Ω_i is replaced by the elliptic equation $\Delta v_i = 0$, then the resulting problem is the quasi-stationary Stefan problem with surface tension, which has also been termed the Mullins–Sekerka problem (or the Hele–Shaw problem with surface tension). Existence, uniqueness, regularity (and global existence in some cases) of classical solutions for the quasi-stationary approximation has recently been investigated in [9,16,17,24–27,33]. Global existence of weak solutions has been established in [86], see also [34,56] for related results.

The challenge of developing efficient and accurate numerical methods for free boundary problems arising from sharp-interface theories has recently driven the development of regularized diffuse-interface, or phase field, theories. It is of utmost importance to have a through understanding of the sharp-interface models in order to evaluate the quality of predictions of the associated phase field models. As remarked in [6], a phase field theory may, in general, possess a variety of sharp-interface limits, and in the absence of a sound sharp-interface theory to serve as a target, the problem of developing a physically meaningful diffuse-interface theory is ill-posed.

(ii) In this paper we consider a general model for phase transitions that is thermodynamically consistent, following the ideas in [6] and [44], see also [36–38] for earlier work. It involves the thermodynamic quantities of absolute temperature, free energy, internal energy, and entropy, and is complemented by constitutive equations for the free energies and the heat fluxes in the bulk regions. An important assumption is that there be no entropy production on the interface. In particular, the interface is assumed to carry no mass and no energy except surface tension.

To be more precise, let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^2, $n \geqq 2$. Ω is occupied by a material that can undergo phase changes: at time t, phase i occupies the subdomain $\Omega_i(t)$ of Ω, respectively, with $i = 1, 2$. We assume that $\partial\Omega_1(t) \cap \partial\Omega = \emptyset$; this means that no *boundary contact* can occur. The closed compact hypersurface $\Gamma(t) := \partial\Omega_1(t) \subset \Omega$ forms the interface between the phases. By the *Stefan problem with surface tension* we mean the following problem: find a family of closed compact hypersurfaces $\{\Gamma(t)\}_{t \geqq 0}$ contained in Ω and an appropriately smooth function $u : \mathbb{R}_+ \times \bar{\Omega} \to \mathbb{R}$ such that

$$\begin{cases} \kappa(u)\partial_t u - \operatorname{div}(\mathrm{d}(u)\nabla u) = 0 & \text{in} \quad \Omega \setminus \Gamma(t) \\ \partial_{\nu_\Omega} u = 0 & \text{on} \quad \partial\Omega \\ [\![u]\!] = 0 & \text{on} \quad \Gamma(t) \\ [\![\psi(u)]\!] + \sigma\mathcal{H} = \gamma(u)V & \text{on} \quad \Gamma(t) \\ [\![\mathrm{d}(u)\partial_\nu u]\!] = (l(u) - \gamma(u)V)V & \text{on} \quad \Gamma(t) \\ u(0) = u_0 & \\ \Gamma(0) = \Gamma_0. & \end{cases} \tag{1.3}$$

Here $u(t)$ denotes the (absolute) temperature, $\nu(t)$ the outer normal field of $\Omega_1(t)$, $V(t)$ the normal velocity of $\Gamma(t)$, $\mathcal{H}(t) = \mathcal{H}(\Gamma(t)) = -\operatorname{div}_{\Gamma(t)}\nu(t)/(n-1)$ the mean curvature of $\Gamma(t)$, and $[\![v]\!] = v_2|_{\Gamma(t)} - v_1|_{\Gamma(t)}$ the jump of a quantity v across $\Gamma(t)$. The sign of the mean curvature $\mathcal{H}$ is chosen to be negative at a point $x \in \Gamma$ if $\Omega_1 \cap B_r(x)$ is convex for some sufficiently small $r > 0$. Thus if Ω_1 is a ball of radius R then $\mathcal{H} = -1/R$ for its boundary Γ.

Several quantities are derived from the free energies $\psi_i(u)$ as follows:

- $\varepsilon_i(u) = \psi_i(u) + u\eta_i(u)$, the internal energy in phase i,
- $\eta_i(u) = -\psi_i'(u)$, the entropy,
- $\kappa_i(u) = \varepsilon_i'(u) = -u\psi_i''(u) > 0$, the heat capacity,
- $l(u) = u[\![\psi'(u)]\!] = -u[\![\eta(u)]\!]$, the latent heat.

Furthermore, $d_i(u) > 0$ denotes the coefficient of heat conduction in Fourier's law, $\gamma(u) \geqq 0$ the coefficient of kinetic undercooling, and $\sigma > 0$ the coefficient of surface tension. As is commonly done, we assume that there exists a unique (constant) *melting temperature* u_{m}, characterized by the equation $[\![\psi(u_{\mathrm{m}})]\!] = 0$. Finally, system (1.3) is to be completed by constitutive equations for the free energies ψ_i in the bulk phases $\Omega_i(t)$.

In the sequel we drop the index i, as there is no danger of confusion; we just keep in mind that the coefficients depend on the phases. The temperature is assumed to be continuous across the interface, as indicated by the condition $[\![u]\!] = 0$ in (1.3). However, the free energy and the conductivities depend on the respective phases, and hence the jumps $[\![\psi(u)]\!]$, $[\![\kappa(u)]\!]$, $[\![\eta(u)]\!]$, $[\![\mathrm{d}(u)]\!]$ are, in general, non-zero at the interface. In this paper we assume that the coefficient of surface tension is constant.

Next we show that the model (1.3) is consistent with the first and second laws of thermodynamics, postulating conservation of energy and growth of entropy. Indeed, the total energy of the system is given by

$$\mathsf{E}(u, \Gamma) = \int_{\Omega\setminus\Gamma} \varepsilon(u)\mathrm{d}x + \frac{1}{n-1}\int_\Gamma \sigma\,\mathrm{d}s, \tag{1.4}$$

and by the transport and surface transport theorem we have for smooth solutions

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{E}(u(t),\Gamma(t)) &= -\int_\Gamma \{[\![\mathrm{d}(u)\partial_\nu u]\!] + [\![\varepsilon(u)]\!]V + \sigma\mathcal{H}V\}\,\mathrm{d}s \\ &= -\int_\Gamma \{[\![\mathrm{d}(u)\partial_\nu u]\!] - (l(u) - \gamma(u)V))V\}\,\mathrm{d}s = 0,\end{aligned}$$

and thus, energy is conserved.

Let us point out that it is essential that $\sigma > 0$ is constant, that is, is independent of temperature. The reason for this lies in the fact that in case $\sigma = \sigma(u)$ depends on the temperature, the surface energy will be $\int_\Gamma \varepsilon_\Gamma(u)\,\mathrm{d}s$ instead of $\int_\Gamma \sigma\,\mathrm{d}s$, where $\varepsilon_\Gamma(u) = \sigma(u) + u\eta_\Gamma(u)$, $\eta_\Gamma(u) = -\sigma'(u)$, and one has to take into account the surface entropy $\int_\Gamma \eta_\Gamma\,\mathrm{d}s$ as well as the balance of surface energy. This means that the Stefan law needs to be replaced by a dynamic boundary condition of the form

$$\kappa_\Gamma(u)\partial_{t,n}u - \mathrm{div}_\Gamma(d_\Gamma(u)\nabla_\Gamma u) = [\![\mathrm{d}\partial_\nu u]\!] - \big(l(u) - \gamma(u)V + l_\Gamma(u)\mathcal{H}\big)V, \quad (1.5)$$

where $\partial_{t,n}$ denotes the time derivative in the normal direction, $\kappa_\Gamma(u) = \varepsilon'_\Gamma(u)$ and $l_\Gamma(u) = u\sigma'(u)$. We intend to study such complex problems elsewhere and restrict our attention here to the case of constant σ.

The fifth equation in (1.3) is usually called the *Stefan law*. It shows that energy is conserved across the interface. The fourth equation is the *Gibbs–Thomson law* (with kinetic undercooling if $\gamma(u) > 0$) which implies, together with Stefan's law, that entropy production on the interface is nonnegative if $\gamma \geqq 0$. In the case where $\gamma \equiv 0$, that is, in the absence of kinetic undercooling, there is no entropy production on the interface. In fact, the total entropy of the system, given by

$$\Phi(u,\Gamma) = \int_{\Omega\setminus\Gamma} \eta(u)\mathrm{d}x, \quad (1.6)$$

satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Phi(u(t),\Gamma(t)) &= \int_\Omega \frac{1}{u^2}\mathrm{d}(u)|\nabla u|^2\,\mathrm{d}x - \int_\Gamma \frac{1}{u}\{[\![\mathrm{d}(u)\partial_\nu u]\!] + u[\![\eta(u)]\!]V\}\,\mathrm{d}s \\ &= \int_\Omega \frac{1}{u^2}\mathrm{d}(u)|\nabla u|^2\,\mathrm{d}x + \int_\Gamma \frac{1}{u}\gamma(u)V^2\,\mathrm{d}s \geqq 0.\end{aligned}$$

In particular, the negative total entropy is a Lyapunov functional for problem (1.3). Even more, $-\Phi$ is a strict Lyapunov functional in the sense that it is strictly decreasing along smooth solutions which are non-constant in time. Indeed, if at some time $t_0 \geqq 0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(u(t_0),\Gamma(t_0)) = \int_\Omega \frac{1}{u^2}\mathrm{d}(u)|\nabla u|^2\,\mathrm{d}x + \int_\Gamma \frac{1}{u}\gamma(u)V^2\,\mathrm{d}s = 0,$$

then $\nabla u(t_0) = 0$ in Ω and $\gamma(u(t_0))V(t_0) = 0$ on $\Gamma(t_0)$. This implies $u(t_0) = \mathrm{const}$ in Ω, hence $\mathcal{H}(t_0) = -[\![\psi(u(t_0))]\!]/\sigma = \mathrm{const}$. Since Ω is bounded, we may conclude that $\Gamma(t_0)$ is a union of finitely many, say m, disjoint spheres of equal radius, that is, $(u(t_0),\Gamma(t_0))$ is an equilibrium. Therefore, the *limit sets* of solutions in the state manifold $\mathcal{SM}_\gamma$, see (3.8)–(3.9) for a definition, are contained in the $(mn+1)$ dimensional manifold of equilibria

$$\mathcal{E} = \Big\{\big(u_*, \cup_{1\leqq l\leqq m} S_{R_*}(x_l)\big) : u_* > 0,\ R_* = \sigma/[\![\psi(u_*)]\!],\ \bar{B}_{R_*}(x_l) \subset \Omega,$$
$$S_{R_*}(x_l) \cap S_{R_*}(x_k) = \emptyset,\ l \neq k\Big\}, \tag{1.7}$$

where $S_{R_*}(x_l)$ and $B_{R_*}(x_l)$ denote the sphere and the ball with radius R_* and center x_l, respectively.

(iii) Another interesting observation is the following. Consider the critical points of the functional $\Phi(u,\Gamma)$ with constraint $\mathsf{E}(u,\Gamma) = \mathsf{E}_0$, say on $C(\bar{\Omega}) \times \mathcal{MH}^2(\Omega)$, see Section 3.1 for the definition of $\mathcal{MH}^2(\Omega)$. Then by the method of Lagrange multipliers, there is $\mu \in \mathbb{R}$ such that at a critical point (u_*, Γ_*) we have

$$\Phi'(u_*, \Gamma_*) + \mu \mathsf{E}'(u_*, \Gamma_*) = 0. \tag{1.8}$$

The derivatives of the functionals are given by

$$\langle \Phi'(u,\Gamma) | (v,h)\rangle = (\eta'(u)|v)_{L_2(\Omega)} - ([\![\eta(u)]\!]|h)_{L_2(\Gamma)},$$

and

$$\langle \mathsf{E}'(u,\Gamma) | (v,h)\rangle = (\varepsilon'(u)|v)_{L_2(\Omega)} - ([\![\varepsilon(u)]\!] + \sigma\mathcal{H}(\Gamma)|h)_{L_2(\Gamma)}.$$

By first setting $h = 0$ and varying v in (1.8) we obtain $\eta'(u_*) + \mu\varepsilon'(u_*) = 0$ in Ω, and then varying h we get

$$[\![\eta(u_*)]\!] + \mu\big([\![\varepsilon(u_*)]\!] + \sigma\mathcal{H}(\Gamma_*)\big) = 0 \quad \text{on } \Gamma_*.$$

The relations $\eta(u) = -\psi'(u)$ and $\varepsilon(u) = \psi(u) - u\psi'(u)$ imply $0 = -\psi''(u_*)(1 + \mu u_*)$, which shows that $u_* = -1/\mu$ is constant in Ω, since $\kappa(u) = -u\psi''(u) > 0$ for all $u > 0$ by assumption. This further implies $[\![\psi(u_*)]\!] + \sigma\mathcal{H}(\Gamma_*) = 0$, that is, the Gibbs–Thomson relation. Since u_* is constant we see that $\mathcal{H}(\Gamma_*)$ is constant, hence Γ_* is a sphere whenever connected, and a union of finitely many disjoint spheres of equal size otherwise. Thus the critical points of the entropy functional for prescribed energy are precisely the equilibria of the problem.

Going further, suppose we have an equilibrium $e_* := (u_*, \Gamma_*)$ where the total entropy has a local maximum with respect to the constraint $\mathsf{E} = \mathsf{E}_0$ constant. Then $\mathcal{D} := [\Phi + \mu\mathsf{E}]''(e_*)$ is negative semi-definite on the kernel of $\mathsf{E}'(e_*)$, where μ is the fixed Lagrange multiplier found above. The kernel of $\mathsf{E}'(e)$ is given by the identity

$$\int_\Omega \kappa(u)v\,\mathrm{d}x - \int_\Gamma \big([\![\varepsilon(u)]\!] + \sigma\mathcal{H}(\Gamma)\big)h\,\mathrm{d}s = 0,$$

which at equilibrium yields

$$\int_\Omega \kappa_* v\,\mathrm{d}x + \int_{\Gamma_*} l_* h\,\mathrm{d}s = 0, \tag{1.9}$$

where $\kappa_* := \kappa(u_*)$ and $l_* := l(u_*)$. On the other hand, a straightforward calculation, with $z = (v, h)$, yields

$$-\langle \mathcal{D}z|z\rangle = \frac{1}{u_*}\left[\frac{1}{u_*}\int_\Omega \kappa_* v^2\,\mathrm{d}x - \sigma\int_{\Gamma_*} h\cdot\mathcal{H}'(\Gamma_*)h\,\mathrm{d}s\right]. \tag{1.10}$$

As κ_* and σ are positive, we see that the form $\langle \mathcal{D}z|z\rangle$ is negative semi-definite as soon as $\mathcal{H}'(\Gamma_*)$ is negative semi-definite. We have

$$\mathcal{H}'(\Gamma_*) = 1/R_*^2 + (1/(n-1))\Delta_{\Gamma_*},$$

where Δ_{Γ_*} denotes the Laplace–Beltrami operator on Γ_* and R_* means the radius of the equilibrium sphere. To derive necessary conditions for an equilibrium e_* to be a local maximum of entropy, we consider two cases.

1. Suppose that Γ_* is not connected, that is, Γ_* is a union of m disjoint spheres Γ_*^k. Set $v = 0$, and let $h = h_k \neq 0$ be constant on Γ_*^k with $\sum_k h_k = 0$. Then the constraint (1.9) holds, and

$$\langle \mathcal{D}z|z\rangle = (\sigma/u_* R_*^2)(|\Gamma_*|/m)\sum_{k=1}^m h_k^2 > 0,$$

hence $\mathcal{D}$ cannot be negative semi-definite in this case. Thus if e_* is an equilibrium with maximal total entropy, then Γ_* must be connected, and hence both phases are connected.

2. Assume that Γ_* is connected. Setting $v = l_*/(\kappa_*|1)_\Omega$ and $h = -1/|\Gamma_*|$, we see that the property that $\mathcal{D}$ is negative semi-definite on the kernel of $\mathsf{E}'(e_*)$ implies

$$\zeta_* := \frac{\sigma u_*(\kappa_*|1)_\Omega}{l_*^2 R_*^2|\Gamma_*|} \leqq 1. \tag{1.11}$$

This is exactly the stability condition found in Theorem 4.5.

In summary, we obtain:

- The equilibria of (1.3) are precisely the critical points of the entropy functional with prescribed energy.
- The entropy functional with prescribed energy does not have a local maximum $e_* = (u_*, \Gamma_*)$ in the case in which Γ_* is not connected.
- A necessary condition for a critical point $e_* = (u_*, \Gamma_*)$ to be a local maximum of the entropy functional with prescribed energy is that Γ_* is connected and that inequality (1.11) holds.

It will be shown in Theorems 4.5 and 5.2 below that

- $(u_*, \Gamma_*) \in \mathcal{E}$ is stable if Γ_* is connected and $\zeta_* < 1$.
- The latter is exactly the case if the reduced energy functional,

$$[u \mapsto \varphi(u) = \mathsf{E}(u, S_{R(u)}(x_0))], \quad R(u) = \sigma/[\![\psi(u)]\!],$$

has a strictly negative derivative at u_*.

- Any solution starting in a neighborhood of a stable equilibrium exists globally and converges to another stable equilibrium exponentially fast.
- $(u_*, \Gamma_*) \in \mathcal{E}$ is always unstable if Γ_* is disconnected, or if $\zeta_* > 1$.

Hence multiple spheres (of the same radius) are always unstable for (1.3). This situation is reminiscent of the onset of *Ostwald ripening*, a process that manifests itself in the way that larger structures grow while smaller ones shrink and disappear. Here we refer to [1–4,35,43,65–72], and the references therein for various aspects and results on Ostwald ripening. In particular, we mention that the authors in [1–4] use the quasi-stationary Stefan problem with surface tension (that is, the Mullins–Sekerka problem) to model Ostwald ripening. Under proper scaling assumptions, the way sphere-like particles evolve is analyzed. Interesting and illuminating connections between various versions of the Stefan problem (mostly the Mullins–Sekerka problem) and Ostwald ripening are given in [43,65–67,69,70,72]. It would be of considerable interest to also pursue the effect of coarsening in the framework of the Stefan problem (1.3).

(iv) Now we want to relate problem (1.3) to the pertinent Stefan problems that have been studied in the mathematical literature so far. For this purpose we linearize $h(u) := [\![\psi(u)]\!]$ near the melting temperature u_m, defined by $h(u_m) = 0$. Then for the relative temperature $v = u - u_m$ we have $h(u) \approx h'(u_m)v$, hence with $l_m = l(u_m)$ and $\gamma_m = \gamma(u_m)$, the Gibbs–Thomson law becomes approximately

$$(l_m/u_m)v + \sigma\mathcal{H} = \gamma_m V. \tag{1.12}$$

This is the classical Gibbs–Thomson law (with kinetic undercooling in case $\gamma_m > 0$). Similarly, assuming that u is close to u_m and V is small, the Stefan law becomes approximately

$$[\![d\partial_\nu v]\!] = l_m V. \tag{1.13}$$

As mentioned above, existence results for the Stefan problem with the classical Gibbs–Thomson law $v = -\alpha\mathcal{H}$ and the classical Stefan law (1.13) in the case where $\kappa_1 = \kappa_2$ can be found in [5,28,32,39,54,59,83–85]. The Stefan problem with the linearized transmission conditions (1.12)–(1.13) in case $\kappa_1 = \kappa_2$ has been studied in [19,83,84,91], see also [48] for the one-phase case.

In the recent publication [40] the author also obtains nonlinear stability of single spheres for the Stefan problem with the linearized transmission conditions (1.12)–(1.13) in the case in which all physical constants are taken to be 1, $\gamma_m = 0$, and the (appropriately modified) stability condition $\zeta_* < 1$ is satisfied. The method relies on higher order energy estimates and requires higher order regularity and compatibility conditions for the initial data, see also the remarks in (v).

Our results go beyond the results in [40] in several significant ways: we obtain existence and uniqueness results for arbitrary initial configurations with only minimal regularity assumptions on the data. We also provide instability results, either in the case of connected equilibria with $\zeta_* > 1$, or in the case of multiple spheres. It should be mentioned that the linearized stability analysis of multiple spheres is considerably more involved than the case of a single sphere. Moreover, we allow for general material laws and we include kinetic undercooling. Lastly, our setting allows for an interpretation of the equilibria in terms of the entropy functional.

It should also be noted that the results and methods of our paper are not restricted to the thermodynamically consistent Stefan problem, but also apply to the case with linearized transmission conditions. In fact, the essential mathematical difficulties encountered in the stability-instability analysis of equilibria are already present in the latter situation.

Linear instability has been observed before in [18] for the particular case in which $\Omega = \mathbb{R}^2$, where suitable boundary conditions for the temperature at infinity are imposed. It is then shown in [18] that equilibria are linearly unstable. This setting formally implies $\zeta_* = \infty$, which is in agreement with the instability condition $\zeta_* > 1$ of this paper.

If $\kappa_1 = \kappa_2 = 0$, then we obtain a thermodynamically consistent quasi-stationary approximation of the Stefan problem with surface tension (and kinetic undercooling). Existence and global existence of classical solutions for the quasi-stationary approximation with (1.12) with $\gamma_{\mathrm{m}} \neq 0$ and the classical Stefan law (1.13) have been investigated in [48,97].

As mentioned before, we assume that the behavior in the bulk phases is described by constitutive equations for the free energies $\psi_i(u)$. A common assumption is that the heat capacities be constant and equal in the respective phases. Then we necessarily have $0 \equiv [\![\kappa]\!] = [\![\varepsilon'(u)]\!] = -u[\![\psi''(u)]\!]$, which implies that the function $h(u) = [\![\psi(u)]\!]$ is linear, that is $h(u) = h_0 + h_1 u$, and then $l(u) = h_1 u$. The melting temperature is given here by $0 < u_{\mathrm{m}} = -h_0/h_1$.

If the heat capacities κ_i are constant in the phases but not necessarily equal, the internal energies depend linearly on the temperature, and the free and the inner energies are of the form

$$\psi_i(u) = a_i + b_i u - \kappa_i u \ln u, \quad \varepsilon_i(u) = a_i + \kappa_i u,$$

hence $h(u) = \alpha + \beta u - \delta u \ln u$, with constants $\alpha, \beta, \delta \in \mathbb{R}$. Concerning existence of equilibria, these special cases will be discussed in more detail in Section 4.

In [55] the author considers a Stefan problem based on thermodynamical principles for the case where the internal energy is given by

$$e = w + \varphi \quad \text{with} \quad w = (1/u_{\mathrm{m}} - 1/u),$$

where the phase function $\varphi : \Omega \setminus \Gamma \to \{0, 1\}$ assumes the distinct values 0 and 1 in the respective bulk phases, and where u and u_{m} denote the absolute and the melting temperature. In this situation, the free energy of the system can be described by

$$\psi_i(u) = 1/u_{\mathrm{m}} - 1/(2u) + \varphi(1 - u/u_{\mathrm{m}}).$$

Setting $d_i(u) = 1/u^2$, the diffusion equation in the bulk phases, expressed for the new variable w, becomes $\partial_t e = \Delta w$. The total energy and total entropy are then given by

$$\mathsf{E}(w, \Gamma) = \int_{\Omega\setminus\Gamma} (w + \varphi)\, \mathrm{d}x + \frac{\sigma|\Gamma|}{n-1}, \quad \Phi(w, \Gamma) = \int_{\Omega\setminus\Gamma} \left\{ -\frac{w^2}{2} + \frac{w+\varphi}{u_{\mathrm{m}}} - \frac{1}{2u_{\mathrm{m}}^2} \right\} \mathrm{d}x,$$

where $|\Gamma|$ denotes the surface area of Γ. Therefore, the function

$$L(w,\Gamma) := \int_{\Omega\setminus\Gamma} \frac{w^2}{2}\,\mathrm{d}x + \frac{\sigma|\Gamma|}{(n-1)u_{\mathrm{m}}} = -\Phi(w,\Gamma) + \frac{\mathsf{E}(w,\Gamma)}{u_{\mathrm{m}}} - \frac{|\Omega|}{2u_m^2},$$

termed total entropy in [55], is a Lyapunov functional for the system.

On a more elementary and ad-hoc level, assuming equal and constant heat capacities $\kappa = \kappa_i$, constant latent heat l_{m}, and constant heat conductivity coefficients d_i, one can assign to the Stefan problem subject to the classical conditions (1.12)–(1.13) an energy and a Lyapunov functional through the relations

$$E(v,\Gamma) = \int_{\Omega\setminus\Gamma} (\kappa v + l_{\mathrm{m}}\varphi)\,\mathrm{d}x, \quad L(v,\Gamma) = \int_{\Omega\setminus\Gamma} \kappa v^2/2\,\mathrm{d}x + \sigma u_{\mathrm{m}}|\Gamma|/(n-1),$$

where $v = u - u_{\mathrm{m}}$ denotes the relative temperature, and where the function φ has the same meaning as above, see also [79] for the case $\kappa_1 \neq \kappa_2$. The Lyapunov functional L plays an important role in the construction of long time weak solutions in [54,55], see also [85]. The authors in [54,55,85] consider the Stefan problem subject to the linearized transmission conditions (1.12)–(1.13) with $\gamma_{\mathrm{m}} = 0$, and they assume equal and constant heat capacities κ_i, constant latent heat, and constant heat conduction coefficients. The weak solutions obtained exist on any given, fixed time interval $(0, T)$ and have the feature that they lead to a sharp interface $\Gamma(t)$, in contrast to the weak solutions previously obtained in [92,93]. A serious drawback of the results in [54,55,85,92,93] is caused by the lack of uniqueness of solutions. This renders further assertions concerning asymptotic properties of solutions rather difficult, if not impossible.

(v) The novelty of our contribution lies in the fact that we consider rather general phase transition models that are thermodynamically consistent. In particular, we allow for different heat capacities, and kinetic undercooling can be included in the model. In the mathematical literature it is commonly assumed that the heat capacities κ_i are equal. However, this assumption is somewhat questionable, as it implies that the internal energies ε_i can differ only by a constant.

We obtain unique strong solutions, but existence is only guaranteed for short time intervals. This, however, is to be expected, as solutions can develop singularities in finite time, say in the way that topological changes in the geometry may occur. We give a complete analysis for the equilibrium states of (1.3), and we investigate the asymptotic behavior of solutions that start out close to equilibria. It is of significant interest to note that the equilibrium states can be characterized as the critical points of the total entropy subject to the constraint that the total energy be conserved. Moreover, we obtain that the equilibrium case where the dispersed phase consists of multiple balls (necessarily of the same radius) always leads to an unstable configuration. As already mentioned, this is reminiscent of the onset of Ostwald ripening. Additionally, we prove that solutions exist globally and have relatively compact orbits, provided they do not exhibit singularities, see Theorem 5.3. It appears that this manuscript is the first work to provide such qualitative results for a thermodynamically consistent Stefan problem.

A major difficulty in the mathematical treatment of the Stefan problem (1.3) is due to the fact that the boundary $\Gamma(t)$, and thus the geometry, is unknown and ever changing. A widely used method to overcome this inherent difficulty is to choose a fixed reference surface Σ and then represent the moving surface $\Gamma(t)$ as the graph of a function in normal direction of Σ. This way, one obtains a time-dependent (unknown) diffeomorphism from Σ onto $\Gamma(t)$, and in a next step this diffeomorphism is extended to a diffeomorphism of fixed reference regions Ω_i^Σ onto the unknown domains $\Omega_i(t)$. The treatment of the free boundary problem (1.3) then proceeds by transforming the equations into a new system of equations defined on the fixed domain $\Omega \setminus \Sigma$ from which both the solution and the parameterizing function have to be determined. In the context of the Stefan problem this approach was first used by Hanzawa [41]. This step in carried out in Section 2.

Section 3 is devoted to results on local well-posedness for problem (1.3), based on the approach in [28] and [21]. We show that solutions do not lose regularity. Thus, solutions give rise to a semiflow in the state manifold $\mathcal{SM}_\gamma$, and this property allows us to use methods from the theory of dynamical systems to further investigate geometric properties of solutions, such as the structure of the ω-limit set, and convergence results for global solutions, see for instance Theorem 5.3.

In Section 4 we discuss equilibria and their linear stability properties. Here we rely on previous work in [79]. However, we should like to point out that the stability results given here are considerably more general than those in [79], where we considered only the situation of a connected disperse phase.

In Section 5 we establish the corresponding stability properties for the nonlinear problem, employing the *generalized principle of linearized stability*, extending the results of [81] to the situation considered here. The main result of this section shows convergence of solutions to an equilibrium which start out near stable equilibria. Moreover, we give a rigorous proof of the instability result. The main difficulty in proving the stability result lies in the fact that equilibria are not isolated, but rather form a manifold, caused by the fact that the equilibrium problem is invariant under translations and scaling. This implies that the standard approach of linearized stability cannot be applied directly; for this reason we need to apply ideas developed in [81]. We emphasize, however, that the situation considered here is much more complicated than in [81], as the compatibility conditions implied by line 4 of Equation (1.3) force us to work in a nonlinear manifold (the state manifold $\mathcal{SM}_\gamma$), rather than in an open subset of a vector space.

In order to prove the stability/instability results, we parametrize $\mathcal{SM}_\gamma$ locally over the tangent space $\tilde{Z}_\gamma$ in a neighborhood of $(0, 0)$. The flow is then decomposed into a part $\tilde{z}$ evolving in the tangent space $\tilde{Z}_\gamma$, and a small component $\bar{z}$ that corresponds to the image of $\tilde{z}$ under the parametrization of $\mathcal{SM}_\gamma$. The flow of $\tilde{z}$ is driven by the linearized operator L_γ and the right-hand side of (5.11), which couples the dynamics of $\tilde{z}$ and $\bar{z}$. The part of the flow in the tangent space $\tilde{Z}_\gamma$ is then further decomposed into a part x, corresponding to the finite dimensional projection of $\tilde{z}$ onto the kernel $N(L_\gamma)$ of L_γ, and a part y in the stable subspace, which basically measures the deviation of $\tilde{z}$ from the manifold of equilibria $\mathcal{E}$ in the stable direction. It is important to note that $N(L_\gamma)$ coincides with the tangent space of the manifold $\mathcal{E}$ of equilibria at a fixed equilibrium e_*. All the equations are coupled, and a contraction mapping argument is employed to obtain the existence of global in time, exponentially decreasing solutions.

The stability result in [40] has been obtained using the method of higher energy estimates as well as suitable orthogonality conditions to deal with the nontrivial kernel $N(L_\gamma)$ of the linearization. A related idea of flow decomposition is used: the flow is decomposed into a finite-dimensional component and a remaining infinite dimensional component which, in a suitable sense, is transversal to the equilibrium manifold $\mathcal{E}$. Related ideas have also been used in the study of long-time asymptotics for the Mullins–Sekerka model [1,27], or in the long-time analysis of some curvature driven flows [89], and of solitons, see for instance the review in [90].

Of ultimate importance is the Lyapunov functional for (1.3), which is given by the negative total entropy $-\Phi(u, \Gamma)$. It takes bounded global-in-time solutions to the set of equilibria, and then by the results of Section 5 and relative compactness of the orbits, any such solution must converge towards an equilibrium in the topology of the state manifold $\mathcal{SM}_\gamma$, provided it comes close to a stable equilibrium.

Our analysis is carried out in the framework of L_p-spaces, with $n+2 < p < \infty$. We expect that it would be enough to require $(n+2)/2 < p < \infty$ (so unfortunately $p > 2$ even in two dimensions!), but for the sake of simplicity we restrict ourselves here to the stronger assumption $p > n+2$. We also expect that a similar analysis can be obtained in the framework of the little Hölder spaces h^α, which would, though, require higher order compatibility conditions.

(vi) Finally, we would like to address open problems and directions of future research. We are confident that our approach based on maximal regularity is flexible and general enough to also investigate more complex models that take *surface energy* into consideration. In fact, the case where the surface tension depends on the temperature has recently been considered in [80]. Of considerable interest, also, is the case where the Gibbs–Thomson law is replaced by

$$[\![\psi(u)]\!] + \sigma\mathcal{H} = \gamma(u)V - \mathrm{div}_\Gamma[\alpha(u)\nabla_\Gamma(V/u)] - u\,\mathrm{div}_\Gamma[\beta(u)\nabla_\Gamma V],$$

with $\alpha, \beta > 0$, see [6] for more background information. The modified Stefan law then reads

$$[\![\mathrm{d}(u)\partial_\nu u]\!] = \Big(l(u) - \gamma(u)V + \mathrm{div}_\Gamma[\alpha(u)\nabla_\Gamma(V/u)] + u\,\mathrm{div}_\Gamma[\beta(u)\nabla_\Gamma V]\Big)V,$$

and the resulting entropy production becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Phi(u(t),\Gamma(t)) &= \int_\Omega \frac{1}{u^2}\mathrm{d}(u)|\nabla u|^2\,\mathrm{d}x + \int_\Gamma \frac{1}{u}\gamma(u)V^2\,\mathrm{d}s \\ &\quad + \int_\Gamma [\alpha(u)|\nabla_\Gamma(V/u)|^2 + \beta(u)|\nabla_\Gamma V|^2]|\,\mathrm{d}s \geqq 0.\end{aligned}$$

Another direction concerns the situation where the densities of the respective phases are different. This case results in the occurrence of so-called *Stefan currents*, and the corresponding models need to also involve the equations of fluid dynamics, see for instance [6]. Additional very interesting problems concern phase transitions in moving viscous fluids, in which case the motion can be modeled by a thermodynamically consistent Stefan problem coupled to the Navier–Stokes equations, see [6,44]. First results in this direction are contained in [76,77]. Of even greater challenge is the case where one fluid is *evaporating*, leading to a phase-transition model which couples the equations of phase transitions to the equations of fluid dynamics with a compressible fluid phase.

As mentioned above, it would be interesting to link the thermodynamically consistent Stefan problem (1.3) to the occurrence of *Ostwald ripening*.

Of considerable interest, also, is a better understanding of the occurrence of *singularities* for solutions of (1.3). A preliminary result ensuring global existence is contained in Theorem 5.3 under rather restrictive assumptions. We conjecture that the only obstruction against global existence is related to the breakdown of the geometry: if no topological changes take place and the curvatures stay bounded, then the solution exists globally. More specifically, we conjecture that assumptions (i), (ii) and (iii) in Theorem 5.3 follow from (iv), provided that at time $t = 0$ we have $u_0 > 0$, and $l(u_0) \neq 0$ in $\bar{\Omega}$ in case $\gamma \equiv 0$.

An additional direction that is of great relevance concerns *triple junctions*, for instance the case when the free surface $\Gamma(t)$ is in contact with the solid container wall. While the situation where $\Gamma(t)$ meets the container wall orthogonally can likely be handled with the methods developed in this paper (by reflection arguments), the case of arbitrary *contact angles* remains a significant challenge. For progress in the case of the Hele–Shaw problem with surface tension we refer to [49,50].

2. Transformation to a Fixed Interface

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial\Omega$ of class C^2, and suppose $\Gamma \subset \Omega$ is a closed hypersurface of class C^2, that is, a C^2-manifold which is the boundary of a bounded domain $\Omega_1 \subset \Omega$. We then set $\Omega_2 = \Omega \setminus \bar{\Omega}_1$. Note that while Ω_2 is connected, Ω_1 may be disconnected. However, Ω_1 consists of finitely many components only, as $\partial\Omega_1 = \Gamma$ by assumption is a manifold, at least of class C^2. Recall that the *second order bundle* of Γ is given by

$$\mathcal{N}^2\Gamma := \{(p, \nu_\Gamma(p), L_\Gamma(p)) : \; p \in \Gamma\}.$$

Note that the Weingarten map L_Γ (also called the shape operator, or the second fundamental tensor) is defined by

$$L_\Gamma(p) = -\nabla_\Gamma \nu_\Gamma(p), \quad p \in \Gamma,$$

where ∇_Γ denotes the surface gradient on Γ. The eigenvalues $\kappa_j(p)$ of $L_\Gamma(p)$ are the principal curvatures of Γ at $p \in \Gamma$, and we have $|L_\Gamma(p)| = \max_j |\kappa_j(p)|$. The mean curvature $\mathcal{H}_\Gamma(p)$ is given by

$$(n-1)\mathcal{H}_\Gamma(p) = \sum_{j=1}^{n-1} \kappa_j(p) = \mathrm{tr} L_\Gamma(p) = -\mathrm{div}_\Gamma \nu_\Gamma(p),$$

where div_Γ means surface divergence. Recall also that the *Hausdorff distance* d_{H} between the two closed subsets $A, B \subset \mathbb{R}^m$ is defined by

$$d_{\mathrm{H}}(A,B) := \max\left\{\sup_{a\in A}\mathrm{dist}(a,B), \sup_{b\in B}\mathrm{dist}(b,A)\right\}.$$

Then we may approximate Γ by a real analytic hypersurface Σ (or merely $\Sigma \in C^3$), in the sense that the Hausdorff distance of the second order bundles of Γ and Σ is as small as we want. More precisely, for each $\eta > 0$ there is a real analytic closed hypersurface such that $d_{\mathrm{H}}(\mathcal{N}^2\Sigma, \mathcal{N}^2\Gamma) \leqq \eta$. If $\eta > 0$ is small enough, then Σ bounds a domain Ω_1^Σ with $\overline{\Omega_1^\Sigma} \subset \Omega$, and we set $\Omega_2^\Sigma = \Omega \setminus \bar{\Omega}_1^\Sigma$.

It is well known that such a hypersurface Σ admits a tubular neighborhood, which means that there is $a > 0$ such that the map

$$\Lambda : \Sigma \times (-a,a) \to \mathbb{R}^n$$
$$\Lambda(p,r) := p + r\nu_\Sigma(p)$$

is a diffeomorphism from $\Sigma \times (-a,a)$ onto $\mathcal{R}(\Lambda)$. The inverse

$$\Lambda^{-1} : \mathcal{R}(\Lambda) \mapsto \Sigma \times (-a,a)$$

of this map is conveniently decomposed as

$$\Lambda^{-1}(x) = (\Pi(x), d_\Sigma(x)), \quad x \in \mathcal{R}(\Lambda).$$

Here $\Pi(x)$ means the nonlinear orthogonal projection of x to Σ and $d_\Sigma(x)$ the signed distance from x to Σ; so $|d_\Sigma(x)| = \mathrm{dist}(x,\Sigma)$ and $d_\Sigma(x) < 0$ iff $x \in \Omega_1^\Sigma$. In particular we have $\mathcal{R}(\Lambda) = \{x \in \mathbb{R}^n : \mathrm{dist}(x,\Sigma) < a\}$.

On the one hand, a is determined by the curvatures of Σ, that is, we must have

$$0 < a < \min\left\{1/|\kappa_j(p)| : j = 1,\dots,n-1,\ p \in \Sigma\right\},$$

where $\kappa_j(p)$ means the principal curvatures of Σ at $p \in \Sigma$. But on the other hand, a is also connected to the topology of Σ, which can be expressed as follows. Since Σ is a compact (smooth) manifold of dimension $n-1$, it satisfies an (interior and exterior) ball condition, which means that there is a radius $r_\Sigma > 0$ such that for each point $p \in \Sigma$ there are $x_j \in \Omega_j^\Sigma$, $j = 1,2$, such that $B_{r_\Sigma}(x_j) \subset \Omega_j^\Sigma$, and $\bar{B}_{r_\Sigma}(x_j) \cap \Sigma = \{p\}$. Choosing r_Σ maximal, we then must also have $a < r_\Sigma$. In the sequel we fix

$$a = \frac{1}{2}\min\left\{r_\Sigma, \frac{1}{|\kappa_j(p)|},\ j = 1,\dots,n-1,\ p \in \Sigma\right\}.$$

For later use we note that the derivatives of $\Pi(x)$ and $d_\Sigma(x)$ are given by

$$\nabla d_\Sigma(x) = \nu_\Sigma(\Pi(x)), \quad \Pi'(x) = M_0(d_\Sigma(x), \Pi(x))P_\Sigma(\Pi(x)),$$

where $P_\Sigma(p) = I - \nu_\Sigma(p) \otimes \nu_\Sigma(p)$ denotes the orthogonal projection onto the tangent space $T_p\Sigma$ of Σ at $p \in \Sigma$, and $M_0(r,p) = (I - rL_\Sigma(p))^{-1}$. Note that $|M_0(r,p)| \leqq 1/(1 - r|L_\Sigma(p)|) \leqq 2$ for all $|r| \leqq a$ and $p \in \Sigma$.

Setting $\Gamma = \Gamma(t)$, we may use the map Λ to parametrize the unknown free boundary $\Gamma(t)$ over Σ by means of a height function $\rho(t, p)$ via

$$\Gamma(t) : [p \mapsto p + \rho(t, p)\nu_\Sigma(p)], \quad p \in \Sigma,\ t \geqq 0,$$

for small $t \geqq 0$, at least. Extend this diffeomorphism to all of $\bar{\Omega}$ by means of

$$\Xi_\rho(t, x) = x + \chi(d_\Sigma(x)/a)\rho(t, \Pi(x))\nu_\Sigma(\Pi(x)) =: x + \theta_\rho(t, x).$$

Here χ denotes a suitable cut-off function; more precisely, $\chi \in \mathcal{D}(\mathbb{R})$, $0 \leqq \chi \leqq 1$, $\chi(r) = 1$ for $|r| < 1/3$, and $\chi(r) = 0$ for $|r| > 2/3$. Note that $\Xi_\rho(t, x) = x$ for $|d_\Sigma(x)| > 2a/3$, and

$$\Xi_\rho^{-1}(t, x) = x - \rho(t, \Pi(x))\nu_\Sigma(\Pi(x)) \quad \text{for } |d_\Sigma(x)| < a/3.$$

In particular,

$$\Xi_\rho^{-1}(t, x) = x - \rho(t, x)\nu_\Sigma(x) \quad \text{for } x \in \Sigma.$$

Setting $v(t, x) = u(t, \Xi_\rho(t, x))$, or $u(t, x) = v(t, \Xi_\rho^{-1}(t, x))$ we have thus transformed the time varying regions $\Omega \setminus \Gamma(t)$ to the fixed domain $\Omega \setminus \Sigma$. This is the direct mapping method, also called Hanzawa transformation.

By means of this transformation, we obtain the following transformed problem.

$$\begin{cases} \kappa(v)\partial_t v + \mathcal{A}(v, \rho)v = \kappa(v)\mathcal{R}(\rho)v & \text{in } \Omega \setminus \Sigma \\ \partial_{\nu_\Omega} v = 0 & \text{on } \partial\Omega \\ [\![v]\!] = 0 & \text{on } \Sigma \\ [\![\psi(v)]\!] + \sigma\mathcal{H}(\rho) = \gamma(v)\beta(\rho)\partial_t\rho & \text{on } \Sigma \\ \{l(v) - \gamma(v)\beta(\rho)\partial_t\rho\}\beta(\rho)\partial_t\rho + \mathcal{B}(v, \rho)v = 0 & \text{on } \Sigma \\ v(0) = v_0,\ \rho(0) = \rho_0. \end{cases} \tag{2.1}$$

Here $\mathcal{A}(v, \rho)$ and $\mathcal{B}(v, \rho)$ denote the transformations of $-\mathrm{div}(d\nabla)$ and $-[\![d\partial_\nu]\!]$, respectively. Moreover, $\mathcal{H}(\rho)$ represents the mean curvature of Γ, $\beta(\rho) = (\nu_\Sigma|\nu_\Gamma(\rho))$, the term $\beta(\rho)\partial_t\rho$ represents the normal velocity V, and

$$\mathcal{R}(\rho)v = \partial_t v - \partial_t u \circ \Xi_\rho.$$

The system (2.1) is a quasi-linear parabolic problem on the domain Ω with fixed interface $\Sigma \subset \Omega$ with a *dynamic boundary condition*, namely the fifth equation which describes the evolution of the interface $\Gamma(t)$.

To elaborate on the structure of this problem in more detail, we calculate

$$\Xi'_\rho = I + \theta'_\rho, \qquad \Xi_\rho'^{-1} = I - [I + \theta'_\rho]^{-1}\theta'_\rho =: I - M_1(\rho)^{\mathsf{T}}$$

and

$$\nabla u \circ \Xi_\rho = [(\Xi_\rho^{-1})'^{\mathsf{T}} \circ \Xi_\rho]\nabla v = (I - M_1(\rho))\nabla v,$$

and for a vector field $q = \bar{q} \circ \Xi_\rho$,

$$(\nabla|\bar{q}) \circ \Xi_\rho = ([(\Xi_\rho^{-1})'^{\mathsf{T}} \circ \Xi_\rho]\nabla|q) = ((I - M_1(\rho))\nabla|q).$$

Further, we have

$$\begin{aligned}\partial_t u \circ \Xi_\rho &= \partial_t v - (\nabla u \circ \Xi_\rho|\partial_t \Xi_\rho) = \partial_t v - ([(\Xi_\rho^{-1})'^{\mathsf{T}} \circ \Xi_\rho]\nabla v|\partial_t \Xi_\rho) \\ &= \partial_t v - (\nabla v|[I + \theta_\rho']^{-1}\partial_t\theta_\rho),\end{aligned}$$

hence

$$\mathcal{R}(\rho)v = (\nabla v|[I + \theta_\rho']^{-1}\partial_t\theta_\rho).$$

With the Weingarten map $L_\Sigma = -\nabla_\Sigma \nu_\Sigma$, we have

$$\begin{aligned}\nu_\Gamma(\rho) &= \beta(\rho)(\nu_\Sigma - \alpha(\rho)), \quad && \alpha(\rho) = M_0(\rho)\nabla_\Sigma\rho, \\ M_0(\rho) &= (I - \rho L_\Sigma)^{-1}, && \beta(\rho) = (1 + |\alpha(\rho)|^2)^{-1/2},\end{aligned}$$

and

$$V = (\partial_t \Xi|\nu_\Gamma) = (\nu_\Sigma|\nu_\Gamma(\rho))\partial_t\rho = \beta(\rho)\partial_t\rho.$$

Employing this notation leads to $\theta_\rho' = 0$ for $|d_\Sigma(x)| > 2a/3$ and

$$\begin{aligned}\theta_\rho'(t,x) = \frac{1}{a}&\chi'(d_\Sigma(x)/a)\rho(t,\Pi(x))\nu_\Sigma(\Pi(x)) \otimes \nu_\Sigma(\Pi(x)) \\ &+ \chi(d_\Sigma(x)/a)[\nu_\Sigma(\Pi(x)) \otimes M_0(d_\Sigma(x))\nabla_\Sigma\rho(t,\Pi(x))] \\ &- \chi(d_\Sigma(x)/a)\rho(t,\Pi(x))L_\Sigma(\Pi(x))M_0(d_\Sigma(x))P_\Sigma(\Pi(x))\end{aligned}$$

for $0 \leqq |d_\Sigma(x)| \leqq 2a/3$. In particular, for $x \in \Sigma$ we have

$$\theta_\rho'(t,x) = \nu_\Sigma(x) \otimes \nabla_\Sigma\rho(t,x) - \rho(t,x)L_\Sigma(x)P_\Sigma(x),$$

and

$$(\theta_\rho')^{\mathsf{T}}(t,x) = \nabla_\Sigma\rho(t,x) \otimes \nu_\Sigma(x) - \rho(t,x)L_\Sigma(x),$$

since $L_\Sigma(x)$ is symmetric and has range in $T_x\Sigma$. Therefore, $[I + \theta_\rho']$ is boundedly invertible, if ρ and $\nabla_\Sigma\rho$ are sufficiently small, and

$$|[I + \theta_\rho']^{-1}| \leqq 2 \quad \text{for } |\rho|_\infty \leqq \frac{1}{4(|\chi'|_\infty/a + 2\max_j|\kappa_j|)}, \quad |\nabla_\Sigma\rho|_\infty \leqq \frac{1}{8}.$$

For the mean curvature $\mathcal{H}(\rho)$ we have

$$\begin{aligned}(n-1)\mathcal{H}(\rho) \; &= \beta(\rho)\{\mathrm{tr}[M_0(\rho)(L_\Sigma + \nabla_\Sigma\alpha(\rho))] \\ &\qquad -\beta^2(\rho)(M_0(\rho)\alpha(\rho)|[\nabla_\Sigma\alpha(\rho)]\alpha(\rho))\},\end{aligned}$$

an expression involving second order derivatives of ρ only linearly. Its linearization at $\rho = 0$ is given by

$$(n-1)\mathcal{H}'(0) = \mathrm{tr}\, L_\Sigma^2 + \Delta_\Sigma.$$

Here Δ_Σ denotes the Laplace–Beltrami operator on Σ. The operator $\mathcal{B}(v, \rho)$ becomes

$$\begin{aligned}\mathcal{B}(v,\rho)v &= -[\![\mathrm{d}(u)\partial_\nu u]\!] \circ \Xi_\rho = -([\![\mathrm{d}(v)(I - M_1(\rho))\nabla v]\!]|\nu_\Gamma)\\ &= -\beta(\rho)([\![\mathrm{d}(v)(I - M_1(\rho))\nabla v]\!]|\nu_\Sigma - \alpha(\rho))\\ &= -\beta(\rho)[\![\mathrm{d}(v)\partial_{\nu_\Sigma} v]\!] + \beta(\rho)([\![\mathrm{d}(v)\nabla v]\!]|(I - M_1(\rho))^{\mathsf{T}}\alpha(\rho)),\end{aligned}$$

since $M_1^{\mathsf{T}}(\rho)\nu_\Sigma = 0$, and finally

$$\begin{aligned}\mathcal{A}(v,\rho)v &= -\operatorname{div}(\mathrm{d}(u)\nabla u) \circ \Xi_\rho = -((I - M_1(\rho))\nabla|\mathrm{d}(v)(I - M_1(\rho))\nabla v)\\ &= -\,\mathrm{d}(v)\Delta v + \mathrm{d}(v)[M_1(\rho) + M_1^{\mathsf{T}}(\rho) - M_1(\rho)M_1^{\mathsf{T}}(\rho)] : \nabla^2 v\\ &\quad - d'(v)|(I - M_1(\rho))\nabla v|^2 + \mathrm{d}(v)((I - M_1(\rho)) : \nabla M_1(\rho)|\nabla v).\end{aligned}$$

We recall that for matrices $A, B \in \mathbb{R}^{n\times n}$, $A : B = \sum_{i,j=1}^n a_{ij}b_{ij} = \operatorname{tr}(AB^{\mathsf{T}})$ denotes the inner product.

Obviously, the leading part of $\mathcal{A}(v, \rho)v$ is $-\mathrm{d}(v)\Delta v$, while the leading part of $\mathcal{B}(v, \rho)v$ is $-\beta(\rho)[\![\mathrm{d}(v)\partial_{\nu_\Sigma} v]\!]$, as $M_1(0) = 0$ and $\alpha(0) = 0$; recall that we may assume ρ small in the C^2-norm. It is important to recognize the quasilinear structure of (2.1): derivatives of highest order only appear linearly in each of the equations.

3. Local Well-Posedness

The basic result for local well-posedness in the absence of kinetic undercooling in an L_p-setting is the following.

Theorem 3.1. ($\gamma \equiv 0$). *Let $p > n+2$, $\gamma = 0$, $\sigma > 0$. Suppose $\psi \in C^3(0,\infty)$, $d \in C^2(0,\infty)$ such that*

$$\kappa(u) = -u\psi''(u) > 0, \quad \mathrm{d}(u) > 0, \quad u \in (0,\infty).$$

Assume the regularity conditions

$$u_0 \in W_p^{2-2/p}(\Omega \setminus \Gamma_0) \cap C(\bar{\Omega}), \quad u_0 > 0, \quad \Gamma_0 \in W_p^{4-3/p},$$

the compatibility conditions

$$\partial_{\nu_\Omega} u_0 = 0, \quad [\![\psi(u_0)]\!] + \sigma\mathcal{H}(\Gamma_0) = 0, \quad [\![d(u_0)\partial_{\nu_{\Gamma_0}} u_0]\!] \in W_p^{2-6/p}(\Gamma_0),$$

and the well-posedness condition

$$l(u_0) \neq 0 \quad \text{on } \Gamma_0.$$

Then there exists a unique L_p-solution for the Stefan problem with surface tension (1.3) on some possibly small but nontrivial time interval $J = [0, \tau]$.

Here the notation $\Gamma_0 \in W_p^{4-3/p}$ means that Γ_0 is a C^2-manifold, such that its (outer) normal field ν_{Γ_0} is of class $W_p^{3-3/p}(\Gamma_0)$. Therefore, the Weingarten tensor $L_{\Gamma_0} = -\nabla_{\Gamma_0}\nu_{\Gamma_0}$ of Γ_0 belongs to $W_p^{2-3/p}(\Gamma_0)$, which embeds into $C^{1+\alpha}(\Gamma_0)$ with $\alpha = 1-(n+2)/p > 0$, since $p > n+2$ by assumption. For the same reason, we also have $u_0 \in C^{1+\alpha}(\bar\Omega)$, and $V_0 \in C^{2\alpha}(\Gamma_0)$. The notion L_p-solution means that (u, Γ) is obtained as the push-forward of an L_p-solution (v, ρ) of the transformed problem (2.1). This class will be discussed below.

There is an analogous result in the presence of kinetic undercooling which reads as follows.

Theorem 3.2. ($\gamma > 0$). *Let $p > n+2$, $\sigma > 0$, and suppose $\psi, \gamma \in C^3(0,\infty)$, $d \in C^2(0,\infty)$ such that*

$$\kappa(u) = -u\psi''(u) > 0, \quad \mathrm{d}(u) > 0, \quad \gamma(u) > 0, \quad u \in (0,\infty).$$

Assume the regularity conditions

$$u_0 \in W_p^{2-2/p}(\Omega\setminus\Gamma_0) \cap C(\bar\Omega), \quad u_0 > 0, \quad \Gamma_0 \in W_p^{4-3/p},$$

and the compatibility conditions

$$\partial_{\nu_\Omega} u_0 = 0, \quad \big([\![\psi(u_0)]\!] + \sigma\mathcal{H}(\Gamma_0)\big)\big(l(u_0) - [\![\psi(u_0)]\!] - \sigma\mathcal{H}(\Gamma_0)\big) = \gamma(u_0)[\![d(u_0)\partial_\nu u_0]\!].$$

Then there exists a unique L_p-solution of the Stefan problem with surface tension and kinetic undercooling (1.3) *on some possibly small but nontrivial time interval $J = [0, \tau]$.*

Proof of Theorems 3.1 and 3.2. (i) *Direct mapping method: Hanzawa transformation.*

As explained in the previous section, we employ a Hanzawa transformation and study the resulting problem (2.1) on the domain Ω with fixed interface Σ.

In the case where $\gamma \equiv 0$, for the L_p-theory, the solution of the transformed problem will belong to the class

$$\begin{aligned} v &\in H_p^1(J; L_p(\Omega)) \cap L_p(J; H_p^2(\Omega\setminus\Sigma)) \hookrightarrow C(J; W_p^{2-2/p}(\Omega\setminus\Sigma)),\\ \rho &\in W_p^{1-1/2p}(J; H_p^2(\Sigma)) \cap L_p(J; W_p^{4-1/p}(\Sigma)) \hookrightarrow C(J; W_p^{4-3/p}(\Sigma)),\\ \partial_t\rho &\in W_p^{1/2-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{2-2/p}(\Sigma)) \hookrightarrow C(J; W_p^{2-6/p}(\Sigma)). \end{aligned} \tag{3.1}$$

See [28] for a proof of the last two embeddings in the case where $\Sigma = \mathbb{R}^n$.

If $\gamma > 0$ we have, moreover,

$$\rho \in W_p^{2-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{4-1/p}(\Sigma)) \hookrightarrow C^1(J; W_p^{2-3/p}(\Sigma)).$$

Note that in both cases, $v \in C(J\times\bar\Omega)$, $v|_{\Omega_j} \in C(J; C^1(\bar\Omega_j))$, $j = 1, 2$. Moreover, $\rho \in C(J; C^3(\Sigma))$ and

$$\partial_t\rho \in C(J; C(\Sigma)) \quad \text{in case } \gamma = 0, \quad \partial_t\rho \in C(J; C^1(\Sigma)) \quad \text{in case } \gamma > 0.$$

We set

$$\begin{aligned}
\mathbb{E}_1(J) &:= \{v \in H_p^1(J; L_p(\Omega)) \cap L_p(J; H_p^2(\Omega \setminus \Sigma) : [\![v]\!] = 0,\ \partial_{\nu_\Omega} v = 0\},\\
\mathbb{E}_2(J) &:= W_p^{3/2-1/2p}(J; L_p(\Sigma)) \cap W_p^{1-1/2p}(J; H_p^2(\Sigma)) \cap L_p(J; W_p^{4-1/p}(\Sigma)),\\
&\quad \gamma \equiv 0,\\
\mathbb{E}_2(J) &:= W_p^{2-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{4-1/p}(\Sigma)),\ \gamma > 0,\\
\mathbb{E}(J) &:= \mathbb{E}_1(J) \times \mathbb{E}_2(J),
\end{aligned}$$

that is, $\mathbb{E}(J)$ denotes the solution space.

Similarly, we define

$$\begin{aligned}
\mathbb{F}_1(J) &:= L_p(J; L_p(\Omega)),\\
\mathbb{F}_2(J) &:= W_p^{1-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{2-1/p}(\Sigma)),\\
\mathbb{F}_3(J) &:= W_p^{1/2-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{1-1/p}(\Sigma)),\\
\mathbb{F}(J) &:= \mathbb{F}_1(J) \times \mathbb{F}_2(J) \times \mathbb{F}_3(J),
\end{aligned}$$

that is, $\mathbb{F}(J)$ represents the space of data. A left subscript zero means vanishing time trace at $t = 0$, whenever it exists. So, for example

$$_0\mathbb{E}_2(J) = \{\rho \in \mathbb{E}_2(J) :\ \rho(0) = \partial_t \rho(0) = 0\}$$

whenever $p > 3$.

Employing the calculations in Section 2 and splitting into the principal linear part and a nonlinear part, we arrive at the following formulation of problem (2.1).

$$\left\{\begin{array}{lll}
\kappa_0(x)\partial_t v - d_0(x)\Delta v = F(v, \rho) & \text{in} & \Omega \setminus \Sigma\\
\partial_{\nu_\Omega} v = 0 & \text{on} & \partial\Omega\\
[\![v]\!] = 0, & \text{on} & \Sigma\\
l_1(t, x)v + \sigma_0 \Delta_\Sigma \rho - \gamma_1(t, x)\partial_t \rho = G(v, \rho) & \text{on} & \Sigma\\
l_0(x)\partial_t \rho - [\![d_0(x)\partial_\nu v]\!] = H(v, \rho) & \text{on} & \Sigma\\
v(0) = v_0,\ \rho(0) = \rho_0. & &
\end{array}\right. \tag{3.2}$$

Here,

$$\begin{aligned}
&\kappa_0(x) = \kappa(v_0(x)),\quad d_0(x) = d(v_0(x)),\quad l_0(x) = l(v_0(x)),\quad \sigma_0 = \frac{\sigma}{n-1},\\
&l_1(t, \cdot) = [\![\psi'(\mathrm{e}^{\Delta_\Sigma t} v_{0\Sigma})]\!],\quad \gamma_1(t, \cdot) = \gamma(\mathrm{e}^{\Delta_\Sigma t} v_{0\Sigma}),
\end{aligned}$$

where $v_{0\Sigma}$ means the restriction of v_0 to Σ. Note that $\kappa_0, d_0 \in W_p^{2-2/p}(\Omega \setminus \Sigma)$, hence these functions are in $C^1(\bar{\Omega}_j)$, $j = 1, 2$. Recall that d and κ may be different in different phases. Further, we have $l_0 \in W_p^{2-3/p}(\Sigma)$ which implies $l_0 \in C^1(\Sigma)$. This is good enough for the space $\mathbb{F}_3(J)$, as C^1-functions are pointwise multipliers for $\mathbb{F}_3(J)$, but it is not good enough for $\mathbb{F}_2(J)$. For this reason, we need to define the extension $v_b := \mathrm{e}^{\Delta_\Sigma t} v_{0\Sigma}$. This function, as well as l_1 and γ_1, belong to $\mathbb{F}_2(J)$, hence are pointwise multipliers for this space, as $\mathbb{F}_2(J)$ and $\mathbb{F}_3(J)$ are Banach algebras with respect to pointwise multiplication, as $p > n+2$.

The nonlinearities F, G, and H are defined as follows.

$$\begin{aligned} F(v,\rho) &= (\kappa_0 - \kappa(v))\partial_t v + (\mathrm{d}(v) - d_0)\Delta v - \mathrm{d}(v)M_2(\rho) : \nabla^2 v \\ &\quad + d'(v)|(I - M_1(\rho))\nabla v|^2 - \mathrm{d}(v)(M_3(\rho)|\nabla v) + \kappa(v)\mathcal{R}(\rho)v, \\ G(v,\rho) &= -([\![\psi(v)]\!] + \sigma\mathcal{H}(\rho)) + l_1 v + \sigma_0\Delta_\Sigma\rho + (\gamma(v)\beta(\rho) - \gamma_1)\partial_t\rho, \\ H(v,\rho) &= [\![(\mathrm{d}(v) - d_0)\partial_\nu v]\!] + (l_0 - l(v))\partial_t\rho - ([\![\mathrm{d}(v)\nabla v]\!]|M_4(\rho)\nabla_\Sigma\rho) \\ &\quad + \gamma(v)\beta(\rho)(\partial_t\rho)^2. \end{aligned} \tag{3.3}$$

Here we have set

$$\begin{aligned} M_2(\rho) &= M_1(\rho) + M_1^{\mathsf{T}}(\rho) - M_1(\rho)M_1^{\mathsf{T}}(\rho), \\ M_3(\rho) &= (I - M_1(\rho)) : \nabla M_1(\rho), \\ M_4(\rho) &= (I - M_1(\rho))^{\mathsf{T}} M_0(\rho). \end{aligned}$$

(ii) *Maximal regularity of the principal linearized problem.*

First we consider the linear problem defined by the left-hand side of (3.2).

$$\begin{cases} \kappa_0(x)\partial_t v - d_0(x)\Delta v = f & \text{in } \ \Omega\backslash\Sigma \\ \partial_{\nu_\Omega} v = 0 & \text{on } \ \partial\Omega \\ [\![v]\!] = 0 & \text{on } \ \Sigma \\ l_1(t,x)v + \sigma_0\Delta_\Sigma\rho - \gamma_1(t,x)\partial_t\rho = g & \text{on } \ \Sigma \\ l_0(x)\partial_t\rho - [\![d_0(x)\partial_\nu v]\!] = h & \text{on } \ \Sigma \\ v(0) = v_0, \ \ \rho(0) = \rho_0. \end{cases} \tag{3.4}$$

This inhomogeneous problem can be solved with maximal regularity; see ESCHER ET AL. [28] for the constant coefficient half-space case with $\gamma \equiv 0$, and DENK ET AL. [21] for the general one-phase case.

Theorem 3.3. ($\gamma \equiv 0$). *Let $p > n+2$, $\sigma > 0$, $\gamma \equiv 0$. Suppose $\kappa_0 \in C(\bar{\Omega}_j)$ and $d_0 \in C^1(\bar{\Omega}_j)$, $j = 1, 2$, $\kappa_0, d_0 > 0$ on $\bar{\Omega}$, $l_0 \in W_p^{2-6/p}(\Sigma)$, and let*

$$l_1 \in W_p^{1-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{2-1/p}(\Sigma))$$

such that $l_0 l_1 > 0$ on $J \times \Sigma$, where $J = [0, t_0]$ is a finite time interval. Then there is a unique solution $z := (v, \rho) \in \mathbb{E}(J)$ of (3.4) if and only if the data (f, g, h) and $z_0 := (v_0, \rho_0)$ satisfy

$$(f, g, h) \in \mathbb{F}(J), \quad z_0 \in [W_p^{2-2/p}(\Omega \setminus \Sigma) \cap C(\bar{\Omega})] \times W_p^{4-3/p}(\Sigma),$$

and the compatibility conditions

$$\partial_{\nu_\Omega} v_0 = 0, \quad l_1(0)v_0 + \sigma_0\Delta_\Sigma\rho_0 = g(0), \quad h(0) + [\![d_0\partial_\nu v_0]\!] \in W_p^{2-6/p}(\Sigma).$$

The solution map $[(f, g, h, z_0) \mapsto z = (v, \rho)]$ is continuous between the corresponding spaces.

Proof. In the one-phase case this result is proved in [21, Example 3.4]. Therefore, we indicate only the necessary modifications for the two-phase case. The localization procedure can be carried out in the same way as in the one-phase case [21], hence we need to consider only the following model problem with constant coefficients where the interface is flat:

$$\begin{cases} \kappa_0 \partial_t v - d\Delta v = f & \text{in} \quad \dot{\mathbb{R}}^n \\ [\![v]\!] = 0 & \text{on} \quad \mathbb{R}^{n-1} \\ l_1 v + \sigma_0 \Delta \rho = g & \text{on} \quad \mathbb{R}^{n-1} \\ l_0 \partial_t \rho - [\![\mathrm{d}\partial_\nu v]\!] = h & \text{on} \quad \mathbb{R}^{n-1} \\ v(0) = v_0, \ \rho(0) = \rho_0. \end{cases}$$

Here $\dot{\mathbb{R}}^n = \mathbb{R}^{n-1} \times \big(\mathbb{R} \setminus \{0\}\big)$, and $\mathbb{R}^{n-1}$ is identified with $\mathbb{R}^{n-1} \times \{0\}$. Reflecting the lower half-plane to the upper, this becomes a problem of the form studied in [21]. As in Example 3.4 of that paper, it is not difficult to verify the necessary Lopatinskii–Shapiro conditions. Then Theorems 2.1 and 2.2 of [21] can be applied, proving the assertion for the model problem. □

Remark 3.4. One might wonder where the somewhat unexpected compatibility condition $h(0) + [\![d_0 \partial_\nu v_0]\!] \in W_p^{2-6/p}(\Sigma)$ in the case $\gamma = 0$ comes from. To illuminate this, note that

$$(h(0) + [\![d_0 \partial_\nu v_0]\!])/l_0 = \partial_t \rho(0)$$

is the trace of $\partial_t \rho$ at time $t = 0$. But by the embedding (3.1) this implies that
$(h(0)+[\![d_0\partial_\nu v_0]\!])/l_0 \in W_p^{2-6/p}(\Sigma)$, which in turn enforces $h(0)+[\![d_0\partial_\nu u_0]\!] \in W_p^{2-6/p}(\Sigma)$.

The main result for problem (3.4) for $\gamma > 0$ is the following theorem.

Theorem 3.5. ($\gamma > 0$). *Let $p > n + 2, \sigma > 0$. Suppose $\kappa_0 \in C(\bar{\Omega}_j)$ and $d_0 \in C^1(\bar{\Omega}_j)$, $j = 1, 2$, $\kappa_0, d_0 > 0$ on $\bar{\Omega}$, $l_0 \in C^1(\Sigma)$, and let*

$$\gamma_1, l_1 \in W_p^{1-1/2p}(J; L_p(\Sigma)) \cap L_p(J; W_p^{2-1/p}(\Sigma)),$$

such that $\gamma_1 > 0$ on $J \times \Sigma$, where $J = [0, t_0]$ is a finite time interval. Then there is a unique solution $z := (v, \rho) \in \mathbb{E}(J)$ of (3.4) if and only if the data (f, g, h) and $z_0 := (v_0, \rho_0)$ satisfy

$$(f, g, h) \in \mathbb{F}(J), \quad z_0 \in [W_p^{2-2/p}(\Omega \setminus \Sigma) \cap C(\bar{\Omega})] \times W_p^{4-3/p}(\Sigma),$$

and the compatibility conditions

$$\partial_{\nu_\Omega} v_0 = 0, \quad (l_0 l_1(0) v_{0|_\Sigma} + l_0 \sigma_0 \Delta_\Sigma \rho_0 - \gamma_1(0)[\![\mathrm{d}\partial_\nu v_0]\!] = \gamma_1(0) h(0) + l_0 g(0).$$

The solution map $[(f, g, h, z_0) \mapsto z = (v, \rho)]$ is continuous between the corresponding spaces.

Proof. The proof of this result is much simpler than for the case $\gamma = 0$. We could follow the strategy in the proof of Theorem 3.3, employing the methods in [21] once more. However, here we want to give a more direct argument that uses the fact that the term $l_0 \partial_t \rho$ is of lower order in the case where $\gamma_1 > 0$. For this purpose, suppose $v_\Sigma := v_{|\Sigma}$ is known. Consider the problem

$$\gamma_1 \partial_t \rho - \sigma_0 \Delta_\Sigma \rho = l_1 v_\Sigma - g, \quad t \in J, \quad \rho(0) = \rho_0.$$

Since the Laplace–Beltrami operator is strongly elliptic, we can solve this problem with maximal regularity to obtain ρ in the proper regularity class. Then we solve the transmission problem

$$\begin{cases} \kappa_0 \partial_t v - d_0 \Delta v = f & \text{in} \quad \Omega \setminus \Sigma \\ \partial_{\nu_\Omega} v = 0 & \text{on} \quad \partial\Omega \\ [\![v]\!] = 0 & \text{on} \quad \Sigma \\ -[\![d_0 \partial_\nu v]\!] = h - l_0 \partial_t \rho & \text{on} \quad \Sigma \\ v(0) = v_0. & \end{cases}$$

Finally, we take the trace of v to obtain an equation for v_Σ of the form

$$v_\Sigma = T v_\Sigma + w,$$

where w is determined by the data alone, and T is a compact operator from $\mathbb{F}_2(J)$ into itself. Here compactness follows from the compact embedding $\mathbb{F}_2(J) \hookrightarrow \mathbb{F}_3(J)$, that is, from the regularity of $\partial_t \rho$ which is higher than needed to solve the transmission problem. Thus $I - T$ is a Fredholm operator with index zero, hence invertible since it is injective by causality. This proves the sufficiency of the conditions on the data. Necessity is a consequence of trace theory. □

Remark 3.6. It is interesting to take a look at the boundary symbol of the linear problem; it is of the form

$$s(\lambda, \xi) = \lambda l^2 + (\lambda\gamma + \sigma_0 |\xi|^2)[\sqrt{\lambda\kappa_1 + d_1|\xi|^2} + \sqrt{\lambda\kappa_2 + d_2|\xi|^2}].$$

Here $\lambda \in \mathbb{C}_+$ denotes the covariable of time t, and $\xi \in \mathbb{R}^{n-1}$ that of the tangential space variable $x' \in \mathbb{R}^{n-1}$. This symbol is invertible for large λ, provided $\gamma > 0$ or $l \neq 0$. Note that in the case $\gamma > 0$ this is a parabolic symbol of order $3/2$ in time t and of order 3 in the space variables x. The term λl^2 is of lower order, thus l does not affect well-posedness. On the other hand, for $l = 0$ and $\gamma = 0$ the boundary symbol is ill-posed, since it admits the zeros $(\lambda, 0)$ with arbitrarily large $\operatorname{Re} \lambda$. If $\gamma = 0$ and $l \neq 0$, then it is well-posed. Note that in this case we have order 1 in time, 3 in space, but also the mixed regularity $1/2$ in time and 2 in space.

(iii) *Reduction to zero initial values.*

It is convenient to reduce the problem to zero initial data and inhomogeneities with vanishing time trace. This can be achieved as follows. We solve the linear problem (3.4) with initial data v_0, ρ_0 and inhomogenities

$$f = 0, \quad g(t) = e^{\Delta_\Sigma t} G(v_0, \rho_0), \quad h(t) = e^{\Delta_\Sigma t} \rho_1 \quad \text{with } \rho_1 = H(v_0, \rho_0).$$

Since the Laplace–Beltrami operator Δ_Σ has maximal L_p-regularity, the fact that $G(v_0, \rho_0) \in W_p^{2-3/p}(\Sigma)$ implies $g \in \mathbb{F}_2(J)$. Similarly, $h \in \mathbb{F}_3(J)$ since $\rho_1 \in W_p^{1-3/p}(\Sigma)$. The compatibility conditions yield $[\![d_0\partial_\nu v_0]\!] + \rho_1 \in W_p^{2-6/p}(\Sigma)$. Therefore, the linear problem has a unique solution $z_* := (v_*, \rho_*)$ with maximal regularity $z_* \in \mathbb{E}(J)$. Then we set $\bar{v} = v - v_*$, $\bar{\rho} = \rho - \rho_*$, and obtain the following problem for $\bar{z} = (\bar{v}, \bar{\rho})$.

$$\begin{cases} \kappa_0(x)\partial_t\bar{v} - d_0(x)\Delta\bar{v} = F(\bar{v} + v_*, \bar{\rho} + \rho_*) & \text{in} \quad \Omega\setminus\Sigma \\ \partial_{\nu_\Omega}\bar{v} = 0 & \text{on} \quad \partial\Omega \\ [\![\bar{v}]\!] = 0 & \text{on} \quad \Sigma \\ l_1(t,x)\bar{v} + \sigma_0\Delta_\Sigma\bar{\rho} - \gamma_1(t,x)\partial_t\bar{\rho} = \bar{G}(\bar{v}, \bar{\rho}; v_*, \rho_*) & \text{on} \quad \Sigma \\ l_0(x)\partial_t\bar{\rho} - [\![d_0(x)\partial_\nu\bar{v}]\!] = \bar{H}(\bar{v}, \bar{\rho}; v_*, \rho_*) & \text{on} \quad \Sigma \\ \bar{v}(0) = 0, \ \bar{\rho}(0) = 0. \end{cases} \tag{3.5}$$

Here we have set

$$\begin{aligned} \bar{G}(\bar{v}, \bar{\rho}; v_*, \rho_*) &= G(\bar{v} + v_*, \bar{\rho} + \rho_*) - \mathrm{e}^{\Delta_\Sigma t}G(v_0, \rho_0), \\ \bar{H}(\bar{v}, \bar{\rho}; v_*, \rho_*) &= H(\bar{v} + v_*, \bar{\rho} + \rho_*) - \mathrm{e}^{\Delta_\Sigma t}H(v_0, \rho_0). \end{aligned}$$

Note that $\bar{G}(0, 0; v_0, \rho_0) = \bar{H}(0, 0; v_0, \rho_0) = 0$ by construction, which ensures time trace zero at $t = 0$.

(iv) *Solution of the nonlinear problem.*

We first concentrate on the case $\gamma \equiv 0$, and rewrite problem (3.5) in abstract form as

$$\mathbb{L}\bar{z} = \mathbb{N}(\bar{z}, z_*),$$

where $\mathbb{L} : {}_0\mathbb{E}(0, t_0) \to {}_0\mathbb{F}(0, t_0)$, defined by

$$\mathbb{L}\bar{z} = \big(\kappa_0\partial_t\bar{v} - d_0\Delta\bar{v}, l_1\bar{v} + \sigma_0\Delta_\Sigma\bar{\rho}, l_0\partial_t\bar{\rho} - [\![d_0\partial_\nu\bar{v}]\!]\big),$$

is an isomorphism by Theorem 3.3. The nonlinearity

$$\mathbb{N} : {}_0\mathbb{E}(0, t_0) \times \mathbb{E}(0, t_0) \to {}_0\mathbb{F}(0, t_0),$$

given by the right-hand side of(3.5), is of class C^1, since the coefficient functions satisfy $\kappa \in C^1$, $d, l \in C^2$, $\psi \in C^3$, and by virtue of the embeddings

$$\mathbb{E}_1(J) \hookrightarrow C(J \times \bar{\Omega}) \cap C(J; C^1(\bar{\Omega}_j)), \ \mathbb{E}_2(J) \hookrightarrow C(J; C^3(\Sigma)) \cap C^1(J; C(\Sigma)).$$

Observe that the constants in these embeddings blow up as $t_0 \to 0$; however, they are uniform in t_0 if one considers the space ${}_0\mathbb{E}(J)$!

We want to apply the contraction mapping principle. For this purpose we consider a closed ball $\mathbb{B}_R(0) \subset {}_0\mathbb{E}(0, \tau)$, where the radius $R > 0$ and the final time $\tau \in (0, t_0]$ are at our disposal. We rewrite the abstract equation $\mathbb{L}\bar{z} = \mathbb{N}(\bar{z}, z_*)$ as the fixed point equation

$$\bar{z} = \mathbb{L}^{-1}\mathbb{N}(\bar{z}, z_*) =: \mathbb{T}(\bar{z}), \quad \bar{z} \in \mathbb{B}_R(0).$$

Since we are working in an L_p-setting, by choosing $\tau = \tau(R)$ small enough we can assure that

$$\|\mathbb{T}(0)\|_{\mathbb{E}(0,\tau)} = \|\mathbb{L}^{-1}\mathbb{N}(0, z_*)\|_{\mathbb{E}(0,\tau)} \leqq R/2.$$

On the other hand, we have

$$\|\mathbb{T}(z_1) - \mathbb{T}(z_2)\|_{\mathbb{E}(0,\tau)} \leqq \|\mathbb{L}^{-1}\|_{\mathcal{B}({}_0\mathbb{F}(0,\tau),{}_0\mathbb{E}(0,\tau))} \times \sup_{\|\bar{z}\|_{{}_0\mathbb{E}(0,\tau)} \leqq R} \|\mathbb{N}'(\bar{z}, z_*)\|_{\mathcal{B}({}_0\mathbb{E}(0,\tau),{}_0\mathbb{F}(0,\tau))} \|z_1 - z_2\|_{\mathbb{E}(0,\tau)},$$

hence $\mathbb{T}(\mathbb{B}_R(0)) \subset \mathbb{B}_R(0)$ and $\mathbb{T}$ is a strict contraction, provided we have

$$\|\mathbb{L}^{-1}\|_{\mathcal{B}({}_0\mathbb{F}(0,\tau),{}_0\mathbb{E}(0,\tau))} \sup_{\|\bar{z}\|_{{}_0\mathbb{E}(0,\tau)} \leqq R} \|\mathbb{N}'(\bar{z}, z_*)\|_{\mathcal{B}({}_0\mathbb{E}(0,\tau),{}_0\mathbb{F}(0,\tau))} \leqq 1/2.$$

For this we observe that

$$\|\mathbb{L}^{-1}\|_{\mathcal{B}({}_0\mathbb{F}(0,\tau),{}_0\mathbb{E}(0,\tau))} \leqq \|\mathbb{L}^{-1}\|_{\mathcal{B}({}_0\mathbb{F}(0,t_0),{}_0\mathbb{E}(0,t_0))} =: C_M < \infty$$

is uniform in $\tau \in (0, t_0)$, since we have vanishing time traces at $t = 0$. So it remains to estimate the Frechét-derivative of $\mathbb{N}$ on the ball $\mathbb{B}_R(0) \subset {}_0\mathbb{E}(0, \tau)$. This is the content of the next proposition, which also covers the case $\gamma > 0$.

Proposition 3.7. *Let $p > n + 2, \sigma \in \mathbb{R}$, and suppose $\psi, \gamma \in C^3(0, \infty)$ and $d \in C^2(0, \infty)$.*

Then $\mathbb{N} : {}_0\mathbb{E}(0, t_0) \times \mathbb{E}(0, t_0) \to {}_0\mathbb{F}(0, t_0)$ is continuously Fréchet-differentiable. There is $\eta > 0$ such that for a given $z_* \in \mathbb{E}(0, t_0)$ with $|\rho_0|_{C^2(\Sigma)} \leqq \eta$, there are continuous functions $\alpha(R) > 0$ and $\beta(\tau) > 0$ with $\alpha(0) = \beta(0) = 0$, such that

$$\|\mathbb{N}'(\bar{z} + z_*)\|_{\mathcal{B}({}_0\mathbb{E}(0,\tau),{}_0\mathbb{F}(0,\tau))} \leqq \alpha(R) + \beta(\tau), \quad \bar{z} \in \mathbb{B}_R \subset {}_0\mathbb{E}(0, \tau).$$

Proof. We may proceed in a way similar to [28, Section 7], where the interface is a graph over $\mathbb{R}^{n-1}$. The additional terms which arise by considering a general geometry are either of lower order or of the form $\tilde{M}(\bar{v}, \bar{\rho})\nabla_\Sigma\bar{\rho}$, where $\tilde{M}(\bar{v}, \bar{\rho})$ is of highest order (see (3.3)), but can be controlled by ensuring that $\nabla_\Sigma\bar{\rho}$ is sufficiently small. The additional terms due to the presence of γ are of highest order, but small. □

Thus, choosing first $R > 0$ and then $\tau > 0$ small enough, $\mathbb{T}$ will be a self-map and a strict contraction on $\mathbb{B}_R(0)$. Concluding, the contraction mapping principle yields a unique fixed point $\bar{z} = \bar{z}(z_*) \in \mathbb{B}_R(0) \subset {}_0\mathbb{E}(0, \tau)$, hence $z = z_* + \bar{z}(z_*)$ is the unique solution of (3.2), that is, of (2.1).

The proof in case $\gamma > 0$ is similar, employing now Theorem 3.5.

Remark 3.8. The assumption $p > n + 2$ simplifies many arguments, since $\mathbb{F}_2(J)$ as well as $\mathbb{F}_3(J)$ are Banach algebras and $\nabla v \in BC(J \times \Omega)$. If we merely assume $p > (n + 2)/2$, then $\mathbb{F}_2(J)$ is still a Banach algebra, but $\mathbb{F}_3(J)$ is not, and ∇v may not be bounded anymore. This leads to much more involved estimates for the nonlinearities.

3.1. Local Semiflows

We denote by $\mathcal{MH}^2(\Omega)$ the closed C^2-hypersurfaces contained in Ω. It can be shown that $\mathcal{MH}^2(\Omega)$ is a C^2-manifold: the charts are the parameterizations over a given hypersurface Σ according to Section 2, and the tangent space consists of the normal vector fields on Σ. We define a metric on $\mathcal{MH}^2(\Omega)$ by means of

$$d_{\mathcal{MH}^2}(\Sigma_1, \Sigma_2) := d_{\mathrm{H}}(\mathcal{N}^2\Sigma_1, \mathcal{N}^2\Sigma_2),$$

where d_{H} denotes the Hausdorff metric on the compact subsets of $\mathbb{R}^n$ introduced in Section 2. In this way $\mathcal{MH}^2(\Omega)$ becomes a Banach manifold of class C^2.

Let $d_\Sigma(x)$ denote the signed distance for Σ as in Section 2. We may then define the *level function* φ_Σ by means of

$$\varphi_\Sigma(x) = \phi(d_\Sigma(x)), \quad x \in \mathbb{R}^n,$$

where

$$\phi(s) = (1 - \chi(s/a)) \operatorname{sgn} s + s\chi(s/a), \quad s \in \mathbb{R}.$$

Then it is easy to see that $\Sigma = \varphi_\Sigma^{-1}(0)$, and $\nabla\varphi_\Sigma(x) = \nu_\Sigma(x)$, for $x \in \Sigma$. Moreover, 0 is an eigenvalue of $\nabla^2\varphi_\Sigma(x)$, and the remaining eigenvalues of $\nabla^2\varphi_\Sigma(x)$ are the principal curvatures of Σ at $x \in \Sigma$.

If we consider the subset $\mathcal{MH}^2(\Omega, r)$ of $\mathcal{MH}^2(\Omega)$, which consists of all closed hypersurfaces $\Gamma \in \mathcal{MH}^2(\Omega)$ such that $\Gamma \subset \Omega$ satisfies a (interior and exterior) ball condition with fixed radius $r > 0$, then the map

$$\Upsilon : \mathcal{MH}^2(\Omega, r) \to C^2(\bar{\Omega}), \quad \Upsilon(\Gamma) := \varphi_\Gamma, \tag{3.6}$$

is an isomorphism of the metric space $\mathcal{MH}^2(\Omega, r)$ onto $\Upsilon(\mathcal{MH}^2(\Omega, r)) \subset C^2(\bar{\Omega})$.

Let $s - (n-1)/p > 2$. Then we define

$$W_p^s(\Omega, r) := \{\Gamma \in \mathcal{MH}^2(\Omega, r) : \varphi_\Gamma \in W_p^s(\Omega)\}. \tag{3.7}$$

In this case the local charts for Γ can be chosen of class W_p^s, as well. A subset $A \subset W_p^s(\Omega, r)$ is said to be (relatively) compact if $\Upsilon(A) \subset W_p^s(\Omega)$ is (relatively) compact.

As an ambient space for the state manifold $\mathcal{SM}_\gamma$ of the Stefan problem with surface tension, we consider the product space $C(\bar{G}) \times \mathcal{MH}^2$, due to continuity of temperature and curvature.

We define the state manifolds $\mathcal{SM}_\gamma, \gamma \geqq 0$, for the Stefan problem (1.3) as follows. For $\gamma = 0$ we set

$$\begin{aligned}\mathcal{SM}_0 := \ &\{(u, \Gamma) \in C(\bar{\Omega}) \times \mathcal{MH}^2 : u \in W_p^{2-2/p}(\Omega \setminus \Gamma),\ \Gamma \in W_p^{4-3/p},\\ &u > 0 \text{ in } \bar{\Omega},\ [\![\psi(u)]\!] + \sigma\mathcal{H} = 0,\ l(u) \neq 0 \text{ on } \Gamma,\ [\![d\partial_\nu u]\!] \in W_p^{2-6/p}(\Gamma)\},\end{aligned} \tag{3.8}$$

and for $\gamma > 0$

$$\begin{aligned}\mathcal{SM}_\gamma := \{(u,\Gamma) \in C(\bar{\Omega}) \times \mathcal{MH}^2 : u \in W_p^{2-2/p}(\Omega \setminus \Gamma),\ \Gamma \in W_p^{4-3/p},\\ u > 0 \text{ in } \bar{\Omega},\ (l(u) - [\![\psi(u)]\!] - \sigma\mathcal{H})([\![\psi(u)]\!] + \sigma\mathcal{H}) = \gamma(u)[\![d\partial_\nu u]\!] \text{ on } \Gamma\}.\end{aligned} \tag{3.9}$$

Charts for these manifolds are obtained by the charts induced by $\mathcal{MH}^2(\Omega)$, followed by a Hanzawa transformation.

Applying Theorem 3.1 or Theorem 3.2, respectively, and re-parametrizing the interface repeatedly, we see that (1.3) yields a local semiflow on $\mathcal{SM}_\gamma$.

Theorem 3.9. *Let $p > n+2, \sigma > 0$ and $\gamma \geqq 0$. Then problem (1.3) generates a local semiflow on the state manifold $\mathcal{SM}_\gamma$. Each solution (u,Γ) exists on a maximal time interval $[0,t_*)$, where $t_* = t_*(u_0,\Gamma_0)$.*

Time weights. For later use we need an extension of the local existence results to spaces with time weights. In particular, we need this extension for a compactness argument in the proof of Theorem 5.3. Given a UMD-Banach space Y and $\mu \in (1/p, 1]$, we define, for $J = (0,t_0)$,

$$K^s_{p,\mu}(J;Y) := \{u \in L_{p,loc}(J;Y) :\ t^{1-\mu}u \in K^s_p(J;Y)\},$$

where $s \geqq 0$ and $K \in \{H,W\}$. It has been shown in [78] that the operator d/dt in $L_{p,\mu}(J;Y)$ with domain

$$D(d/dt) = {}_0H^1_{p,\mu}(J;Y) = \{u \in H^1_{p,\mu}(J;Y) :\ u(0) = 0\}$$

is sectorial and admits an H^∞-calculus with angle $\pi/2$. However, it does not generate a C_0-semigroup unless $\mu = 1$. This is the main tool for extending the results for the linear problem, that is, Theorems 3.3 and 3.5, to the time weighted setting, where the solution space $\mathbb{E}(J)$ is replaced by

$$\mathbb{E}_\mu(J) = \mathbb{E}_{\mu,1}(J) \times \mathbb{E}_{\mu,2}(J),$$

with

$$\begin{aligned}&\mathbb{E}_{\mu,1}(J) = \{v \in H^1_{p,\mu}(J;L_p(\Omega)) \cap L_{p,\mu}(J;H^2_p(\Omega\setminus\Sigma)) : [\![v]\!] = 0,\ \partial_{\nu_\Omega} v = 0\},\\ &\mathbb{E}_{\mu,2}(J) := W^{3/2-1/2p}_{p,\mu}(J;L_p(\Sigma)) \cap W^{1-1/2p}_{p,\mu}(J;H^2_p(\Sigma)) \cap L_{p,\mu}(J;W^{4-1/p}_p(\Sigma)),\\ &\quad \gamma \equiv 0,\\ &\mathbb{E}_{\mu,2}(J) := W^{2-1/2p}_{p,\mu}(J;L_p(\Sigma)) \cap L_{p,\mu}(J;W^{4-1/p}_p(\Sigma)),\ \gamma > 0.\end{aligned}$$

In a similar way, the space of data is defined by

$$\begin{aligned}\mathbb{F}_{\mu,1}(J) &:= L_{p,\mu}(J;L_p(\Omega)),\\ \mathbb{F}_{\mu,2}(J) &:= W^{1-1/2p}_{p,\mu}(J;L_p(\Sigma)) \cap L_{p,\mu}(J;W^{2-1/p}_p(\Sigma)),\\ \mathbb{F}_{\mu,3}(J) &:= W^{1/2-1/2p}_{p,\mu}(J;L_p(\Sigma)) \cap L_{p,\mu}(J;W^{1-1/p}_p(\Sigma)),\\ \mathbb{F}_\mu(J) &:= \mathbb{F}_{\mu,1}(J) \times \mathbb{F}_{\mu,2}(J) \times \mathbb{F}_{\mu,3}(J).\end{aligned}$$

The trace spaces for v and ρ for $p > 3$ are then given by

$$v_0 \in W_p^{2\mu-2/p}(\Omega \setminus \Sigma), \quad \rho_0 \in W_p^{2+2\mu-3/p}(\Sigma), \quad \rho_1 \in W_p^{4\mu-2-6/p}(\Sigma), \tag{3.10}$$

where for the last trace - which is of relevance only in case $\gamma \equiv 0$—we need, in addition, $\mu > 1/2 + 3/2p$. Note that the embeddings

$$\mathbb{E}_{\mu,1}(J) \hookrightarrow C(J \times \bar{\Omega}) \cap C(J; C^1(\bar{\Omega}_j)), \quad \mathbb{E}_{\mu,2}(J) \hookrightarrow C(J; C^3(\Sigma))$$

require $\mu > 1/2+(n+2)/2p$, which is feasible since $p > n+2$ by assumption. This restriction is needed for the estimation of the nonlinearities, that is, Proposition 3.7 remains valid for $\mu \in (1/2 + (n+2)/2p, 1)$.

The assertions for the linear problem remain valid for this μ, replacing $\mathbb{E}(J)$ by $\mathbb{E}_\mu(J)$, $\mathbb{F}(J)$ by $\mathbb{F}_\mu(J)$, for initial data subject to (3.10). This relies on the fact mentioned above that $d/\mathrm{d}t$ admits a bounded H^∞-calculus with angle $\pi/2$ in the spaces $L_{p,\mu}(J; Y)$. Therefore the main results in Denk et al. [21] remain valid for $\mu \in (1/p, 1)$. This has recently been established in [60,61]. As a consequence of these considerations we have the following result.

Corollary 3.10. *Let* $p > n+2$, $\mu \in (1/2 + (n+2)/2p, 1]$, $\sigma > 0$*, and suppose that* $\psi, \gamma \in C^3(0,\infty)$, $d \in C^2(0,\infty)$ *such that* $\gamma \equiv 0$ *or* $\gamma(u) > 0$, $u \in (0,\infty)$*, and*

$$\kappa(u) = -u\psi''(u) > 0, \quad \mathrm{d}(u) > 0, \quad u \in (0,\infty).$$

Assume the regularity conditions

$$u_0 \in W_p^{2\mu-2/p}(\Omega \setminus \Gamma_0) \cap C(\bar{\Omega}), \quad u_0 > 0, \quad \Gamma_0 \in W_p^{2+2\mu-3/p},$$

and the compatibility conditions $\partial_{\nu_\Omega} u_0 = 0$ *and*

(a) $[\![\psi(u_0)]\!] + \sigma\mathcal{H}(\Gamma_0) = 0$, $[\![d(u_0)\partial_\nu u_0]\!] \in W_p^{4\mu-2-6/p}(\Gamma_0)$, *as well as the* well-posedness condition $l(u_0) \neq 0$ *on* Γ_0*, in case* $\gamma \equiv 0$.

(b) $\big([\![\psi(u_0)]\!] + \sigma\mathcal{H}(\Gamma_0)\big)\big(l(u_0) - [\![\psi(u_0)]\!] - \sigma\mathcal{H}(\Gamma_0)\big) = \gamma(u_0)[\![d(u_0)\partial_\nu u_0]\!]$ *in case* $\gamma > 0$,

Then the transformed problem (2.1) admits a unique solution $z = (v,\rho) \in \mathbb{E}_\mu(0,\tau)$ *for some nontrivial time interval* $J = [0,\tau]$*. The solution depends continuously on the data. For each* $\delta > 0$ *the solution belongs to* $\mathbb{E}(\delta,\tau)$*, that is, it regularizes instantly.*

4. Equilibria

Suppose (u_*, Γ_*) is an equilibrium for (1.3). Then $\partial_t u_* \equiv 0$ as well as $V_* \equiv 0$, and we obtain

$$\begin{cases} \mathrm{div}(d(u_*)\nabla u_*) = 0 & \text{in} \quad \Omega \setminus \Gamma_* \\ \partial_{\nu_\Omega} u_* = 0 & \text{on} \quad \partial\Omega \\ [\![u_*]\!] = 0 & \text{on} \quad \Gamma_* \\ [\![\psi(u_*)]\!] + \sigma\mathcal{H}(\Gamma_*) = 0 & \text{on} \quad \Gamma_* \\ [\![d(u_*)\partial_\nu u_*]\!] = 0 & \text{on} \quad \Gamma_*. \end{cases} \tag{4.1}$$

This yields $u_* = \text{const}$, hence $\mathcal{H}(\Gamma_*) = -[\![\psi(u_*)]\!]/\sigma$ is constant, as well. If Γ_* is connected (and Ω is bounded), this implies that Γ_* is a sphere $S_{R_*}(x_0)$ with radius $R_* = \sigma/[\![\psi(u_*)]\!]$. Thus there is an $(n+1)$-parameter family of equilibria

$$\mathcal{E} := \{(u_*, S_{R_*}(x_0)) : u_* > 0,\ 0 < R_* = \sigma/[\![\psi(u_*)]\!],\ \bar{B}_{R_*}(x_0) \subset \Omega\}.$$

Otherwise, Γ_* is the union of finitely many, say m, nonintersecting spheres of equal radius. It will be shown in the proof of Theorem 4.5(vii) that $\mathcal{E}$ is a C^1-manifold of dimension $(mn+1)$ in $W_p^2(\Omega \setminus \Gamma_*) \times W_p^{4-1/p}(\Gamma_*)$.

4.1. Conservation of Energy

As we have just seen, the equilibria of (1.3) are constant temperature, and the dispersed phase consists of finitely many non-intersecting balls with the same radius. To determine u and R, taking into account conservation of energy, we have to solve the system

$$\begin{aligned} \mathsf{E}(u, R) &:= |\Omega_1|\varepsilon_1(u) + |\Omega_2|\varepsilon_2(u) + \frac{\sigma}{n-1}|\Sigma| = \mathsf{E}_0, \\ &[\![\psi(u)]\!] + \sigma\mathcal{H} = 0. \end{aligned}$$

In order not to overburden the notation, we use (u, R) instead of (u_*, R_*). The constant E_0 means the initial total energy in the system. Since $\mathcal{H} = -\sigma/R$ we may eliminate R by the second equation $R = \sigma/[\![\psi(u)]\!]$, and we are left with a single equation for the temperature u:

$$\varphi(u) \coloneqq \mathsf{E}(u, R(u)) = |\Omega|\varepsilon_2(u) - \frac{m\omega_n}{n} R^n(u)[\![\varepsilon(u)]\!] + \frac{\sigma m\omega_n}{n-1} R^{n-1}(u) = \varphi_0, \quad (4.2)$$

with $\varphi_0 = \mathsf{E}_0$. Note that only the temperature range $[\![\psi(u)]\!] > 0$ is relevant due to the requirement $R > 0$, and with

$$R_{\mathrm{m}} = \sup\{R > 0 : \ \Omega \text{ contains } m \text{ disjoint balls of radius } R\}$$

we must also have $R < R_{\mathrm{m}}$, that is, with $h(u) = [\![\psi(u)]\!]$

$$h(u) > \frac{\sigma}{R_{\mathrm{m}}}.$$

With $\varepsilon(u) = \psi(u) - u\psi'(u)$, that is, $[\![\varepsilon(u)]\!] = h(u) - uh'(u)$, we may rewrite $\varphi(u)$ as

$$\varphi(u) = |\Omega|\varepsilon_2(u) + c_n\Big(\frac{1}{h(u)^{n-1}} + (n-1)u\frac{h'(u)}{h(u)^n}\Big),$$

where we have set $c_n = m\frac{\omega_n}{n(n-1)}\sigma^n$.

Next, with

$$R'(u) = -\frac{\sigma h'(u)}{h^2(u)} = -\frac{h'(u)R^2(u)}{\sigma},$$

we obtain

$$\begin{aligned}
\varphi'(u) &= |\Omega|\varepsilon_2'(u) - [\![\varepsilon'(u)]\!]|\Omega_1| + m\omega_n(\frac{\sigma}{R(u)} - [\![\varepsilon(u)]\!])R^{n-1}(u)R'(u) \\
&= |\Omega|\kappa_2(u) - [\![\kappa(u)]\!]|\Omega_1| + m\omega_n u h'(u)R^{n-1}(u)R'(u) \\
&= (\kappa(u)|1)_{L_2(\Omega)} - uh'(u)|\Sigma|\frac{h'(u)R^2(u)}{\sigma} \\
&= \left\{\frac{\sigma u(\kappa(u)|1)_{L_2(\Omega)}}{l^2(u)R^2(u)|\Sigma|} - 1\right\}\frac{l^2(u)R^2(u)|\Sigma|}{\sigma u},
\end{aligned}$$

with $l(u) = uh'(u)$. It will turn out that in the case of connected phases the term in the parentheses determines whether an equilibrium is stable: it is stable if $\varphi'(u) < 0$ and unstable if $\varphi'(u) > 0$; see Theorem 5.2 below.

In general it is not a simple task to analyze the equation for the temperature,

$$\varphi(u) = |\Omega|\varepsilon_2(u) + c_n\left(\frac{1}{h(u)^{n-1}} + (n-1)u\frac{h'(u)}{h(u)^n}\right) = \varphi_0,$$

unless more properties of the functions $\varepsilon_2(u)$, and in particular of $h(u)$, are known. A natural assumption is that h has exactly one positive zero $u_{\rm m} > 0$, the melting temperature. Therefore we look at two examples.

Example 4.1. Suppose that the heat capacities are identical, that is, $[\![\kappa]\!] \equiv 0$. This implies

$$uh''(u) = u[\![\psi''(u)]\!] = -[\![\kappa(u)]\!] \equiv 0,$$

which means that $h(u) = h_0 + h_1 u$ is linear. The melting temperature, then, is $0 < u_{\rm m} = -h_0/h_1$, hence we have two cases.

Case 1. $h_0 < 0, h_1 > 0$; this means $l(u_{\rm m}) > 0$. In this case, the relevant temperature range is $u > u_{\rm m}$, as h is positive there. We assume now that ε_2 is increasing and convex. As $u \to u_{\rm m}+$ we have $h(u) \to 0$, hence $\varphi(u) \to \infty$, and also $\varphi(u) \to \infty$ for $u \to \infty$ since $\varepsilon_2(u)$ is increasing and convex. Further, we have

$$\begin{aligned}
\varphi'(u) &= |\Omega|\varepsilon_2'(u) - n(n-1)c_n\frac{h_1^2 u}{(h_0+h_1u)^{n+1}}, \\
\varphi''(u) &= |\Omega|\varepsilon_2''(u) + n(n-1)c_n h_1^2\frac{-h_0 + nh_1u}{(h_0+h_1u)^{n+2}} > 0,
\end{aligned}$$

which shows that $\varphi(u)$ is strictly convex for $u > u_{\rm m}$. Thus $\varphi(u)$ has a unique minimum $u_0 > u_{\rm m}$, $\varphi(u)$ is decreasing for $u_{\rm m} < u < u_0$ and increasing for $u > u_0$. Thus there are precisely two equilibrium temperatures $u_*^+ \in (u_0, \infty)$ and $u_*^- \in (u_{\rm m}, u_0)$, provided $\varphi_0 > \varphi(u_0)$, and none if $\varphi_0 < \varphi(u_0)$. The smaller temperature leads to stable equilibria while the larger leads to unstable ones.

Case 2. $h_0 > 0$, $h_1 < 0$; this means $l(u_{\rm m}) < 0$. In this case, the relevant temperature range is $u < u_{\rm m}$, as h is positive there. As $u \to u_{\rm m}-$ we have $h(u) \to 0^+$ hence $\varphi(u) \to -\infty$, and as $u \to 0^+$ we have $\varphi(u) \to \varphi(0) = |\Omega|\varepsilon_2(0) + c_n/h_0^{n-1}$, assuming that $\varepsilon_2(0) := \lim_{u\to 0^+} \varepsilon_2(u)$ exists. Further, for u sufficiently close to zero, $\varphi'(u)$ is positive, since $\kappa_2 = \varepsilon_2' > 0$, and $\varphi'(u) \to -\infty$ as $u \to u_{\rm m}-$. Therefore, $\varphi'(u)$ admits at least one zero in $(0, u_{\rm m})$. But there may be more than one unless $\varepsilon_2(u)$ is concave, so let us assume this. Let $u_0 \in (0, u_{\rm m})$ denote the absolute maximum of $\varphi(u)$ in $(0, u_{\rm m})$. Then there is exactly one equilibrium temperature $u_* \in (u_0, u_{\rm m})$ if $\varphi_0 < \varphi(0)$ and it is stable; there are exactly two equilibria $u_*^- \in (0, u_0)$ and $u_*^+ \in (u_0, u_{\rm m})$ if $\varphi(0) < \varphi_0 < \varphi(u_0)$, the first one is unstable, the second is stable. If $\varphi_0 > \varphi(u_0)$, there are no equilibria.

Note that in both cases these equilibrium temperatures give rise to equilibria only if the corresponding radius $R(u)$ is smaller than $R_{\rm m}$.

Example 4.2. Suppose that the internal energies $\varepsilon_i(u)$ are linearly increasing, that is,

$$\varepsilon_i(u) = a_i + \kappa_i u, \quad i = 1, 2,$$

where $\kappa_i > 0$, and now $[\![\kappa]\!] \neq 0$. The identity $\varepsilon_i = \psi_i - u\psi_i'$ then leads to

$$\psi_i(u) = a_i + b_i u - \kappa_i u \ln u, \quad i = 1, 2,$$

where the constants b_i are arbitrary. This yields with $\alpha = [\![a]\!]$, $\beta = [\![b]\!]$ and $\delta = [\![\kappa]\!]$

$$h(u) = \alpha + \beta u - \delta u \ln u.$$

Scaling the temperature by $u = u_0 w$ with $\beta - \delta \ln u_0 = 0$ and scaling h, we may assume $\beta = 0$ and $\delta = \pm 1$. Then we have to investigate the equation $\varphi(w) = \varphi_1$, where

$$\varphi(w) = cw + \{\frac{1}{h^{n-1}(w)} + (n-1)w\frac{h'(w)}{h^n(w)}\}, \quad h(w) = \pm(\alpha + w \ln w),$$

with $c > 0$ and $\alpha, \varphi_1 \in \mathbb{R}$. The requirement of existence of a melting temperature $w_{\rm m} > 0$, that is, a zero of $h(w)$ leads to the restriction $\alpha \leqq 1/e$.

Actually, the requirement that the melting temperature be unique, that is, that h have exactly one positive zero, implies $\alpha < 0$. Indeed, for $\alpha \in (0, 1/e)$ there is a second zero $w_- > 0$ of h, and h is positive in $(0, w_-)$. Equilibrium temperatures in this range would not make sense physically.

Here, also, we have to distinguish between two cases: that of a plus-sign where the relevant temperature range is $w > w_{\rm m}$, and that of a minus-sign where the range is $(0, w_{\rm m})$. Note that h is convex in the first case and concave in the second.

Case 1. For the derivatives, in the first case we get

$$\varphi'(w) = c + (n-1)\left\{\frac{h(w) - nw(h'(w))^2}{h^{n+1}(w)}\right\},$$

$$\varphi''(w) = n(n-1)\frac{h'(w)}{h^{n+2}(w)}\Big\{(n+1)w(h'(w))^2 - h(w)(3 + h'(w))\Big\}.$$

We have $\varphi(w) \to \infty$ for $w \to \infty$ and for $w \to w_{\rm m}+$, hence $\varphi(w)$ has a global minimum u_0 in $(u_{\rm m}, \infty)$. Further, $\varphi''(w) > 0$ in $(w_{\rm m}, \infty)$, hence the minimum is unique and there are precisely two equilibrium temperatures $w_*^- \in (w_{\rm m}, w_0)$ and $w_*^+ \in (w_0, \infty)$, provided $\varphi_1 > \varphi(w_0)$, the first one is stable, the second unstable.

To prove convexity of φ we write

$$(n+1)w(h'(w))^2 - h(w)(3 + h'(w)) = (n-1)w(h'(w))^2 + f(w),$$

where

$$f(w) = 2w(h'(w))^2 - h(w)(3 + h'(w)) = 2w(1 + \ln w)^2 - (\alpha + w \ln w)(4 + \ln w).$$

We then have $f(w_{\rm m}) = 2w_{\rm m}(1 + \ln w_{\rm m})^2 > 0$, and

$$f'(w) = (1 + \ln w)^2 + 1 - \alpha/w > 1 - \alpha/w \geqq 0,$$

for $\alpha \leqq 1/e < w_{\rm m} \leqq w$. Let us illustrate the sign in h with the water-ice system, ignoring the density jump of water at freezing temperature. Suppose that Ω_2 consists of ice and Ω_1 of water. In this case we have $\kappa_1 > \kappa_2$, hence $\delta < 0$, which implies the plus-sign for h. Here we obtain $w_*^\pm > w_{\rm m}$, that is, the ice is overheated. Equilibria exist only if ϕ_1 is large enough, which means that there is enough energy in the system. If the energy in the system is very large then the stable equilibrium temperature w_*^- comes close to the melting temperature $w_{\rm m}$; then $R(w)$ will become large, eventually larger than R^*. This excludes equilibria in Ω; the physical interpretation is that everything will eventually melt. On the other hand, if Ω_1 consists of ice and Ω_2 of water, we have the minus sign, which we want to consider next. Here we expect under-cooling of the water-phase, existence of equilibria only for low values of energy, and if the energy in the system is too small, everything will freeze.

Case 2. Assume the minus-sign for h and let $\alpha < 0$. Then the relevant temperature range is $(0, w_{\rm m})$. Here we have $\varphi(w) \to -\infty$ as $w \to w_{\rm m}-$ and $\varphi(w) \to 1/|\alpha|^{n-1} > 0$ as $w \to 0^+$.

To investigate concavity of φ in the interval $(0, w_{\rm m})$, we recompute the derivatives of φ:

$$\varphi'(w) = c - (n-1)\left\{\frac{1}{h^n(w)} + n\frac{w(h'(w))^2}{h^{n+1}(w)}\right\},$$
$$\varphi''(w) = n(n-1)\frac{h'(w)}{h^{n+2}(w)}\left\{(n+1)w(h'(w))^2 + h(w)(3 - h'(w))\right\}.$$

Setting $w_+ = 1/e$, for $w \in (w_+, w_{\rm m})$ we have $h(w) > 0$ and $h'(w) < 0$, hence $\varphi''(w) < 0$. On the other hand, for $w \in (0, w_+)$, both $h(w)$ and $h'(w)$ are positive. Then we rewrite

$$(n+1)w(h'(w))^2 + 3h(w) - h(w)h'(w) = (n-1)w(1 + \ln w)^2 + f(w),$$

where

$$\begin{aligned} f(w) &= 2w(h'(w))^2 + h(w)(3 - h'(w)) \\ &= 2w(1 + \ln w)^2 - (\alpha + w \ln w)(4 + \ln w), \\ f'(w) &= (1 + \ln w)^2 + 1 - \alpha/w > 0, \end{aligned}$$

provided $\alpha \leqq 0$. This shows that f is increasing, $f(w) \to -\infty$ as $w \to 0^+$, and $f(1/e^3) = 11/e^3 - \alpha > 0$. On the other hand, the function $w(1+\ln w)^2$ is increasing in $(0, 1/e^3)$, hence $\varphi''(w)$ has a unique zero $w_- \in (0, 1/e^3)$. Therefore, φ is concave in $(0, w_-) \cup (w_+, w_{\mathrm{m}})$ and convex in (w_-, w_+), and φ' has a minimum at w_- and a maximum at w_+. Observe that $\varphi'(w) < c$, $\varphi'(w) \to -\infty$ for $w \to w_{\mathrm{m}}-$ and $\varphi'(0) = c - (n-1)/|\alpha|^n < \varphi'(w_+)$. Therefore, φ' may have no, one, two, or three zeros in $(0, w_{\mathrm{m}})$, depending on the value of $c > 0$. However, if $c > 0$ is large enough, then φ' has only one zero, w_1, which lies in (w_+, w_{m}). In this case φ is increasing in $(0, w_1)$ and decreasing in (w_1, w_{m}), hence for $\varphi_1 \in (\varphi(0), \varphi(w_1))$ there are precisely two equilibrium temperatures: the smaller leads to unstable, the larger to a stable equilibrium. If $\varphi_1 < \varphi(0)$, there is a unique equilibrium which is stable, and in the case where $\varphi_1 > \varphi(w_1)$ there is none. However, in general, there may be up to four equilibrium temperatures.

4.2. Linearization at Equilibria

The linearization at an equilibrium (u_*, Γ_*) with $R_* = \sigma/[\![\psi(u_*)]\!]$, reads

$$\begin{cases} \kappa_* \partial_t v - d_* \Delta v = f & \text{in} \quad \Omega \setminus \Gamma_* \\ \partial_{\nu_\Omega} v = 0 & \text{on} \quad \partial\Omega \\ [\![v]\!] = 0 & \text{on} \quad \Gamma_* \\ (l_*/u_*)v + \sigma A_* \rho - \gamma_* \partial_t \rho = g & \text{on} \quad \Gamma_* \\ l_* \partial_t \rho - [\![d_* \partial_\nu v]\!] = h & \text{in} \quad \Gamma_* \\ v(0) = v_0, \ \rho(0) = \rho_0. \end{cases} \tag{4.3}$$

Here

$$\kappa_* = \kappa(u_*), d_* = d(u_*), l_* = l(u_*), \gamma_* = \gamma(u_*), \quad A_* = \frac{1}{n-1}\left(\frac{n-1}{R_*^2} + \Delta_*\right),$$

where Δ_* denotes the *Laplace–Beltrami* operator on Γ_*.

We note that if $l_* = 0$ and $\gamma_* = 0$, then the problem is not well-posed. On the other hand, if $l_* \neq 0$ and $\gamma_* = 0$, then the operator $-L_0$ defined by

$$\begin{aligned} D(L_0) = \big\{&(v, \rho) \in [H_p^2(\Omega \setminus \Gamma_*) \cap C(\bar\Omega)] \times W_p^{4-1/p}(\Gamma_*) : \\ &\partial_{\nu_\Omega} v = 0, \ (l_*/u_*)v + \sigma A_* \rho = 0, \ [\![d_* \partial_\nu v]\!] \in W_p^{2-2/p}(\Gamma_*)\big\}, \\ L_0(u, \rho) = &((-d_*/\kappa_*)\Delta v, -[\![(d_*/l_*)\partial_\nu v]\!]), \end{aligned} \tag{4.4}$$

generates an analytic C_0-semigroup with maximal regularity in

$$X_0 := L_p(\Omega) \times W_p^{2-2/p}(\Gamma_*).$$

More precisely, we have the following result.

Theorem 4.3. *Let $3 < p < \infty, \sigma > 0$, suppose $\gamma_* = 0$ and let $l_* \neq 0$. Then for each finite interval $J = [0, t_0]$, there is a unique solution $z = (v, \rho) \in \mathbb{E}(J)$ of (4.3) if and only if the data (f, g, h) and $z_0 = (v_0, \rho_0)$ satisfy*

$$(f, g, h) \in \mathbb{F}(J), \quad z_0 \in [W_p^{2-2/p}(\Omega \setminus \Gamma_*) \cap C(\bar{\Omega})] \times W_p^{4-3/p}(\Gamma_*)$$

and the compatibility conditions

$$\partial_{\nu_\Omega} v_0 = 0, \quad (l_*/u_*)v_0 + \sigma A_* \rho_0 = g(0), \quad h(0) + [\![d_* \partial_\nu v_0]\!] \in W_p^{2-6/p}(\Gamma_*).$$

The operator $-L_0$ defined above generates an analytic C_0-semigroup in X_0 with maximal regularity of type L_p.

In case $\gamma_* > 0$, similar assertions are valid for L_γ in

$$X_\gamma := L_p(\Omega) \times W_p^{2-1/p}(\Gamma_*),$$

where

$$\begin{aligned} D(L_\gamma) = \big\{(v, \rho) \in [H_p^2(\Omega \setminus \Gamma_*) \cap C(\bar{\Omega})] \times W_p^{4-1/p}(\Gamma_*) :\\ \partial_{\nu_\Omega} v = 0,\ (l_*^2/u_*)v + l_* \sigma A_* \rho = \gamma_* [\![d_* \partial_\nu v]\!]\big\},\\ L_\gamma(v, \rho) = ((-d_*/\kappa_*)\Delta v, -(\sigma/\gamma_*)A_* \rho - (l_*/u_* \gamma_*)v). \end{aligned} \tag{4.5}$$

The main result on the problem (4.3) for $\gamma_* > 0$ is the following.

Theorem 4.4. *Let $3 < p < \infty$, and suppose $\sigma, \gamma_* > 0$. Then for each finite interval $J = [0, t_0]$, there is a unique solution $z = (v, \rho) \in \mathbb{E}(J)$ of (4.3) if and only if the data (f, g, h) and $z_0 = (v_0, \rho_0)$ satisfy*

$$(f, g, h) \in \mathbb{F}(J), \quad z_0 \in [W_p^{2-2/p}(\Omega \setminus \Gamma_*) \cap C(\bar{\Omega})] \times W_p^{4-3/p}(\Gamma_*)$$

and the compatibility condition

$$\partial_{\nu_\Omega} v_0 = 0, \quad (l_*^2/u_*)v_0 + l_* \sigma A_* \rho_0 - \gamma_* [\![\mathrm{d} \partial_\nu v_0]\!] = l_* g(0) + \gamma_* h(0).$$

The operator $-L_\gamma$ defined above generates an analytic C_0-semigroup in X_γ with maximal regularity of type L_p.

Proof. (Proof of Theorem 4.3 and Theorem 4.4) These results, up to the last assertions, are special cases of Theorems 3.3 and 3.5, respectively, in Section 3. In addition, since the Cauchy problem for L_γ has maximal L_p-regularity, we conclude in both cases by [74, Proposition 1.2] that $-L_\gamma$ generates an analytic C_0-semigroup in X_γ. Recall that the spaces $\mathbb{E}(J)$ are different for $\gamma = 0$ and $\gamma > 0$. □

4.3. The Eigenvalue Problem

By compact embedding, the spectrum of L_γ consists only of countably many discrete eigenvalues of finite multiplicity and is independent of p. The eigenvalue problem reads as follows

$$\begin{cases} \kappa_*\lambda v - d_*\Delta v = 0 & \text{in} \quad \Omega\setminus\Gamma_* \\ \partial_{\nu_\Omega} v = 0 & \text{on} \quad \partial\Omega \\ [\![v]\!] = 0 & \text{on} \quad \Gamma_* \\ (l_*/u_*)v + \sigma A_*\rho - \gamma_*\lambda\rho = 0 & \text{on} \quad \Gamma_* \\ l_*\lambda\rho - [\![d_*\partial_\nu v]\!] = 0 & \text{on} \quad \Gamma_*. \end{cases} \tag{4.6}$$

Assume, first, that Γ_* is connected. As shown in [79], $\lambda = 0$ is always an eigenvalue, and $N(L_\gamma)$ is independent of $\gamma_* \geqq 0$, κ_* and d_*. We have

$$N(L_\gamma) = \operatorname{span}\left\{\left(\frac{\sigma u_*}{l_* R_*^2}, -1\right), (0, Y_1), \dots, (0, Y_n)\right\}, \tag{4.7}$$

where the functions Y_j denote the *spherical harmonics of degree one*, normalized by $(Y_j|Y_k)_{L_2(\Gamma_*)} = \delta_{jk}$. $N(L_\gamma)$ is isomorphic to the tangent space of $\mathcal{E}$ at $(u_*, \Gamma_*) \in \mathcal{E}$.

Let $\lambda \neq 0$ be an eigenvalue with eigenfunction $(v, \rho) \neq 0$. Then (4.6) yields

$$\lambda\left\{\left|\sqrt{\kappa_*}v\right|^2_{L_2(\Omega)} - \sigma u_*(A_*\rho|\rho)_{L_2(\Gamma_*)}\right\} + \left|\sqrt{d_*}\nabla v\right|^2_{L_2(\Omega)} + \gamma_* u_*|\lambda|^2|\rho|^2_{L_2(\Gamma_*)} = 0.$$

Since A_* is selfadjoint in $L_2(\Gamma_*)$, this identity shows that all eigenvalues of L_γ are real. Decomposing $v = v_0 + \bar{v}$, $\rho = \rho_0 + \bar{\rho}$, with $(\kappa_*|v_0)_{L_2(\Omega)} = (\rho_0|1)_{L_2(\Gamma_*)} = 0$, this identity can be rewritten as

$$\lambda\left\{\left|\sqrt{\kappa_*}v_0\right|^2_{L_2(\Omega)} - \sigma u_*(A_*\rho_0|\rho_0)_{L_2(\Gamma_*)} + \lambda u_*\gamma_*|\rho_0|^2_{L_2(\Gamma_*)}\right\} + \left|\sqrt{d_*}\nabla v_0\right|^2_{L_2(\Omega)}$$
$$+ \left[\lambda\gamma_* u_* + l_*^2|\Gamma_*|/(\kappa_*|1)_{L_2(\Omega)} - \sigma u_*/R_*^2\right]\lambda\bar{\rho}^2|\Gamma_*| = 0.$$

In the case in which Γ_* is connected, the bracket determines whether there is a positive eigenvalue.

If $\Gamma_* = \bigcup_{1\leqq l\leqq m}\Gamma_*^l$ consists of $m > 1$ spheres Γ_*^l of equal radius, then

$$N(L_\gamma) = \operatorname{span}\left\{\left(\frac{\sigma u_*}{l_* R_*^2}, -1\right), (0, Y_1^l), \dots, (0, Y_n^l) : 1 \leqq l \leqq m\right\}, \tag{4.8}$$

where the functions Y_j^l denote the *spherical harmonics of degree one* on Γ_*^l (and $Y_j^l \equiv 0$ on $\bigcup_{i\neq l}\Gamma_*^i$), normalized by $(Y_j^l|Y_k^l)_{L_2(\Gamma_*^l)} = \delta_{jk}$. $N(L_\gamma)$ is isomorphic to the tangent space of $\mathcal{E}$ at $(u_*, \Gamma_*) \in \mathcal{E}$, as will be shown in Theorem 4.5, below.

Theorem 4.5. *Let $\sigma > 0$, $\gamma_* \geqq 0$, $l_* \neq 0$, and assume that the interface Γ_* consists of $m \geqq 1$ components. Let*

$$\zeta_* = \frac{\sigma u_*(\kappa_*|1)_{L_2(\Omega)}}{l_*^2 R_*^2|\Gamma_*|}, \tag{4.9}$$

and let φ be defined as in (4.2). Then

(i) $\varphi'(u_*) = (\zeta_* - 1)l_*^2 R_*^2 |\Gamma_*|/(\sigma u_*)$.
(ii) 0 *is an eigenvalue of* $-L_\gamma$ *with geometric multiplicity* $(mn + 1)$.
(iii) 0 *is semi-simple if* $\zeta_* \neq 1$.
(iv) *If* Γ_* *is connected and* $\zeta_* \leqq 1$, *then all eigenvalues of* $-L_\gamma$ *are negative, except for* 0.
(v) *If* $\zeta_* > 1$, *then there are precisely m positive eigenvalues of* $-L_\gamma$,
(vi) *If* $\zeta_* \leqq 1$ *then* $-L_\gamma$ *has precisely* $m - 1$ *positive eigenvalues.*
(vii) $N(L_\gamma)$ *is isomorphic to the tangent space* $T_{(u_*,\Gamma_*)}\mathcal{E}$ *of* $\mathcal{E}$ *at* $(u_*, \Gamma_*) \in \mathcal{E}$.

Remark 4.6. (a) The result is also true if $l_* = 0$ and $\gamma_* \neq 0$. In this case $\varphi'(u_*) = (\kappa_*|1)_{L_2(\Omega)} > 0$ and $\zeta_* = \infty$, hence the equilibrium is always unstable.
(b) Note that ζ_* depends neither on d, nor on the undercooling coefficient γ.
(c) For the *Mullins–Sekerka problem*, that is, for $\kappa \equiv 0$, we have $\zeta_* \equiv 0$, in accordance with the result obtained in [27].
(d) It is shown in [79] that in case $\zeta_* = 1$ and Γ_* connected, the eigenvalue 0 is no longer semi-simple: its algebraic multiplicity rises by 1. This is also true if Γ_* is disconnected.

Proof. (**Proof of Theorem 4.5**) For the case where Γ_* is connected, this result is proved in [79]. The assertions (i)–(iii) also remain valid in the disconnected case. However, the proof of [79, Theorem 2.1(e)], addressing instability, is not completely correct, as it relies on the assertions [79, Proposition 3.2(b) and Proposition 5.1(c)] which are incorrect. (We remark, though, that the instability result of [79, Theorem 1.3] is, indeed, valid.) Here we give a modified proof for [79, Theorem 2.1(e)] which also applies in the case where Γ_* is not connected.

It thus remains to prove the assertions in (v), (vi), and (vii). If the stability condition $\zeta_* \leqq 1$ does not hold or if Γ_* is disconnected, then there is always a positive eigenvalue. To prove this we proceed as follows. Suppose $\lambda > 0$ is an eigenvalue, and that ρ is known; solve the elliptic transmission problem

$$\begin{cases} \kappa_*\lambda v - d_*\Delta v = 0 & \text{in} \quad \Omega \setminus \Gamma_* \\ \partial_{\nu_\Omega} v = 0 & \text{on} \quad \partial\Omega \\ [\![v]\!] = 0 & \text{on} \quad \Gamma_* \\ -[\![d_*\partial_\nu v]\!] = h & \text{on} \quad \Gamma_* \end{cases} \tag{4.10}$$

to get $v = S_\lambda h$, with S_λ being the solution operator. Then, taking the trace at Γ_* we obtain $v|_{\Gamma_*} = N_\lambda h$, where N_λ denotes the Neumann-to-Dirichlet operator for the transmission problem (4.10). Setting $h = -\lambda l_* \rho$ this implies, with the linearized Gibbs–Thomson law, the equation

$$[(l_*^2/u_*)\lambda N_\lambda + \gamma_*\lambda]\rho - \sigma A_*\rho = 0. \tag{4.11}$$

$\lambda > 0$ is an eigenvalue of $-L_\gamma$ if and only if (4.11) admits a nontrivial solution. We consider this problem in $L_2(\Gamma_*)$. Then A_* is selfadjoint and

$$-\sigma(A_*g|g)_{L_2(\Gamma_*)} \geqq -\frac{\sigma}{R_*^2}|g|^2_{L_2(\Gamma_*)}.$$

On the other hand, we will see below that N_λ is selfadjoint and positive semi-definite on $L_2(\Gamma_*)$. Moreover, since A_* has compact resolvent, the operator

$$B_\lambda := [(l_*^2/u_*)\lambda N_\lambda + \gamma_*\lambda] - \sigma A_*$$

has compact resolvent as well, for each $\lambda > 0$. Therefore the spectrum of B_λ consists only of eigenvalues which, in addition, are real. We intend to prove that in cases where either Γ_* is disconnected or the stability condition does not hold, B_{λ_0} has 0 as an eigenvalue, for some $\lambda_0 > 0$.

We will need the following result on the Neumann-to-Dirichlet operator N_λ. We denote by e the function which is identical to one on Γ_*.

Proposition 4.7. *The Neumann-to-Dirichlet operator N_λ for problem* (4.10) *has the following properties in $L_2(\Gamma_*)$.*

(i) *If v denotes the solution of* (4.10), *then*

$$(N_\lambda h|h)_{L_2(\Gamma_*)} = \lambda \left|\sqrt{\kappa_*}v\right|^2_{L_2(\Omega)} + \left|\sqrt{d_*}\nabla v\right|^2_{L_2(\Omega)}, \quad \lambda > 0,\ h \in L_2(\Gamma_*).$$

(ii) *For each $\alpha \in (0, 1/2)$ and $\lambda_0 > 0$, there is a constant $C > 0$ such that*

$$(N_\lambda h|h)_{L_2(\Gamma_*)} \geqq \frac{\lambda^\alpha}{C}|N_\lambda h|^2_{L_2(\Gamma_*)}, \quad h \in L_2(\Gamma_*),\ \lambda \geqq \lambda_0.$$

In particular, N_λ is injective, and

$$|N_\lambda|_{\mathcal{B}(L_2(\Gamma_*))} \leqq \frac{C}{\lambda^\alpha}, \quad \lambda \geqq \lambda_0.$$

(iii) *On $L_{2,0}(\Gamma_*) = \{\theta \in L_2(\Gamma_*) : (\theta|\mathsf{e})_{L_2(\Gamma_*)} = 0\}$, we even have*

$$(N_\lambda h|h)_{L_2(\Gamma_*)} \geqq \frac{(1+\lambda)^\alpha}{C}|N_\lambda h|^2_{L_2(\Gamma_*)}, \quad h \in L_{2,0}(\Gamma_*),\ \lambda > 0,$$

and

$$|N_\lambda|_{\mathcal{B}(L_{2,0}(\Gamma_*),L_2(\Gamma_*))} \leqq \frac{C}{(1+\lambda)^\alpha}, \quad \lambda > 0.$$

In particular, for $\lambda = 0$, (4.10) *is solvable if and only if $(h|\mathsf{e})_{\Gamma_*} = 0$, and then the solution is unique up to a constant.*

Proof. (**Proof of Proposition 4.7**) The first assertion follows from the divergence theorem. The second and third assertions are consequences of trace theory, combined with Poincaré's inequality. The last assertion is a standard statement in the theory of elliptic transmission problems. We refer to [79]. □

Proof of Theorem 4.5, continued: (a) Suppose, first, that Γ_* is connected. Consider $h = \mathsf{e}$. Then with $c_* := l_*^2/u_* \geqq 0$, we have

$$(B_\lambda \mathsf{e}|\mathsf{e})_{L_2(\Gamma_*)} = c_*\lambda(N_\lambda \mathsf{e}|\mathsf{e})_{L_2(\Gamma_*)} + \lambda\gamma_*|\mathsf{e}|^2_{L_2(\Gamma_*)} - \frac{\sigma}{R_*^2}|\mathsf{e}|^2_{L_2(\Gamma_*)}.$$

We compute the limit $\lim_{\lambda\to 0}\lambda(N_\lambda \mathsf{e}|\mathsf{e})_{L_2(\Gamma_*)}$ as follows. First solve the problem

$$\begin{cases} -d_*\Delta v = -\kappa_* a_0 & \text{in} \quad \Omega\setminus\Gamma_* \\ \partial_{\nu_\Omega} v = 0 & \text{on} \quad \partial\Omega \\ [\![v]\!] = 0 & \text{on} \quad \Gamma_* \\ -[\![d_*\partial_\nu v]\!] = \mathsf{e} & \text{on} \quad \Gamma_*, \end{cases} \tag{4.12}$$

where $a_0 = |\Gamma_*|/(\kappa_*|1)_{L_2(\Omega)}$, which is solvable since the necessary compatibility condition holds. Let v_0 denote the solution which satisfies the normalization condition $(\kappa_*|v_0)_{L_2(\Omega)} = 0$. Then $v_\lambda := S_\lambda \mathsf{e} - v_0 - a_0/\lambda$ satisfies the problem

$$\begin{cases} \kappa_*\lambda v_\lambda - d_*\Delta v_\lambda = -\kappa_*\lambda v_0 & \text{in} \quad \Omega\setminus\Gamma_* \\ \partial_{\nu_\Omega} v = 0 & \text{on} \quad \partial\Omega \\ [\![v_\lambda]\!] = 0 & \text{on} \quad \Gamma_* \\ -[\![d_*\partial_\nu v_\lambda]\!] = 0 & \text{on} \quad \Gamma_*. \end{cases} \tag{4.13}$$

By the normalization $(\kappa_*|v_0)_{L_2(\Omega)} = 0$, we see that v_λ is bounded in $W_2^2(\Omega\setminus\Gamma_*)$ as $\lambda\to 0$. Hence we have

$$\lim_{\lambda\to 0}\lambda N_\lambda \mathsf{e} = \lim_{\lambda\to 0}[\lambda v_\lambda|_{\Gamma_*} + \lambda v_0|_{\Gamma_*} + a_0] = a_0 = |\Gamma_*|/(\kappa_*|1)_{L_2(\Omega)}.$$

This then implies

$$\lim_{\lambda\to 0}(B_\lambda \mathsf{e}|\mathsf{e})_{L_2(\Gamma_*)} = c_*\frac{|\Gamma_*|^2}{(\kappa_*|1)_{L_2(\Omega)}} - \frac{\sigma}{R_*^2}|\Gamma_*| < 0,$$

if the stability condition does not hold, that is, if $\zeta_* > 1$.

(b) Next, suppose that Γ_* is disconnected. If Γ_* consists of m components Γ_*^k, $k = 1,\ldots,m$, we set $\mathsf{e}_k = 1$ on Γ_*^k and zero elsewhere. Let $h = \sum_k a_k\mathsf{e}_k \neq 0$ with $\sum_k a_k = 0$, hence $Q_0 h = h$, where Q_0 is the canonical projection onto $L_{2,0}(\Gamma_*)$,

$$Q_0 h = h - \frac{(h|\mathsf{e})_{L_2(\Gamma_*)}}{|\Gamma_*|}.$$

Then

$$\lim_{\lambda\to 0}\lambda N_\lambda h = \lim_{\lambda\to 0}\lambda N_\lambda Q_0 h = 0,$$

since $N_\lambda Q_0$ is bounded as $\lambda\to 0$. This implies

$$\lim_{\lambda\to 0}(B_\lambda h|h)_{L_2(\Gamma_*)} = -\frac{\sigma}{R_*^2}\sum_k |\Gamma_*^k|a_k^2 < 0.$$

(c) Next, we consider the behavior of $(B_\lambda h|h)_{L_2(\Gamma_*)}$ as $\lambda \to \infty$. We want to show that B_λ is positive semi-definite for large λ. For this purpose we introduce the projections P and Q by

$$Ph = c_m \sum_{k=1}^{m} (h|\mathsf{e}_k)_{L_2(\Gamma_*)}\mathsf{e}_k, \quad Q = I - P,$$

where $c_m = m/|\Gamma_*|$ in the case where Γ_* has m components. Recall that $|\Gamma_*^k| = |\Gamma_*|/m$ for $k = 1, \dots, m$. Then with $h_k = (h|\mathsf{e}_k)_{L_2(\Gamma_*)}$

$$\begin{aligned}
|(N_\lambda Ph|Qh)_{L_2(\Gamma_*)}| &\leqq c_m \sum_k |h_k|\,|(N_\lambda Qh|\mathsf{e}_k)_{L_2(\Gamma_*)}| \\
&\leqq C \sum_k |h_k||N_\lambda Qh|_{L_2(\Gamma_*)} \leqq C\lambda^{-\alpha/2} \\
&\quad \times \sum_k |h_k|(N_\lambda Qh|Qh)^{1/2}_{L_2(\Gamma_*)} \\
&\leqq C\lambda^{-\alpha/2}\left[\sum_k |h_k|^2 + m(N_\lambda Qh|Qh)_{L_2(\Gamma_*)}\right] \\
&\leqq C\lambda^{-\alpha/2}\left[|Ph|^2_{L_2(\Gamma_*)} + (N_\lambda Qh|Qh)_{L_2(\Gamma_*)}\right],
\end{aligned}$$

for $\lambda > 0$, and C standing for a generic positive constant, which may change from line to line. Hence for $\lambda \geqq \lambda_0$, with λ_0 sufficiently large, we have

$$\begin{aligned}
(N_\lambda h|h)_{L_2(\Gamma_*)} &= (N_\lambda Qh|Qh)_{L_2(\Gamma_*)} + 2(N_\lambda Qh|Ph)_{L_2(\Gamma_*)} + (N_\lambda Ph|Ph)_{L_2(\Gamma_*)} \\
&\geqq \frac{1}{2}(N_\lambda Qh|Qh)_{L_2(\Gamma_*)} + (N_\lambda Ph|Ph)_{L_2(\Gamma_*)} - \frac{C}{\lambda_0^{\alpha/2}}|Ph|^2_{L_2(\Gamma_*)}.
\end{aligned}$$

This implies

$$\begin{aligned}
(B_\lambda h|h)_{L_2(\Gamma_*)} &= c_*\lambda(N_\lambda h|h)_{L_2(\Gamma_*)} + \gamma_*\lambda|h|^2_{L_2(\Gamma_*)} - \sigma(A_*h|h)_{L_2(\Gamma_*)} \\
&\geqq \frac{c_*\lambda}{2}(N_\lambda Qh|Qh)_{L_2(\Gamma_*)} + c_*\lambda(N_\lambda Ph|Ph)_{L_2(\Gamma_*)} \\
&\quad - \sigma(A_*Qh|Qh)_{L_2(\Gamma_*)} - c|Ph|^2_{L_2(\Gamma_*)}.
\end{aligned}$$

Since N_λ is positive semi-definite and $-A_*Q$ also has this property, we need to prove only $\lambda(N_\lambda Ph|Ph)_{L_2(\Gamma_*)} \to \infty$ as $\lambda \to \infty$.

To prove this, as before, we estimate

$$|(N_\lambda \mathsf{e}_i|\mathsf{e}_j)_{L_2(\Gamma_*)}| \leqq C|N_\lambda \mathsf{e}_i|_{L_2(\Gamma_*)} \leqq \tilde{C}\lambda_0^{-\alpha/2}(N_\lambda \mathsf{e}_i|\mathsf{e}_i)^{1/2}_{L_2(\Gamma_*)},$$

and choosing λ_0 sufficiently large, this yields

$$(N_\lambda Pg|Pg)_{L_2(\Gamma_*)} \geqq c_0\left[\min_i (N_\lambda \mathsf{e}_i|\mathsf{e}_i)_{L_2(\Gamma_*)} - \frac{C}{\lambda_0^{\alpha/2}}\right]|Pg|^2_{L_2(\Gamma_*)}.$$

Therefore, it is sufficient to show

$$\lim_{\lambda\to\infty} \lambda(N_\lambda \mathsf{e}_k|\mathsf{e}_k)_{L_2(\Gamma_*)} = \infty, \quad k = 1, \ldots, m. \tag{4.14}$$

So suppose, on the contrary, that $\lambda_j(N_{\lambda_j} g|g)_{L_2(\Gamma_*)}$ is bounded, for some $g = \mathsf{e}_k$ and some sequence $\lambda_j \to \infty$. Then the corresponding solution v_j of (4.10) is such that $\lambda_j v_j$ is bounded in $L_2(\Omega)$, hence has a weakly convergent subsequence. Without loss of generality, $\lambda_j v_j \to v_\infty$ weakly in $L_2(\Omega)$. Fix a test function $\psi \in \mathcal{D}(\Omega\setminus\Gamma_*)$. Then

$$\begin{aligned}\lambda_j(\kappa_* v_j|\psi)_{L_2(\Omega)} &= (d_*\Delta v_j|\psi)_{L_2(\Omega)} = (v_j|d_*\Delta\psi)_{L_2(\Omega)} \\ &= (\lambda_j v_j|d_*\Delta\psi)_{L_2(\Omega)}/\lambda_j \to 0\end{aligned}$$

as $j \to \infty$, hence $v_\infty = 0$ in $L_2(\Omega)$. On the other hand, we have

$$\begin{aligned}0 < |\Gamma_*|/m &= \int_{\Gamma_*} g \,\mathrm{d}s = \int_{\Gamma_*} -[\![d_*\partial_\nu v_j]\!]\,\mathrm{d}s \\ &= \int_\Omega d_*\Delta v_j \,\mathrm{d}x = \lambda_j \int_\Omega \kappa_* v_j \,\mathrm{d}x \to \int_\Omega \kappa_* v_\infty \,\mathrm{d}x,\end{aligned}$$

hence v_∞ is nontrivial, a contradiction. This implies that (4.14) is valid, provided $l_* > 0$.

On the other hand, in the case where $l_* = 0$ we have $\gamma_* > 0$, hence $\lambda\gamma_*|g|^2_{L_2(\Gamma_*)} \to \infty$, so, also, in this case B_λ is positive semi-definite for large λ.

(d) Summarizing, we have shown that B_λ is not positive semi-definite for small $\lambda > 0$ if either Γ_* is not connected or the stability condition does not hold, and B_λ is always positive semi-definite for large λ. Set

$$\lambda_0 = \sup\{\lambda > 0 : \ B_\mu \text{ is not positive semi-definite for each } \mu \in (0, \lambda]\}.$$

Since B_λ has compact resolvent, B_λ has a negative eigenvalue for each $\lambda < \lambda_0$. This implies that 0 is an eigenvalue of B_{λ_0}, thereby proving that $-L_\gamma$ admits the positive eigenvalue λ_0.

Moreover, we have also shown that

$$B_0 h = \lim_{\lambda\to 0} c_*\lambda N_\lambda h - \sigma A_* h = c_*|\Gamma_*|/(\kappa_*|1)_{L_2(\Omega)} P_0 h - \sigma A_* h,$$

where $P_0 h := (I - Q_0)h = (h|\mathsf{e})_{L_2(\Gamma_*)}/|\Gamma_*|$. Therefore, B_0 has the eigenvalue $c_*|\Gamma_*|/(\kappa_*|1)_{L_2(\Omega)} - \sigma/R_*^2$ with eigenfunction e, and in case $m > 1$ it also possesses the eigenvalue $-\sigma/R_*^2$ with precisely $m-1$ linearly independent eigenfunctions of the form $\sum_k a_k \mathsf{e}_k$ with $\sum_k a_k = 0$. This implies that $-L_\gamma$ has exactly m positive eigenvalues if the stability condition does not hold, and $m - 1$ otherwise.

(e) It remains to show assertion (vii). Suppose for the moment that Γ_* consists of a single sphere of radius $R_* = \sigma/[\![\psi(u_*)]\!]$, centered at the origin of $\mathbb{R}^n$. Suppose $\mathcal{S} \subset \Omega$ is a sphere that is sufficiently close to Γ_*. Denote by $(z_1, \ldots, z_n)$ the coordinates of its center and let z_0 be such that $\sigma/[\![\psi(u_* + z_0)]\!]$ corresponds to its radius.

Then, by [27, Section 6], the sphere $\mathcal{S}$ can be parametrized over Γ_* by the distance function

$$\rho(z) = \sum_{j=1}^{n} z_j Y_j - R_* + \sqrt{\left(\sum_{j=1}^{n} z_j Y_j\right)^2 + (\sigma/[\![\psi(u_* + z_0)]\!])^2 - \sum_{j=1}^{n} z_j^2}.$$

Denoting by O a sufficiently small neighborhood of 0 in $\mathbb{R}^{n+1}$, the mapping

$$[z \mapsto \Psi(z) := (u_* + z_0, \rho(z))] : O \to W_p^2(\Omega) \times W_p^{4-1/p}(\Gamma_*)$$

is C^1 (in fact C^k if ψ is C^k), and the derivative at 0 is given by

$$\Psi'(0)h = \big(1, -\sigma[\![\psi'(u_*)]\!]/[\![\psi(u_*)]\!]^2\big)h_0 + \left(0, \sum_{j=1}^{n} h_j Y_j\right), \quad h \in \mathbb{R}^{n+1}.$$

Noting that $\sigma[\![\psi'(u_*)]\!]/[\![\psi(u_*)]\!]^2 = l_* R_*^2/(\sigma u_*)$, we can conclude that near (u_*, Γ_*) the set $\mathcal{E}$ of equilibria is a C^1-manifold in $W_p^2(\Omega) \times W_p^{4-1/p}(\Gamma_*)$ of dimension $n+1$, and that the tangent space $T_{(u_*,\Gamma_*)}(\mathcal{E})$ coincides with $N(L_\gamma)$, see (4.7). It is now easy to see that this result remains valid for the case of m spheres of the same radius R_*. The dimension of $\mathcal{E}$ is then given by $(mn+1)$, as mn parameters are needed to locate their respective centers, and one additional parameter is needed to track the temperature.

5. Nonlinear Stability and Instability of Equilibria

Before we discuss nonlinear stability of equilibria, we need to establish some preliminaries. The first observation is that the equations near an equilibrium $(u_*, \Gamma_*) \in \mathcal{E}$ can be restated as

$$\begin{cases} \kappa_* \partial_t v - d_* \Delta v = F_*(v, \rho) & \text{in} \quad \Omega \setminus \Gamma_* \\ \partial_{\nu_\Omega} v = 0 & \text{on} \quad \partial\Omega \\ [\![v]\!] = 0 & \text{on} \quad \Gamma_* \\ (l_*/u_*)v + \sigma A_* \rho - \gamma_* \partial_t \rho = G_*(v, \rho) & \text{on} \quad \Gamma_* \\ l_* \partial_t \rho - [\![d_* \partial_\nu v]\!] = H_*(v, \rho) & \text{on} \quad \Gamma_* \\ v(0) = v_0, \ \rho(0) = \rho_0. \end{cases}, \tag{5.1}$$

where

$$\begin{aligned} F_*(v, \rho) = {} & (\kappa_* - \kappa(u_* + v))\partial_t v + (d(u_* + v) - d_*)\Delta v + d(u_* + v)M_2(\rho) : \nabla^2 v \\ & - d'(u_* + v)|(I - M_1(\rho))\nabla v|^2 + d(u_* + v)(M_3(\rho)|\nabla v) \\ & + \kappa(u_* + v)\mathcal{R}(\rho)(u_* + v), \\ G_*(v, \rho) = {} & -([\![\psi(u_* + v)]\!] + \sigma\mathcal{H}(\rho)) + (l_*/u_*)v + \sigma A_* \rho \\ & + (\gamma(u_* + v)\beta(\rho) - \gamma_*)\partial_t \rho, \\ H_*(v, \rho) = {} & [\![(d(u_* + v) - d_*)\partial_\nu v]\!] + (l_* - l(u_* + v))\partial_t \rho \\ & - ([\![d(u_* + v)\nabla v]\!]|M_4(\rho)\nabla_\Sigma \rho) + \gamma(u_* + v)\beta(\rho)(\partial_t \rho)^2, \end{aligned}$$

see Sections 2 and 3 for the definition of $M_j(\rho)$, $j = 1, \dots, 4$. Here we replace $\partial_t \rho$ in the nonlinearities $G_*(v, \rho)$ and $H_*(v, \rho)$ by the following expressions:

$$\partial_t \rho = \frac{1}{l(u_* + v)} \big([\![d(u_* + v)\partial_\nu v]\!] + ([\![d(u_* + v)\nabla v]\!] | M_4(\rho)\nabla_\Sigma \rho) \big) \quad \text{if} \quad \gamma = 0,$$
$$\partial_t \rho = \frac{1}{\beta(\rho)\gamma(u_* + v)} \big([\![\psi(u_* + v)]\!] + \sigma \mathcal{H}(\rho) \big) \quad \text{if} \quad \gamma > 0.$$

From the equilibrium equation $[\![\psi(u_*)]\!] + \sigma\mathcal{H}(0) = 0$ follows that the nonlinearities satisfy $F_*(0,0) = G_*(0,0) = H_*(0,0) = 0$. Moreover, we have $F_*'(0,0) = G_*'(0,0) = H_*'(0,0) = 0$.

The state manifold for problem (5.1) near the equilibrium (u_*, Γ_*) can then be described by

$$\begin{aligned} \mathcal{SM}_0 = \big\{ (v, \rho) \in W_p^{2-2/p}(\Omega \setminus \Gamma_*) \times W_p^{4-3/p}(\Gamma_*) :\ \partial_{\nu_\Omega} v = 0,\ [\![v]\!] = 0, \\ (l_*/u_*)v + \sigma A_* \rho = G_*(v, \rho),\ [\![d_* \partial_\nu v]\!] + H_*(v, \rho) \in W_p^{2-6/p}(\Gamma_*) \big\}, \end{aligned} \tag{5.2}$$

for $\gamma_* = 0$, in case $l_* \neq 0$ (otherwise the linear problem is not well-posed), and

$$\begin{aligned} \mathcal{SM}_\gamma = \big\{ (v, \rho) \in W_p^{2-2/p}(\Omega \setminus \Gamma_*) \times W_p^{4-3/p}(\Gamma_*) :\ \partial_{\nu_\Omega} v = 0,\ [\![v]\!] = 0, \\ (l_*^2/u_*)v + l_*\sigma A_* \rho - \gamma_* [\![d_* \partial_\nu v]\!] = l_* G_*(v, \rho) + \gamma_* H_*(v, \rho) \big\}, \end{aligned} \tag{5.3}$$

in case $\gamma_* > 0$.

We would like to parametrize these manifolds over their tangent spaces at $(0, 0)$, given by

$$\begin{aligned} \tilde{Z}_0 = \big\{ (\tilde{v}, \tilde{\rho}) \in [W_p^{2-2/p}(\Omega \setminus \Gamma_*) \cap C(\bar{\Omega})] \times W_p^{4-3/p}(\Gamma_*) : \\ \partial_{\nu_\Omega} \tilde{v} = 0,\ (l_*/u_*)\tilde{v} + \sigma A_* \tilde{\rho} = 0,\ [\![d_* \partial_\nu \tilde{v}]\!] \in W_p^{2-6/p}(\Gamma_*) \big\}, \end{aligned} \tag{5.4}$$

respectively, for $\gamma_* > 0$

$$\begin{aligned} \tilde{Z}_\gamma = \big\{ (\tilde{v}, \tilde{\rho}) \in [W_p^{2-2/p}(\Omega \setminus \Gamma_*) \cap C(\bar{\Omega})] \times W_p^{4-3/p}(\Gamma_*) : \\ \partial_{\nu_\Omega} \tilde{v} = 0,\ (l_*^2/u_*)\tilde{v} + l_*\sigma A_* \tilde{\rho} - \gamma_* [\![d_* \partial_\nu \tilde{v}]\!] = 0 \big\}. \end{aligned} \tag{5.5}$$

Note that the norm in $\tilde{Z}_\gamma$ for $\gamma = 0$ is given by

$$|(\tilde{v}, \tilde{\rho})|_{\tilde{Z}_0} = |\tilde{v}|_{W_p^{2-2/p}} + |\tilde{\rho}|_{W_p^{4-3/p}} + |[\![d_* \partial_\nu \tilde{v}]\!]|_{W_p^{2-6/p}},$$

while for $\gamma > 0$ it is given by $|(\tilde{v}, \tilde{\rho})|_{\tilde{Z}_\gamma} = |\tilde{v}|_{W_p^{2-2/p}} + |\tilde{\rho}|_{W_p^{4-3/p}}$.

Is should be observed that $\tilde{Z}_\gamma$ is a linear space. The parametrization of $\mathcal{SM}_\gamma$ over the tangent space $\tilde{Z}_\gamma$ will facilitate the use of maximal regularity results for the stability/instability analysis.

In order to determine a parameterization, we consider the linear problem

$$\begin{cases} \kappa_*\omega v - d_*\Delta v = 0 & \text{in} \quad \Omega\setminus\Gamma_* \\ \partial_{\nu_\Omega} v = 0 & \text{on} \quad \partial\Omega \\ [\![v]\!] = 0 & \text{on} \quad \Gamma_* \\ (l_*/u_*)v + \sigma A_*\rho - \gamma_*\omega\rho = g & \text{on} \quad \Gamma_* \\ l_*\omega\rho - [\![d_*\partial_\nu v]\!] = h & \text{on} \quad \Gamma_*. \end{cases} \tag{5.6}$$

We have the following result.

Proposition 5.1. *Suppose* $p > 3$, $\gamma_* \geqq 0$, $l_* \neq 0$ *in the case where* $\gamma_* = 0$, *and* $\omega > 0$ *is sufficiently large. Then problem* (5.6) *admits a unique solution* (v, ρ) *with regularity*

$$v \in W_p^{2-2/p}(\Omega\setminus\Gamma_*), \quad \rho \in W_p^{4-3/p}(\Gamma_*)$$

if and only if the data (g, h) *satisfy*

$$g \in W_p^{2-3/p}(\Gamma_*), \quad h \in W_p^{1-3/p}(\Gamma_*).$$

The solution map $[(g, h) \mapsto (v, \rho)]$ *is continuous in the corresponding spaces.*

Proof. This purely elliptic problem can be solved in the same way as the corresponding linear parabolic problems, see Theorems 4.3 and 4.4. □

For the parametrization we pick $\omega > 0$ sufficiently large. Given $\tilde{z} = (\tilde{v}, \tilde{\rho}) \in \tilde{Z}_\gamma$ sufficiently small, we can solve the auxiliary problem

$$\begin{cases} \kappa_*\omega\bar{v} - d_*\Delta\bar{v} = 0 & \text{in} \quad \Omega\setminus\Gamma_* \\ \partial_{\nu_\Omega}\bar{v} = 0 & \text{on} \quad \partial\Omega \\ [\![\bar{v}]\!] = 0 & \text{on} \quad \Gamma_* \\ (l_*/u_*)\bar{v} + \sigma A_*\bar{\rho} - \gamma_*\omega\bar{\rho} = G_*(\tilde{v}+\bar{v}, \tilde{\rho}+\bar{\rho}) & \text{on} \quad \Gamma_* \\ l_*\omega\bar{\rho} - [\![d_*\partial_\nu\bar{v}]\!] = H_*(\tilde{v}+\bar{v}, \tilde{\rho}+\bar{\rho}) & \text{on} \quad \Gamma_* \end{cases} \tag{5.7}$$

by means of the implicit function theorem, employing Proposition 5.1. This yields a unique solution $\bar{z} = (\bar{v}, \bar{\rho}) = \phi(\tilde{z}) \in W_p^{2-2/p}(\Omega\setminus\Gamma_*) \times W_p^{4-3/p}(\Gamma_*)$ with a C^1-function ϕ such that $\phi(0) = 0$ as well as $\phi'(0) = 0$. One readily verifies that $z = \tilde{z} + \phi(\tilde{z}) \in \mathcal{SM}_\gamma$. To prove surjectivity of this map, given $(v, \rho) \in \mathcal{SM}_\gamma$, we solve the linear problem

$$\begin{cases} \kappa_*\omega\bar{v} - d_*\Delta\bar{v} = 0 & \text{in} \quad \Omega\setminus\Gamma_* \\ \partial_{\nu_\Omega}\bar{v} = 0 & \text{on} \quad \partial\Omega \\ [\![\bar{v}]\!] = 0 & \text{on} \quad \Gamma_* \\ (l_*/u_*)\bar{v} + \sigma A_*\bar{\rho} - \gamma_*\omega\bar{\rho} = G_*(v, \rho) & \text{on} \quad \Gamma_* \\ l_*\omega\bar{\rho} - [\![d_*\partial_\nu\bar{v}]\!] = H_*(v, \rho) & \text{on} \quad \Gamma_* \end{cases} \tag{5.8}$$

and set $\tilde{z} = z - \bar{z}$. Then $\tilde{z} \in \tilde{Z}_\gamma$ and $\bar{z} = \phi(\tilde{z})$, hence the map $[\tilde{z} \mapsto \tilde{z} + \phi(\tilde{z})]$ is also surjective near 0. We have thus obtained a local parametrization of $\mathcal{SM}_\gamma$ near zero over the tangent space $\tilde{Z}_\gamma$.

Next we derive a similar decomposition for the solutions of problem (5.1). Let $z_0 = (\tilde{z}_0, \phi(\tilde{z}_0)) \in \mathcal{SM}_\gamma$ be given and let $z \in \mathbb{E}(a)$, where we set

$$\mathbb{E}(a) := \mathbb{E}([0, a]), \tag{5.9}$$

to be the solution of (5.1) with initial value z_0. Then we would like to devise a decomposition $z = z_\infty + \tilde{z} + \bar{z}$, where $\tilde{z}(t) \in \tilde{Z}_\gamma$ for all $t \in [0, a]$, and where $z_\infty = \tilde{z}_\infty + \phi(\tilde{z}_\infty)$ is an equilibrium for (5.1). In order to achieve this, we consider the coupled systems of equations

$$\begin{cases} \kappa_*\omega\bar{v} + \kappa_*\partial_t\bar{v} - d_*\Delta\bar{v} = F_*(z_\infty + \tilde{z} + \bar{z}) - F_*(z_\infty) \\ \partial_{\nu_\Omega}\bar{v} = 0 \\ [\![\bar{v}]\!] = 0 \\ (l_*/u_*)\bar{v} + \sigma A_*\bar{\rho} - \gamma_*(\partial_t\bar{\rho} + \omega\bar{\rho}) = G_*(z_\infty + \tilde{z} + \bar{z}) - G_*(z_\infty) \\ l_*\omega\bar{\rho} + l_*\partial_t\bar{\rho} - [\![d_*\partial_\nu\bar{v}]\!] = H_*(z_\infty + \tilde{z} + \bar{z}) - H_*(z_\infty) \\ \bar{z}(0) = \phi(\tilde{z}_0) - \phi(\tilde{z}_\infty), \end{cases} \tag{5.10}$$

and

$$\begin{cases} \kappa_*\partial_t\tilde{v} - d_*\Delta\tilde{v} = \kappa_*\omega\bar{v} \\ \partial_{\nu_\Omega}\tilde{v} = 0 \\ [\![\tilde{v}]\!] = 0 \\ (l_*/u_*)\tilde{v} + \sigma A_*\tilde{\rho} - \gamma_*\partial_t\tilde{\rho} = -\gamma_*\omega\bar{\rho} \\ l_*\partial_t\tilde{\rho} - [\![d_*\partial_\nu\tilde{v}]\!] = l_*\omega\bar{\rho} \\ \tilde{z}(0) = \tilde{z}_0 - \tilde{z}_\infty. \end{cases} \tag{5.11}$$

It should be mentioned that $F_*(z_\infty) = 0$, as can be seen from the equilibrium equation for (5.1) and the fact that $v_\infty =$ constant for $z_\infty = (v_\infty, \rho_\infty)$. For reasons of symmetry and consistency we will, nevertheless, include this term.

Equations (5.10)–(5.11) can be rewritten in the more condensed form

$$\begin{aligned} \mathbb{L}_{\gamma,\omega}\bar{z} &= N(z_\infty + \tilde{z} + \bar{z}) - N(z_\infty), \qquad & \bar{z}(0) &= \phi(\tilde{z}_0) - \phi(\tilde{z}_\infty), \\ \dot{\tilde{z}} + L_\gamma\tilde{z} &= \omega\bar{z}, & \tilde{z}(0) &= \tilde{z}_0 - \tilde{z}_\infty, \end{aligned} \tag{5.12}$$

where we use the abbreviation $\mathbb{L}_{\gamma,\omega}$ to denote the linear operator on the left-hand side of (5.10), and N to denote the nonlinearities on the right-hand side of (5.10), respectively.

We are now ready to formulate the main theorem of this section.

Theorem 5.2. *Suppose $\sigma > 0$, $\gamma_* = \gamma(u_*) \geqq 0$ and $l_* = l(u_*) \neq 0$ in the case where $\gamma_* = 0$. Then in the topology of the state manifold $\mathcal{SM}_\gamma$, we have:*

(a) *$(u_*, \Gamma_*) \in \mathcal{E}$ is stable if Γ_* is connected and $\zeta_* < 1$.*
Any solution starting in a neighborhood of such a stable equilibrium exists globally and converges to another stable equilibrium exponentially fast.

(b) *$(u_*, \Gamma_*) \in \mathcal{E}$ is unstable if Γ_* is disconnected or if $\zeta_* > 1$.*
Any solution starting and staying in a neighborhood of such an unstable equilibrium converges to another unstable equilibrium exponentially fast.

Proof. (a) We begin with the case that (u_*, Γ_*) is linearly stable. Then according to Theorem 4.5 we have $X_\gamma = N(L_\gamma) \oplus R(L_\gamma)$. Let P^c denote the projection onto $X^c_\gamma := N(L_\gamma)$ along $X^s_\gamma := R(L_\gamma)$ and $P^s = I - P^c$ the complementary projection onto $R(L_\gamma)$. We parametrize the set of equilibria $\mathcal{E}$ near 0 over $N(L_\gamma)$ via the C^1-map $[\mathsf{x} \mapsto \mathsf{x} + \psi(\mathsf{x}) + \phi(\mathsf{x} + \psi(\mathsf{x}))]$ such that $\psi(0) = \psi'(0) = 0$ and $\phi(0) = \phi'(0) = 0$. It follows from the equilibrium equation associated to (5.1) (recall that $F_*(z_e)$ vanishes for any equilibrium z_e), and from the definition of ϕ that the mapping ψ is determined by the equation

$$L^s_\gamma \psi(\mathsf{x}) = P^s \omega \phi(\mathsf{x} + \psi(\mathsf{x})), \quad \mathsf{x} \in B_{X^c_\gamma}(r). \tag{5.13}$$

Since L^s_γ is invertible on X^s_γ, $\psi \in C^1(B_{X^c_\gamma}(r), D(L^s_\gamma))$ is well-defined by the implicit function theorem and $\psi(0) = \psi'(0) = 0$.

For $\mathsf{x}_\infty \in X^c_\gamma$ sufficiently small we set $z_\infty := \mathsf{x}_\infty + \psi(\mathsf{x}_\infty) + \phi(\mathsf{x}_\infty + \psi(\mathsf{x}_\infty))$. Then z_∞ is an equilibrium for (5.1) and we will now consider the decomposition $z = z_\infty + \tilde{z} + \bar{z}$ introduced in (5.10)–(5.11), or (5.12), respectively. With the ansatz

$$\tilde{z} = \mathsf{x} + \psi(\mathsf{x}_\infty + \mathsf{x}) - \psi(\mathsf{x}_\infty) + \mathsf{y}, \quad (\mathsf{x}, \mathsf{y}) \in X^c_\gamma \times X^s_\gamma, \tag{5.14}$$

for $\mathsf{x}, \mathsf{x}_\infty \in X^c_\gamma$ small enough, the second line in (5.12) becomes

$$\begin{cases} \dot{\mathsf{x}} = P^c \omega \bar{z}, & \mathsf{x}(0) = \mathsf{x}_0 - \mathsf{x}_\infty, \\ \dot{\mathsf{y}} + L^s_\gamma \mathsf{y} = S(\mathsf{x}_\infty, \mathsf{x}, \bar{z}), & \mathsf{y}(0) = \mathsf{y}_0, \end{cases} \tag{5.15}$$

where

$$S(\mathsf{x}_\infty, \mathsf{x}, \bar{z}) = P^s \omega \bar{z} - \psi'(\mathsf{x}_\infty + \mathsf{x}) P^c \omega \bar{z} - L^s_\gamma[\psi(\mathsf{x}_\infty + \mathsf{x}) - \psi(\mathsf{x}_\infty)],$$

and

$$\tilde{z}_0 = \mathsf{x}_0 + \psi(\mathsf{x}_0) + \mathsf{y}_0, \quad (\mathsf{x}_0, \mathsf{y}_0) \in X^c_\gamma \times (X^s_\gamma \cap \tilde{Z}_\gamma). \tag{5.16}$$

Next we show that the system of equations (5.15) admits a unique global solution $(\mathsf{x}_\infty, \mathsf{x}, \mathsf{y})$, where the functions (x, y) are exponentially decaying, provided $\bar{z}$ is exponentially decaying and $(\mathsf{x}_0, \mathsf{y}_0)$ is sufficiently small. For this let us first introduce some more notation. For $\delta \geqq 0$ we set

$$\mathbb{E}_i(\mathbb{R}_+, \delta) := \{v : \mathrm{e}^{\delta t} v \in \mathbb{E}_i(\mathbb{R}_+)\}, \quad i = 1, 2,$$
$$\mathbb{F}_j(\mathbb{R}_+, \delta) := \{v : \mathrm{e}^{\delta t} v \in \mathbb{F}_j(\mathbb{R}_+)\}, \quad j = 1, 2, 3,$$

endowed with the norms

$$\|v\|_{\mathbb{E}_i(\mathbb{R}_+, \delta)} = \|\mathrm{e}^{\delta t} v\|_{\mathbb{E}_i(\mathbb{R}_+)},$$
$$\|v\|_{\mathbb{F}_j(\mathbb{R}_+, \delta)} = \|\mathrm{e}^{\delta t} v\|_{\mathbb{F}_j(\mathbb{R}_+)}.$$

The spaces $\mathbb{E}(\mathbb{R}_+, \delta)$ and $\mathbb{F}(\mathbb{R}_+, \delta)$ are then defined analogously as in Section 3. We also need the space

$$\mathbb{X}(\mathbb{R}_+, \delta) := H^1_p(\mathbb{R}_+, \delta; X_\gamma) \cap L_p(\mathbb{R}_+, \delta; D(L_\gamma)), \tag{5.17}$$

where $H^k_p(\mathbb{R}_+, \delta; E)$ denotes all functions $v : \mathbb{R}_+ \to E$ such that $e^{\delta t} v \in H^k_p(\mathbb{R}_+; E)$, with E a given Banach space. Finally, let

$$\mathbb{B}_1(r, \delta) : = \{(\mathsf{x}_0, \mathsf{y}_0, \bar{z}) \in X^c_\gamma \times (X^s_\gamma \cap \tilde{Z}_\gamma) \times \mathbb{E}(\mathbb{R}_+, \delta) : |(\mathsf{x}_0, \mathsf{y}_0)|_{\tilde{Z}_\gamma} < r\}.$$

For given $(\mathsf{x}_0, \mathsf{y}_0, \bar{z}) \in \mathbb{B}_1(r_0, \delta)$, with r_0 sufficiently small, we set

$$\begin{aligned} \mathsf{x}_\infty &:= \mathsf{x}_0 + \int_0^\infty P^c \omega \bar{z}(\tau)\, d\tau, \\ \mathsf{x} &:= -\int_t^\infty P^c \omega \bar{z}(\tau)\, d\tau, \\ \mathsf{y} &:= \left(\frac{d}{dt} + L^s_\gamma, \mathrm{tr}\right)^{-1} \big(S(\mathsf{x}_\infty, \mathsf{x}, \bar{z}), \mathsf{y}_0\big). \end{aligned} \tag{5.18}$$

Here we used the notation tr $w := w(0)$. It should be observed that the functions $(\mathsf{x}_\infty, \mathsf{x})$ occurring in the third line of (5.18) are defined through the first two lines in (5.18). We now set

$$\mathfrak{S}(\mathsf{x}_0, \mathsf{y}_0, \bar{z}) := (\mathsf{x}_\infty, \mathsf{x}, \mathsf{y}), \quad (\mathsf{x}_0, \mathsf{y}_0, \bar{z}) \in \mathbb{B}_1(r_0, \delta), \tag{5.19}$$

where r_0 is chosen sufficiently small.

Next we will show that there exists a number $\delta_0 > 0$ such that for any $\delta \in [0, \delta_0]$ the mapping $\mathfrak{S}$ has the following properties:

$$\mathfrak{S} \in C\big(\mathbb{B}_1(r_0, \delta), X^c_\gamma \times \mathbb{X}^c(\mathbb{R}_+, \delta) \times \mathbb{X}^s(\mathbb{R}_+, \delta)\big), \quad \mathfrak{S}(0) = 0, \tag{5.20}$$

where

$$\mathbb{X}^c(\mathbb{R}_+, \delta) := H^1_p(\mathbb{R}_+, \delta; X^c_\gamma), \mathbb{X}^s(\mathbb{R}_+, \delta) := H^1_p(\mathbb{R}_+, \delta; X^s_\gamma) \cap L_p(\mathbb{R}_+, \delta; D(L^s_\gamma)).$$

Writing $\mathfrak{S} = (\mathfrak{S}_1, \mathfrak{S}_2, \mathfrak{S}_3)$ we readily observe that

$$\mathfrak{S}_1 \in C^\infty(\mathbb{B}_1(r_0, \delta), X^c_\gamma), \quad \mathfrak{S}_1(0) = 0. \tag{5.21}$$

For $g \in L_p(\mathbb{R}_+, \delta; X^c_\gamma)$, let $(Kg)(t) := \int_t^\infty g(\tau)\, d\tau$ and note that

$$e^{\delta t}(Kg)(t) = \int_t^\infty e^{\delta(t-\tau)} e^{\delta\tau} g(\tau)\, d\tau.$$

Young's inequality for convolution integrals readily yields

$$K \in \mathcal{B}\big(L_p(\mathbb{R}_+, \delta; X^c_\gamma), H^1_p(\mathbb{R}_+, \delta; X^c_\gamma)\big),$$

and this shows that $\mathfrak{S}_2 \in \mathbb{X}^c(\mathbb{R}_+, \delta)$. Hence we have

$$\mathfrak{S}_2 \in C^\infty\big(\mathbb{B}_1(r_0, \delta), \mathbb{X}^c(\mathbb{R}_+, \delta)\big), \quad \mathfrak{S}_2(0) = 0. \tag{5.22}$$

Concerning the function $\mathfrak{S}_3$, we know from Theorem 4.5(v) that $s(-L_\gamma^s)$, the spectral bound of $(-L_\gamma^s)$, is negative. Fixing $\delta_0 > 0$ with $s(-L_\gamma^s) < -\delta_0$, it follows from semigroup theory and the L_p-maximal regularity results stated in Theorem 4.3 and Theorem 4.4 that

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + L_\gamma^s, \mathrm{tr}\right)^{-1} \in \mathcal{B}\big(L_p(\mathbb{R}_+, \delta; X_\gamma^s) \times \tilde{Z}_\gamma^s, \ \mathbb{X}^s(\mathbb{R}_+, \delta)\big), \quad \delta \in [0, \delta_0], \tag{5.23}$$

where $\tilde{Z}_\gamma^s = X_\gamma^s \cap \tilde{Z}_\gamma$. This in conjunction with (5.21)–(5.22) and the definition of S implies

$$\mathfrak{S}_3 \in C\big(\mathbb{B}_1(r_0, \delta), \mathbb{X}^s(\mathbb{R}_+, \delta)\big), \quad \mathfrak{S}_3(0) = 0. \tag{5.24}$$

Combining (5.21)–(5.24) then yields (5.20).

For given $(\mathsf{x}_0, \mathsf{y}_0, \bar{z}) \in \mathbb{B}_1(r_0, \delta)$ let $(\mathsf{x}_\infty, \mathsf{x}, \mathsf{y}) = \mathfrak{S}(\mathsf{x}_0, \mathsf{y}_0, \bar{z})$. Then we have

$$\begin{aligned}\mathsf{x}(t) &= -\int_t^\infty P^c\omega\bar{z}(\tau)\,\mathrm{d}\tau = -\int_0^\infty P^c\omega\bar{z}(\tau)\,\mathrm{d}\tau + \int_0^t P^c\omega\bar{z}(\tau)\,\mathrm{d}\tau \\ &= \mathsf{x}_0 - \mathsf{x}_\infty + \int_0^t P^c\omega\bar{z}(\tau)\,\mathrm{d}\tau,\end{aligned}$$

thus showing that x solves the first equation in (5.15). In summary, we have shown that $(\mathsf{x}_\infty, \mathsf{x}, \mathsf{y}) = \mathfrak{S}(\mathsf{x}_0, \mathsf{y}_0, \bar{z})$ is for every $(\mathsf{x}_0, \mathsf{y}_0, \bar{z}) \in \mathbb{B}_1(r_0, \delta)$ the unique solution of (5.15) in $X_\gamma^c \times \mathbb{X}^c(\mathbb{R}_+, \delta) \times \mathbb{X}^s(\mathbb{R}_+, \delta)$, where $\delta \in [0, \delta_0]$.

Setting

$$\begin{aligned}\tilde{z} &= \tilde{\mathfrak{Z}}(\mathsf{x}_0, \mathsf{y}_0, \bar{z}) := \mathsf{x} + \psi(\mathsf{x}_\infty + \mathsf{x}) - \psi(\mathsf{x}_\infty) + \mathsf{y}, \\ z_\infty &= \mathfrak{Z}_\infty(\mathsf{x}_0, \mathsf{y}_0, \bar{z}) := \mathsf{x}_\infty + \psi(\mathsf{x}_\infty) + \phi(\mathsf{x}_\infty + \psi(\mathsf{x}_\infty))\end{aligned} \tag{5.25}$$

for $(\mathsf{x}_\infty, \mathsf{x}, \mathsf{y}) = \mathfrak{S}(\mathsf{x}_0, \mathsf{y}_0, \bar{z})$, we see that

$$\tilde{\mathfrak{Z}} \in C(\mathbb{B}_1(r_0, \delta), \mathbb{X}(\mathbb{R}_+, \delta)), \quad \tilde{\mathfrak{Z}}(0) = 0,$$

and

$$\mathfrak{Z}_\infty \in C(\mathbb{B}_1(r_0, \delta), Z_\infty), \quad \mathfrak{Z}_\infty(0) = 0, \tag{5.26}$$

where $Z_\infty = [W_p^2(\Omega \setminus \Gamma_*) \cap C(\bar{\Omega})] \times W_p^{4-1/p}(\Gamma_*)$. It then follows from the derivation of (5.14)–(5.15) that

$$(z_\infty, \tilde{z}) = (\mathfrak{Z}(\mathsf{x}_0, \mathsf{y}_0, \bar{z}), \tilde{\mathfrak{Z}}(\mathsf{x}_0, \mathsf{y}_0, \bar{z}))$$

is, for every given $(\mathsf{x}_0, \mathsf{y}_0, \bar{z}) \in \mathbb{B}_1(r_0, \delta)$, the unique (global) solution of (5.11) with $\tilde{z}$ in the regularity class $\mathbb{X}(\mathbb{R}_+, \delta)$. In a next step we shall show that $\tilde{z}$ in fact has better regularity properties, namely

$$\tilde{\mathfrak{Z}} \in C(\mathbb{B}_1(r_0, \delta), \mathbb{E}(\mathbb{R}_+, \delta)), \quad \tilde{\mathfrak{Z}}(0) = 0. \tag{5.27}$$

In order to see this, let us first consider the case $\gamma \equiv 0$ (which implies $\gamma_* = 0$). From the fourth line of (5.11), the fact that $\tilde{z} \in \mathbb{X}(\mathbb{R}_+, \delta)$, and

$$[v \mapsto v|_{\Gamma_*}] \in \mathcal{B}(\mathbb{X}(\mathbb{R}_+, \delta), W_p^{1-1/2p}(\mathbb{R}_+, \delta; L_p(\Gamma_*))),$$

follows

$$\tilde{\rho} = (\mu - \sigma A_*)^{-1}((l_*/u_*)\tilde{v} + \mu\tilde{\rho}) \in W_p^{1-1/2p}(\mathbb{R}_+, \delta; H_p^2(\Gamma_*)),$$

where μ is in the resolvent set of σA_*. From the fifth line of (5.11), the fact that $(\tilde{z}, \bar{z}) \in \mathbb{X}(\mathbb{R}_+, \delta) \times \mathbb{E}(\mathbb{R}_+, \delta)$, and trace theory for $\tilde{v}$ follows

$$l_*\partial_t\tilde{\rho} = [\![d_*\partial_\nu\tilde{v}]\!] + l_*\omega\bar{\rho} \in W_p^{1/2-1/2p}(\mathbb{R}_+, \delta; L_p(\Gamma_*)),$$

implying that $\tilde{\rho} \in W_p^{3/2-1/2p}(\mathbb{R}_+, \delta; L_p(\Gamma_*))$. Hence (5.27) holds for $\gamma = 0$.

If $\gamma > 0$ (and thus $\gamma_* > 0$), we use the embedding

$$H_p^1(\mathbb{R}_+, \delta; W_p^{2-1/p}(\Gamma_*)) \cap L_p(\mathbb{R}_+, \delta; W_p^{4-1/p}(\Gamma_*)) \hookrightarrow W_p^{1-1/2p}(\mathbb{R}_+, \delta; H_p^2(\Gamma_*))$$

and the fourth equation in (5.11) to conclude that $\tilde{\rho} \in W_p^{2-1/2p}(\mathbb{R}_+, \delta; L_p(\Gamma_*))$. Hence (5.27) holds in this case, as well.

Let us now turn our attention to equation (5.10), or equivalently, the first line of (5.12). In a similar way as in the proof of [53, Proposition 10] (extra consideration is needed in order to deal with the additional terms involving z_∞) one verifies that the mapping

$$[(z_\infty, z) \mapsto N(z_\infty + z) - N(z_\infty)] : \mathbb{U}(\delta) \to \mathbb{F}(\mathbb{R}_+, \delta)$$

is C^1 and vanishes together with its Fréchet derivative at (0, 0). Here $\mathbb{U}(\delta)$ denotes an open neighborhood of (0, 0) in $Z_\infty \times \mathbb{E}(\mathbb{R}_+, \delta)$. Let

$$\mathbb{B}(r, \delta) = \{(\mathsf{x}_0, \mathsf{y}_0, \bar{z}) \in X_\gamma^c \times (X_\gamma^s \cap \tilde{Z}_\gamma) \times \mathbb{E}(\mathbb{R}_+, \delta) : |(\mathsf{x}_0, \mathsf{y}_0, \bar{z})|_{[\tilde{Z}_\gamma]^2 \times \mathbb{E}(\mathbb{R}_+, \delta)} < r_0\},$$

and let $\mathrm{ext}_\delta \in \mathcal{B}\big(W_p^{2-2/p}(\Omega \setminus \Gamma_*) \cap C(\overline{\Omega})) \times W_p^{4-3/p}(\Gamma_*), \mathbb{E}(\mathbb{R}_+, \delta)\big)$ be an appropriate extension operator with $(\mathrm{ext}_\delta w_0)(0) = w_0$.

For $(\mathsf{x}_0, \mathsf{y}_0, \bar{z}) \in \mathbb{B}(r_0, \delta)$, with r_0 sufficiently small, we define

$$M(\mathsf{x}_0, \mathsf{y}_0, \bar{z}) := N(z_\infty + \tilde{z} + \mathrm{ext}_\delta(\phi(\tilde{z}_0) - \phi(\tilde{z}_\infty) - \bar{z}(0)) + \bar{z}) - N(z_\infty).$$

It follows from (5.25)–(5.27) that $M \in C(\mathbb{B}(r_0, \delta), \mathbb{F}(\mathbb{R}_+, \delta))$, $M(0, 0, 0) = 0$, and $D_3 M(0, 0, 0) = 0$. Moreover,

$$M(\mathsf{x}_0, \mathsf{y}_0, \bar{z})(0) = N(z_0) - N(z_\infty), \quad (\mathsf{x}_0, \mathsf{y}_0, \bar{z}) \in \mathbb{B}(r_0, \delta), \tag{5.28}$$

where we recall that $\bar{z}(0) = \tilde{z}_0 - \tilde{z}_\infty$, $z_0 = \tilde{z}_0 + \phi(\tilde{z}_0)$, and $z_\infty = \tilde{z}_\infty + \phi(\tilde{z}_\infty)$.

Finally, for $(\mathsf{x}_0, \mathsf{y}_0, \bar{z}) \in \mathbb{B}(r_0, \delta)$ let

$$K(\mathsf{x}_0, \mathsf{y}_0, \bar{z}) := (\mathbb{L}_{\gamma,\omega}, \mathrm{tr})^{-1}(M(\mathsf{x}_0, \mathsf{y}_0, \bar{z}), \phi(\tilde{z}_0) - \phi(\tilde{z}_\infty)). \tag{5.29}$$

It follows from (5.28) and the definition of ϕ that the functions

$$(M(\mathsf{x}_0, \mathsf{y}_0, \bar{z}), \phi(\tilde{z}_0) - \phi(\tilde{z}_\infty))$$

satisfy the necessary compatibility conditions, whenever $(\mathsf{x}_0, \mathsf{y}_0, \bar{z}) \in \mathbb{B}(r_0, \delta)$. Slight modifications of the results in [21] then imply that $K : \mathbb{B}(r_0, \delta) \to \mathbb{E}(\mathbb{R}_+, \delta)$ is well-defined, provided ω is large enough (and δ is in $[0, \delta_0]$ with δ_0 as above).

From the properties of the mappings N, ψ and ϕ, the definition of $\tilde{z}_0$ and $\tilde{z}_\infty$ (recall that $\tilde{z}_0 = \mathsf{x}_0 + \psi(\mathsf{x}_0) + \mathsf{y}_0$, $\tilde{z}_\infty = \mathsf{x}_\infty + \psi(\mathsf{x}_\infty)$), and the contraction mapping theorem, it follows that K, defined in (5.29), has for each $(\mathsf{x}_0, \mathsf{y}_0)$ sufficiently small a unique fixed point

$$\bar{z} = \bar{z}(\mathsf{x}_0, \mathsf{y}_0) \in \mathbb{E}(\mathbb{R}_+, \delta),$$

and that the mapping $[(\mathsf{x}_0, \mathsf{y}_0) \mapsto \bar{z}(\mathsf{x}_0, \mathsf{y}_0)]$ is continuous and vanishes at $(0, 0)$. By construction it follows that $\bar{z} = \bar{z}(\mathsf{x}_0, \mathsf{y}_0)$ solves

$$\mathbb{L}_{\gamma,\omega}\bar{z} = N(z_\infty + \tilde{z} + \bar{z}) - N(z_\infty), \quad \bar{z}(0) = \phi(\tilde{z}_0) - \phi(\tilde{z}_\infty).$$

In summary, we have shown that for each $z_0 \in \mathcal{SM}_\gamma$ small enough, there exists

$$(z_\infty, \tilde{z}, \bar{z}) \in Z_\infty \times \mathbb{E}(\mathbb{R}_+, \delta) \times \mathbb{E}(\mathbb{R}_+, \delta)$$

such that $z = z_\infty + \tilde{z} + \bar{z}$ is the unique global solution of (5.1). In particular, we have shown that for every $z_0 \in \mathcal{SM}_\gamma$ small enough there exists a unique equilibrium $z_\infty = z_\infty(z_0)$ such that the solution of (5.1) exists for all $t \geqq 0$ and converges to z_∞ in $\mathcal{SM}_\gamma$ at an exponential rate.

(b) Now we consider the linearly unstable case; we first show that the equilibrium 0 is unstable for the nonlinear equation (5.1). Using the same notation as in part (i) we consider the system of equations

$$\begin{aligned} \mathbb{L}_{\gamma,\omega}\bar{z} &= N(\tilde{z} + \bar{z}), & \bar{z}(0) &= \phi(\tilde{z}_0), \\ \dot{\tilde{z}} + L_\gamma \tilde{z} &= \omega \bar{z}, & \tilde{z}(0) &= \tilde{z}_0. \end{aligned} \tag{5.30}$$

Given $\alpha \in \mathbb{R}$, one verifies (by considerations similar to [53, Proposition 10]) that there is a nondecreasing function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\eta(r) \to 0$ as $r \to 0$, and

$$\|\mathrm{e}^{\alpha t} N(z)\|_{\mathbb{F}(a)} \leqq \eta(r) \|\mathrm{e}^{\alpha t} z\|_{\mathbb{E}(a)}, \quad \mathrm{e}^{\alpha t} z \in \mathbb{E}(a), \tag{5.31}$$

whenever $|z(t)|_{Z_\gamma} \leqq r$ for $0 \leqq t \leqq a$. Here $a > 0$ is an arbitrary fixed number, $\mathbb{E}(a) := \mathbb{E}([0, a])$ and $\mathbb{F}(a) := \mathbb{F}([0, a])$. For later use we note that

$$\mathbb{E}(a) \hookrightarrow L_p([0, a]; X_\gamma), \tag{5.32}$$

where the embedding constant is independent of a.

Let σ^+ be the collection of all positive eigenvalues of $(-L_\gamma)$ and let P^+ be the spectral projection related to the spectral set σ^+. Additionally, let $P^- := I - P^+$ and $X_\gamma^\pm := P^\pm(X_\gamma)$. Then X_γ^+ is finite dimensional and we obtain the decomposition

$$X = X_\gamma^+ \oplus X_\gamma^-, \quad L_\gamma = L_\gamma^+ \oplus L_\gamma^-.$$

We note that $\sigma(-L_\gamma^+) = \sigma^+$ and $\sigma(-L_\gamma^-) \subset [\operatorname{Re} z \leqq 0]$, where $\sigma(-L_\gamma^\pm)$ denotes the spectrum of $(-L_\gamma^\pm)$, respectively. Let λ_* be the smallest positive eigenvalue of $(-L_\gamma^+)$ and choose positive numbers κ, μ such that $[\kappa - \mu, \kappa + \mu] \subset (0, \lambda_*)$. We recall that the spectrum of $(-L_\gamma)$ consists of real eigenvalues, so that the strip

$[\kappa - \mu \leqq \operatorname{Re} z \leqq \kappa + \mu]$ does not contain any spectral values of $(-L_\gamma)$. Therefore, there exists a constant $M \geqq 1$ such that

$$|e^{-L_\gamma^- t}| \leqq M e^{(\kappa-\mu)t}, \quad |e^{L_\gamma^+ t}| \leqq M e^{-(\kappa+\mu)t}, \quad t \geqq 0. \tag{5.33}$$

Suppose now, by contradiction, that the equilibrium 0 is stable for (5.1). Then for every $r > 0$ there is a number $\delta > 0$ such that (5.1) admits a global solution $z \in \mathbb{E}(\mathbb{R}_+)$ with $|z(t)| \leqq r$ for all $t \geqq 0$ whenever $z_0 \in \bar{B}_\delta(0)$.

In the following we will use the decomposition $z = \tilde{z} + \bar{z}$, where $(\tilde{z}, \bar{z})$ is the solution of the linear system (5.30). (The function $z = \tilde{z} + \bar{z}$ is known, so that the first equation has a unique solution $\bar{z}$. With $\bar{z}$ determined, $\tilde{z} = z - \bar{z}$ is the unique solution of the second equation.) The functions $P^\pm \tilde{z}$ satisfy

$$\frac{d}{dt} P^\pm \tilde{z} + L_\gamma^\pm P^\pm \tilde{z} = P^\pm \omega \bar{z}, \quad P^\pm \tilde{z}(0) = P^\pm \tilde{z}_0. \tag{5.34}$$

Next we shall show that $P^+ \tilde{z}$ is given by the formula

$$P^+ \tilde{z}(t) = -\int_t^\infty e^{-L_\gamma^+ (t-\tau)} P^+ \omega \bar{z} \, d\tau, \quad t \geqq 0. \tag{5.35}$$

Given any $a > 0$ it follows from $|P^+ \tilde{z}(t)|_{X_\gamma^+} \leqq r$ that

$$\|e^{-\kappa t} P^+ \tilde{z}\|_{L_p([0,a]; X_\gamma^+)} \leqq r \left(\int_0^a e^{-\kappa p t} \, dt \right)^{1/p} \leqq C(\kappa, p) r. \tag{5.36}$$

From the relation

$$\frac{d}{dt} e^{-\kappa t} P^+ \tilde{z} = (-\kappa - L_\gamma^+) e^{-\kappa t} P^+ \tilde{z} + e^{-\kappa t} P^+ \omega \bar{z}, \tag{5.37}$$

(5.36)–(5.37) and (5.32) follows

$$\|e^{-\kappa t} P^+ \tilde{z}\|_{\mathbb{X}(a)} \leqq C_1 \big(r + \|e^{-\kappa t} \bar{z}\|_{\mathbb{E}(a)} \big), \tag{5.38}$$

with a universal constant C_1. Here $\mathbb{X}(a)$ is defined as in (5.17), with the difference that $\mathbb{R}_+$ is replaced by the interval $[0, a]$ and $\delta = 0$. We also recall that X_γ^+ is finite dimensional, so that the spaces X_γ^+ and $D(L_\gamma^+)$ coincide (and therefore carry equivalent norms). From semigroup theory, maximal regularity, (5.33)–(5.34) and (5.32), it follows that

$$\begin{aligned} \|e^{-\kappa t} P^- \tilde{z}\|_{\mathbb{X}(a)} &\leqq M \big(|P^- \tilde{z}_0| + \|e^{-\kappa t} P^- \omega \bar{z}\|_{L_p([0,a]; X_\gamma)} \big) \\ &\leqq M \big(|P^- \tilde{z}_0| + C_2 \|e^{-\kappa t} \bar{z}\|_{\mathbb{E}(a)} \big). \end{aligned} \tag{5.39}$$

Combining (5.38)–(5.39) results in

$$\|e^{-\kappa t} \tilde{z}\|_{\mathbb{X}(a)} \leqq C_3 \big(r + |P^- \tilde{z}_0| + \|e^{-\kappa t} \bar{z}\|_{\mathbb{E}(a)} \big), \tag{5.40}$$

where C_3 is a universal constant. Similarly to part (a), we can infer from the equation for $\tilde{z}$ that

$$\|e^{-\kappa t} \tilde{z}\|_{\mathbb{E}(a)} \leqq c (\|e^{-\kappa t} \tilde{z}\|_{\mathbb{X}(a)} + \|e^{-\kappa t} \bar{z}\|_{\mathbb{E}(a)}), \tag{5.41}$$

and this implies

$$\|\mathrm{e}^{-\kappa t}\tilde{z}\|_{\mathbb{E}(a)} \leqq C_4\big(r + |P^-\tilde{z}_0| + \|\mathrm{e}^{-\kappa t}\bar{z}\|_{\mathbb{E}(a)}\big) \tag{5.42}$$

with $C_4 = c(1 + C_3)$.

On the other hand, we obtain from the equation for $\bar{z}$ and (5.31)

$$\begin{aligned}\|\mathrm{e}^{-\kappa t}\bar{z}\|_{\mathbb{E}(a)} &\leqq \bar{C}\big(|\phi(\tilde{z}_0)| + \|\mathrm{e}^{-\kappa t}N(\tilde{z}+\bar{z})\|_{\mathbb{E}(a)}\big)\\ &\leqq \bar{C}\big(|\phi(\tilde{z}_0)| + \eta(r)(\|\mathrm{e}^{-\kappa t}\tilde{z}\|_{\mathbb{E}(a)} + \|\mathrm{e}^{-\kappa t}\bar{z}\|_{\mathbb{E}(a)})\big).\end{aligned}$$

If r is chosen small enough such that $\bar{C}\eta(r) \leqq 1/2$ then

$$\|\mathrm{e}^{-\kappa t}\bar{z}\|_{\mathbb{E}(a)} \leqq 2\bar{C}\big(|\phi(\tilde{z}_0)| + \eta(r)\|\mathrm{e}^{-\kappa t}\tilde{z}\|_{\mathbb{E}(a)}\big). \tag{5.43}$$

We can, at last, combine (5.42)–(5.43) to the result

$$\|\mathrm{e}^{-\kappa t}\tilde{z}\|_{\mathbb{E}(a)} + \|\mathrm{e}^{-\kappa t}\bar{z}\|_{\mathbb{E}(a)} \leqq C_5\big(r + |P^-\tilde{z}_0| + |\phi(\tilde{z}_0)|\big), \tag{5.44}$$

provided r is chosen small enough so that $2(1 + C_4)\bar{C}\eta(r) \leqq 1/2$. Since all estimates are independent of a we conclude that $\mathrm{e}^{-\kappa t}z \in \mathbb{E}(\mathbb{R}_+)$. From (5.44) and Hölder's inequality it follows that

$$\begin{aligned}&\mathrm{e}^{-\kappa t}\int_t^\infty |\mathrm{e}^{-L_\gamma^+(t-\tau)}P^+\omega\bar{z}(\tau)|_{X_\gamma^+}\,\mathrm{d}\tau\\ &\quad\leqq M\left(\int_t^\infty \mathrm{e}^{\mu p'(t-\tau)}\,\mathrm{d}\tau\right)^{1/p'}\|\mathrm{e}^{-\kappa\tau}\omega\bar{z}\|_{L_p(\mathbb{R}_+:X_\gamma)} \leqq C\|\mathrm{e}^{-\kappa t}\bar{z}\|_{\mathbb{E}(\mathbb{R}_+)} < \infty,\end{aligned}$$

thus showing that the integral $\int_t^\infty \mathrm{e}^{-L_\gamma^+(t-\tau)}P^+\omega\bar{z}\,\mathrm{d}\tau$ exists in X_γ^+ for every $t \geqq 0$. Moreover, its norm in X_γ^+ grows no faster than the exponential function $\mathrm{e}^{\kappa t}$.

It follows from the variation of parameters formula that

$$\mathrm{e}^{L_\gamma^+ t}\left(P^+\tilde{z}(t) + \int_t^\infty \mathrm{e}^{-L_\gamma^+(t-\tau)}P^+\omega\bar{z}(\tau)\,\mathrm{d}\tau\right) = P^+\tilde{z}_0 + \int_0^\infty \mathrm{e}^{L_\gamma^+\tau}P^+\omega\bar{z}(\tau)\,\mathrm{d}\tau,$$

and the estimate

$$\left|\mathrm{e}^{L_\gamma^+ t}(P^+\tilde{z}(t) + \int_t^\infty \mathrm{e}^{-L_\gamma^+(t-\tau)}P^+\omega\bar{z}\,\mathrm{d}\tau)\right|_{X_\gamma^+} \leqq M\mathrm{e}^{-(\kappa+\mu)t}(r + Ce^{\kappa t}), \quad t \geqq 0,$$

then shows that $P^+\tilde{z}_0 + \int_0^\infty \mathrm{e}^{L_\gamma^+\tau}P^+\omega\bar{z}\,\mathrm{d}\tau = 0$. Thus the representation (5.35) holds as claimed. With this established, we obtain from Young's inequality for convolution integrals

$$\|\mathrm{e}^{-\kappa t}P^+\tilde{z}(t)\|_{L_p(\mathbb{R}_+,X_\gamma^+)} \leqq M\mu^{-1}\|\mathrm{e}^{-\kappa t}P^+\omega\bar{z}\|_{L_p(\mathbb{R}_+;X_\gamma^+)}.$$

It then follows from (5.32) and (5.37) that

$$\|\mathrm{e}^{-\kappa t}P^+\tilde{z}\|_{\mathbb{X}(\mathbb{R}_+)} \leqq C\|\mathrm{e}^{-\kappa t}\bar{z}\|_{\mathbb{E}(\mathbb{R}_+)}. \tag{5.45}$$

We can now imitate the estimates in (5.39)–(5.43), with the interval $[0, a]$ replaced by $\mathbb{R}_+$, to conclude that

$$\|e^{-\kappa t}\tilde{z}\|_{\mathbb{E}(\mathbb{R}_+)} + \|e^{-\kappa t}\bar{z}\|_{\mathbb{E}(\mathbb{R}_+)} \leqq C_6\big(|P^-\tilde{z}_0| + |\phi(\tilde{z}_0)|\big). \tag{5.46}$$

This, in combination with (5.33), (5.35), and Hölder's inequality, yields the estimate

$$\begin{aligned}|P^+\tilde{z}_0|_{X_\gamma^+} &\leqq M\int_0^\infty e^{-\mu\tau}|e^{-\kappa\tau}P^+\omega\bar{z}|_{X_\gamma^+}\,d\tau\\ &\leqq C\|e^{-\kappa t}P^+\omega\bar{z}\|_{L_p(\mathbb{R}_+;X_\gamma^+)} \leqq C\big(|P^-\tilde{z}_0| + |\phi(\tilde{z}_0)|\big).\end{aligned}$$

By decreasing δ if necessary, we can assume that $C|\phi(\tilde{z}_0)| \leqq 1/2(|P^+\tilde{z}_0|+|P^-\tilde{z}_0|)$ for all $\tilde{z}_0 \in B_\delta(0)$. (Recall that $\phi(0) = \phi'(0) = 0$.) Hence

$$|P^+\tilde{z}_0|_{\tilde{Z}_\gamma} \leqq C_7|P^-\tilde{z}_0|_{\tilde{Z}_\gamma}, \quad \tilde{z}_0 \in B_\delta(0), \tag{5.47}$$

with a uniform constant C_7, and this shows that 0 cannot be stable for (5.1).

It remains to show the last assertion of Theorem 5.2(b). For this we consider the projection $P^u = I - P^c - P^s$, which projects onto X_γ^u, the unstable subspace of X_γ associated with the (finitely many) unstable eigenvalues. As in part (a) we will show that there exists an equilibrium z_∞ such that any solution that stays in a small neighborhood of 0 converges to $z_\infty = z_\infty(z_0)$ exponentially fast as $t \to \infty$. Using the decomposition $\mathsf{y} = \mathsf{y}_s + \mathsf{y}_u$, we obtain as in (a) the following system of equations:

$$\begin{cases}\dot{\mathsf{x}} = P^c\omega\bar{z}, & \mathsf{x}(0) = \mathsf{x}_0 - \mathsf{x}_\infty,\\ \dot{\mathsf{y}}_s + L_\gamma^s\,\mathsf{y}_s = S_s(\mathsf{x}_\infty, \mathsf{x}, \bar{z}), & \mathsf{y}_s(0) = \mathsf{y}_0^s,\\ \dot{\mathsf{y}}_u + L_\gamma^u\,\mathsf{y}_u = S_u(\mathsf{x}_\infty, \mathsf{x}, \bar{z}), & \mathsf{y}_u(0) = \mathsf{y}_0^u,\end{cases} \tag{5.48}$$

with

$$S_j(\mathsf{x}_\infty, \mathsf{x}, \bar{z}) = P^j\omega\bar{z} - \psi_j'(\mathsf{x}_\infty + \mathsf{x})P^c\omega\bar{z} - L_\gamma^j[\psi_j(\mathsf{x}_\infty + \mathsf{x}) - \psi_j(\mathsf{x}_\infty)],$$

where $j \in \{s, u\}$, and where the functions ψ_j are defined similarly as in (5.13).

Suppose we have a global solution $z \in \mathbb{E}(\mathbb{R}_+)$ of (5.1) with $z(0) = z_0 \in \mathcal{SM}_\gamma$ which satisfies $|z|_{\tilde{Z}_\gamma} \leqq r$, where $r > 0$ is sufficiently small. By arguments similar to those above (the presence of the function S_u does not cause any principal difficulties), we infer that

$$\mathsf{y}_u(t) = -\int_t^\infty e^{-L_\gamma^u(t-\tau)}S_u(\mathsf{x}_\infty, \mathsf{x}, \bar{z})\,d\tau, \quad t \geqq 0. \tag{5.49}$$

For $(\mathsf{x}_0, \mathsf{y}_0^s, \bar{z}) \in \mathbb{B}_1(r_0, \delta)$, with r_0 sufficiently small, we set

$$\begin{aligned}\mathsf{x}_\infty &:= \mathsf{x}_0 + \int_0^\infty P^c\omega\bar{z}(\tau)\,d\tau,\\ \mathsf{x}(t) &:= -\int_t^\infty P^c\omega\bar{z}(\tau)\,d\tau,\\ \mathsf{y}_s &:= \left(\frac{d}{dt} + L_\gamma^s, \mathrm{tr}\right)^{-1}\big(S_s(\mathsf{x}_\infty, \mathsf{x}, \bar{z}), \mathsf{y}_0^s\big),\\ \mathsf{y}_u(t) &:= -\int_t^\infty e^{-L_\gamma^u(t-\tau)}S_u(\mathsf{x}_\infty, \mathsf{x}, \bar{z})\,d\tau.\end{aligned} \tag{5.50}$$

As in part (a) we conclude that (5.50) admits for each $(\mathsf{x}_0, \mathsf{y}_0^s, \bar{z}) \in \mathbb{B}_1(r_0, \delta)$, with r_0 sufficiently small, a unique solution

$$(\mathsf{x}_\infty, \mathsf{x}, \mathsf{y}_s, \mathsf{y}_u) = \mathfrak{S}(\mathsf{x}_0, \mathsf{y}_0^s, \bar{z}) \in X_\gamma^c \times \mathbb{X}^c(\mathbb{R}_+, \delta) \times \mathbb{X}^s(\mathbb{R}_+, \delta) \times \mathbb{X}^u(\mathbb{R}_+, \delta).$$

Following the arguments of part (a) then renders a solution

$$\mathfrak{Z}(\mathsf{x}_0, \mathsf{y}_0^s) = z_\infty + \mathsf{x} + \psi(\mathsf{x} + \mathsf{x}_\infty) - \psi(\mathsf{x}_\infty) + \mathsf{y}_s + \mathsf{y}_u + \bar{z}$$

of (5.1) with $z_0 = \mathsf{x}_0 + \psi(\mathsf{x}_0) + \mathsf{y}_0^u + \mathsf{y}_0^s + \phi(\mathsf{x}_0 + \psi(\mathsf{x}_0) + \mathsf{y}_0^u + \mathsf{y}_0^s)$, where y_0^u is determined by

$$\mathsf{y}_0^u = -\int_0^\infty \mathrm{e}^{L_\gamma^u \tau} S_u(\mathsf{x}_\infty, \mathsf{x}, \bar{z})\, \mathrm{d}\tau. \tag{5.51}$$

The solution $\mathfrak{Z}(\mathsf{x}_0, \mathsf{y}_0^s)$ converges exponentially fast toward the equilibrium z_∞. In addition, we have shown that the initial value z_0 necessarily lies on the stable manifold belonging to z_∞, determined by the relation (5.51).

Due to uniqueness of (local) solutions to (5.1), the solution $\mathfrak{Z}(\mathsf{x}_0, \mathsf{y}_0^s)$ coincides with the given global solution z, and the proof of part (b) is now complete. □

Global existence and convergence. There are several obstructions against global existence for the Stefan problem (1.3):

- *regularity*: the norms of either $u(t)$, $\Gamma(t)$, and, in addition, $[\![\mathrm{d}\partial_\nu u(t)]\!]$ in the case where $\gamma \equiv 0$, become unbounded;
- *well-posedness*: in the case where $\gamma \equiv 0$ the well-posedness condition $l(u) \neq 0$ may become violated; or u may become 0;
- *geometry*: the topology of the interface changes;
  or the interface touches the boundary of Ω;
  or the interface contracts to a point.

Note that the compatibility conditions $[\![\psi(u)]\!] + \sigma\mathcal{H} = 0$ in the case where $\gamma \equiv 0$, and

$$(l(u) - [\![\psi(u)]\!] - \sigma\mathcal{H})([\![\psi(u)]\!] + \sigma\mathcal{H}) = \gamma(u)[\![\mathrm{d}\partial_\nu u]\!]$$

in the case where $\gamma > 0$, are preserved by the semiflow.

Let (u, Γ) be a solution in the state manifold $\mathcal{SM}_\gamma$. By a *uniform ball condition* we mean the existence of a radius $r_0 > 0$ such that for each t, at each point $x \in \Gamma(t)$ there exist centers $x_i \in \Omega_i(t)$ such that $B_{r_0}(x_i) \subset \Omega_i$ and $\Gamma(t) \cap \bar{B}_{r_0}(x_i) = \{x\}$, $i = 1, 2$. Note that this condition bounds the curvature of $\Gamma(t)$, prevents it from shrinking to a point, from touching the outer boundary $\partial\Omega$, and from undergoing topological changes.

With this property, combining the semiflow for (1.3) with the Lyapunov functional and compactness, we obtain the following result.

Theorem 5.3. *Let $p > n + 2$, $\sigma > 0$, suppose $\psi, \gamma \in C^3(0, \infty)$, $d \in C^2(0, \infty)$ such that either $\gamma \equiv 0$ or $\gamma(u) > 0$ on $(0, \infty)$, and assume*

$$\kappa(u) = -u\psi''(u) > 0, \quad \mathrm{d}(u) > 0, \quad u \in (0, \infty).$$

Suppose that (u, Γ) is a solution of (1.3) in the state manifold $\mathcal{SM}_\gamma$ on its maximal time interval $[0, t_*)$. Assume the following on $[0, t_*)$:

(i) $|u(t)|_{W_p^{2-2/p}} + |\Gamma(t)|_{W_p^{4-3/p}} \leqq M < \infty$;
(ii) $|[\![d(u(t))\partial_\nu u(t)]\!]|_{W_p^{2-6/p}} \leqq M < \infty$ *in case* $\gamma \equiv 0$;
(iii) $|l(u(t))| \geqq 1/M$ *in case* $\gamma \equiv 0$;
(iv) $u(t) \geqq 1/M$;
(v) $\Gamma(t)$ *satisfies a uniform ball condition.*

Then $t_* = \infty$, *that is, the solution exists globally. If its limit set contains a stable equilibrium* $(u_\infty, \Gamma_\infty) \in \mathcal{E}$, *that is,* $\varphi'(u_\infty) < 0$, *then it converges in* $\mathcal{SM}_\gamma$ *to this equilibrium. On the contrary, if* $(u(t), \Gamma(t))$ *is a global solution in* $\mathcal{SM}_\gamma$ *which converges to an equilibrium* (u_*, Γ_*) *with* $l(u_*) \neq 0$ *in case* $\gamma \equiv 0$ *in* $\mathcal{SM}_\gamma$ *as* $t \to \infty$, *then properties* (i)-(v) *are valid.*

Proof. Assume that assertions (i)–(v) are valid. Then $\Gamma([0, t_*)) \subset W_p^{4-3/p}(\Omega, r)$ is bounded, hence relatively compact in $W_p^{4-3/p-\varepsilon}(\Omega, r)$. (See (3.7) for the definition of $W_p^s(\Omega, r)$.)

Thus we may cover this set by finitely many balls with centers Σ_k real analytic in such a way that $\text{dist}_{W_p^{4-3/p-\varepsilon}}(\Gamma(t), \Sigma_j) \leqq \delta$ for some $j = j(t), t \in [0, t_*)$. Let $J_k = \{t \in [0, t_*) : j(t) = k\}$. Using for each k a Hanzawa-transformation Ξ_k, we see that the pull backs $\{u(t, \cdot) \circ \Xi_k : t \in J_k\}$ are bounded in $W_p^{2-2/p}(\Omega \setminus \Sigma_k)$, hence relatively compact in $W_p^{2-2/p-\varepsilon}(\Omega \setminus \Sigma_k)$. Then, employing Corollary 3.10, we obtain solutions (u^1, Γ^1) with initial configurations $(u(t), \Gamma(t))$ in the state manifold on a common time interval, say $(0, \tau]$, and by uniqueness we have

$$(u^1(\tau), \Gamma^1(\tau)) = (u(t + \tau), \Gamma(t + \tau)).$$

Continuous dependence, then, implies relative compactness of $(u(\cdot), \Gamma(\cdot))$ in $\mathcal{SM}_\gamma$. In particular, $t_* = \infty$ and the orbit $(u, \Gamma)(\mathbb{R}_+) \subset \mathcal{SM}_\gamma$ is relatively compact. The negative total entropy is a strict Lyapunov functional, hence the limit set $\omega(u, \Gamma) \subset \mathcal{SM}_\gamma$ of a solution is contained in the set $\mathcal{E}$ of equilibria. By compactness, $\omega(u, \Gamma) \subset \mathcal{SM}_\gamma$ is non-empty, hence the solution comes close to $\mathcal{E}$ and stays there; then we may apply the convergence result Theorem 5.2. The converse is proved by a compactness argument. □

Remark 5.4. We believe that the extra assumption $\varphi'(u_\infty) < 0$ in Theorem 5.3 can be replaced by $\varphi'(u_\infty) \neq 0$. However, to prove this requires more technical effort, and we refrain from doing this here.

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References

1. Alikakos, N., Fusco, G.: Ostwald ripening for dilute systems under quasistationary dynamics. *Commun. Math. Phys.* **238**, 429–479 (2003)
2. Alikakos, N., Fusco, G., Karali, G.: The effect of the geometry of the particle distribution in Ostwald ripening. *Commun. Math. Phys.* **238**, 481–488 (2003)

3. ALIKAKOS, N., FUSCO, G., KARALI, G.: Ostwald ripening in two dimensions-the rigorous derivation of the equations from the Mullins–Sekerka dynamics. *J. Differ. Equ.* **205**, 1–49 (2004)
4. ALIKAKOS, N., FUSCO, G., KARALI, G.: Continuum limits of particles interacting via diffusion. *Abstr. Appl. Anal.* **2004**, 215–237 (2004)
5. ALMGREN, F., WANG, L.: Mathematical existence of crystal growth with Gibbs--Thomson curvature effects. *J. Geom. Anal.* **10**, 1–100 (2000)
6. ANDERSON, D.M., CERMELLI, P., FRIED, E., GURTIN, M.E., MCFADDEN, G.B.: General dynamical sharp-interface conditions for phase transformations in viscous heat-conducting fluids. *J. Fluid Mech.* **581**, 323–370 (2007)
7. ATHANASOPOULOS, I., CAFFARELLI, L., SALSA, S.: Regularity of the free boundary in parabolic phase-transition problems. *Acta Math.* **176**, 245–282 (1996)
8. ATHANASOPOULOS, I., CAFFARELLI, L., SALSA, S.: Phase transition problems of parabolic type: flat free boundaries are smooth. *Comm. Pure Appl. Math* **51**, 77–112 (1998)
9. BAZALII, B.V.:Stefan problem for the Laplace equation with regard to the curvature of the free boundary. *Ukrainian Math. J.* **49**, 1465–1484 (1997)
10. CAFFARELLI, L.A.: The regularity of free boundaries in higher dimensions. *Acta Math.* **139**, 155–184 (1977)
11. CAFFARELLI, L.A.: Some aspects of the one-phase Stefan problem. *Indiana Univ. Math. J.* **27**, 73–77 (1978)
12. CAFFARELLI, L.A., EVANS, L.C.: Continuity of the temperature in the two-phase Stefan problem. *Arch. Rational Mech. Anal.* **81**, 199–220 (1983)
13. CAFFARELLI, L.A., FRIEDMAN, A.: Continuity of the temperature in the Stefan problem. *Indiana Univ. Math. J.* **28**, 53–70 (1979)
14. CAGINALP, G.: An analysis of a phase field model of a free boundary. *Arch. Rational Mech. Anal.* **92**, 205–245 (1986)
15. CHALMERS, B.: *Principles of solidification*. Krieger, Huntington (1977)
16. CHEN, X.: The Hele–Shaw problem and area-preserving curve-shortening motion. *Arch. Rational Mech. Anal.* **123**, 117–151 (1993)
17. CHEN, X., HONG, J., YI, F.: Existence, uniqueness, and regularity of classical solutions of the Mullins–Sekerka problem. *Comm. Partial Differ. Equ.* **21**, 1705–1727 (1996)
18. CHEN, X., JONES, J., TROY, W.: Linear stability of a solid ball in an undercooled liquid. *J. Math. Anal. Appl.* **193**, 859–888 (1995)
19. CHEN, X., REITICH, F.: Local existence and uniqueness of solutions of the Stefan problem with surface tension and kinetic undercooling. *J. Math. Anal. Appl.* **164**, 350–362 (1992)
20. DENK, R., HIEBER, M., PRÜSS, J.: *$\mathcal{R}$-boundedness, Fourier multipliers, and problems of elliptic and parabolic type*, AMS Memoirs 788, Providence (2003)
21. DENK, R., PRÜSS, J., ZACHER, R.: Maximal L_p-regularity of parabolic problems with boundary conditions of relaxation type. *J. Funct. Anal.* **255**, 3149–3187 (2008)
22. DIBENEDETTO, E.: Regularity properties of the solution of an n-dimensional two-phase Stefan problem. *Boll. Un. Mat. Ital.* **Suppl.** 129–152 (1980)
23. DIBENEDETTO, E.: Continuity of weak solutions to certain singular parabolic equations. *Ann. Mat. Pura Appl. (4)* **130**, 131–176 (1982)
24. ESCHER, J., SIMONETT, G.: On Hele–Shaw models with surface tension. *Math. Res. Lett.* **3**, 467–474 (1996)
25. ESCHER, J., SIMONETT, G.: Classical solutions for the quasi-stationary Stefan problem with surface tension. *Differential equations, asymptotic analysis, and mathematical-physics*, (Potsdam, 1996), Math. Res., vol. 100, Akademie Verlag, Berlin, pp. 98–104, 1997
26. ESCHER, J., SIMONETT, G.: Classical solutions for Hele–Shaw models with surface tension. *Adv. Differ. Equ.* **2**, 619–642 (1997)
27. ESCHER, J., SIMONETT, G.: A center manifold analysis for the Mullins–Sekerka model. *J. Differ. Equ.* **143**, 267–292 (1998)
28. ESCHER, J., PRÜSS, J., SIMONETT, G.: Analytic solutions for a Stefan problem with Gibbs–Thomson correction. *J. Reine Angew. Math.* **563**, 1–52 (2003)

29. Friedman, A.: The Stefan problem in several space variables. *Trans. Am. Math. Soc.* **133**, 51–87 (1968)
30. Friedman, A.: *Variational Principles and Free-Boundary Problems*. Wiley, New York, 1982
31. Friedman, A., Kinderlehrer, D.: A one phase Stefan problem. *Indiana Univ. Math. J.* **24**, 1005–1035 (1975)
32. Friedman, A., Reitich, F.: The Stefan problem with small surface tension. *Trans. Am. Math. Soc.* **328**, 465–515 (1991)
33. Friedman, A., Reitich, F.: Nonlinear stability of a quasi-static Stefan problem with surface tension: a continuation approach. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **30**, 341–403 (2001)
34. Garcke, H., Sturzenhecker, T.: The degenerate multi-phase Stefan problem with Gibbs–Thomson law. *Adv. Math. Sci. Appl.* **8**, 929–941 (1998)
35. Glasner, K., Otto, F., Rump, T., Slepcev, D.: Ostwald ripening of droplets: the role of migration. *Eur. J. Appl. Math.* **20**, 1–67 (2009)
36. Gurtin, M.E.: On the two phase problem with interfacial energy and entropy. *Arch. Rational Mech. Anal.* **96**, 199–241 (1986)
37. Gurtin, M.E.: Toward a nonequilibrium thermodynamics of two-phase materials. *Arch. Rational Mech. Anal.* **100**, 275–312 (1988)
38. Gurtin, M.E.: Multiphase thermomechanics with interfacial structure. I. Heat conduction and the capillary balance law. *Arch. Rational Mech. Anal.* **104**, 195–221 (1988)
39. Hadzić, M., Guo, Y.: Stability in the Stefan problem with surface tension (I). *Comm. Partial Differ. Equ.* **35**, 201–244 (2010)
40. Hadzić, M.: Orthogonality conditions and asymptotic stability in the Stefan problem with surface tension. *Arch. Rational Mech. Anal.* **203**, 719–745 (2012)
41. Hanzawa, E.: Classical solutions of the Stefan problem. *Tôhoku Math. J. (2)* **33**, 297–335 (1981)
42. Hartman, P.: *Crystal growth: An introduction*. North-Holland, Amsterdam, 1973
43. Hönig, A., Niethammer, B., Otto, F.: On first-order corrections to the LSW theory. II. Finite systems. *J. Stat. Phys.* **119**, 123–164 (2005)
44. Ishii, M., Takashi, H.: *Thermo-fluid dynamics of two-phase flow.* Springer, New York, 2006
45. Kamenomostskaja, S.L.: On Stefan's problem. *Math. Sbornik* **53**, 485–514 (1965)
46. Kinderlehrer, D., Nirenberg, L.: Regularity in free boundary problems. *Ann. Scuola Norm. Sup. Pisa (4)* **4**, 373–391 (1977)
47. Kinderlehrer, D., Nirenberg, L.: The smoothness of the free boundary in the one phase Stefan problem. *Commun. Pure Appl. Math.* **31**, 257–282 (1978)
48. Kneisel, C.: Über das Stefan-Problem mit Oberflächenspannung und thermischer Unterkühlung. *PhD thesis*, University of Hannover, Germany, 2007
49. Knüpfer, H., Masmoudi, N.: Well-posedness and uniform bounds for a nonlocal third order evolution operator on an infinite wedge (2012) (Preprint)
50. Knüpfer, H., Masmoudi, N.: Darcy's flow with prescribed contact angle: Well-posedness and lubrication approximation. *arXiv:1204.2278*
51. Ladyženskaja, O.A., Solonnikov, V.A., Ural'ceva, N.N.: *Linear and quasilinear equations of parabolic type*. Translations of Mathematical Monographs, vol. 23, American Mathematical Society, Providence, 1968
52. Langer, J.S.: Instabilities and pattern formation in crystal growth. *Rev. Mod. Phys.* **52**, 1–28 (1980)
53. Latushkin, Y., Prüss, J., Schnaubelt, R.: Stable and unstable manifolds for quasilinear parabolic systems with fully nonlinear boundary conditions. *J. Evol. Equ.* **6**, 537–576 (2006)
54. Luckhaus, S.: Solutions for the two-dimensional Stefan problem with the Gibbs–Thomson law for melting temperature. *Eur. J. Appl. Math.* **1**, 101–111 (1990)
55. Luckhaus, S.: The Stefan problem with surface tension. *Variational and free boundary problems*, pp. 153–157, IMA Math. Appl., vol. 53, Springer, New York, 1993

56. LUCKHAUS, S., STURZENHECKER, T.: Implicit time discretization for the mean curvature flow equation. *Calc. Var. Partial Differ. Equ.* **3**, 253–271 (1995)
57. MEIRMANOV, A.M.: On the classical solution of the multidimensional Stefan problem for quasilinear parabolic equations. *Math. Sb.* **112**, 170–192 (1980)
58. MEIRMANOV, A.M.: *The Stefan Problem*, De Gruyter, Berlin, 1992
59. MEIRMANOV, A.M.: The Stefan problem with surface tension in the three dimensional case with spherical symmetry: non-existence of the classical solution. *Eur. J. Appl. Math.* **5**, 1–20 (1994)
60. MEYRIES, M. *Maximal regularity in weighted spaces, nonlinear boundary conditions, and global attractors.* PhD thesis, Karlsruhe Institute of Technology, Germany, 2010
61. MEYRIES, M., SCHNAUBELT, R.: Maximal regularity with temporal weights for parabolic problems with inhomogeneous boundary conditions (2012) (Preprint)
62. Matano, H.: Asymptotic behavior of the free boundaries arising in one phase Stefan problems in multi-dimensional spaces. *Lect. Notes Num. Appl. Anal.* **5**, 133–151 (1982)
63. MULLINS, W.W., SEKERKA, R.F.: Morphological stability of a particle growing by diffusion or heat flow. *J. Appl. Phys.* **34**, 323–329 (1963)
64. MULLINS, W.W., SEKERKA, R.F.: Stability of a planar interface during solidification of a dilute binary alloy. *J. Appl. Phys.* **35**, 444–451 (1964)
65. NIETHAMMER, B.: Derivation of the LSW theory for Ostwald ripening by homogenization methods. *Arch. Rational Mech. Anal.* **147**, 119–178 (1999)
66. NIETHAMMER, B.: The LSW model for Ostwald ripening with kinetic undercooling. *Proc. R. Soc. Edinburgh Sect. A* **130**, 1337–1361 (2000)
67. NIETHAMMER, B.: The mathematics of Ostwald ripening. *Geometric analysis and nonlinear partial differential equations,* pp. 649–663, Springer, Berlin, 2003
68. NIETHAMMER, B.: The influence of encounters on domain coarsening [Plenary lecture presented at the 80th Annual GAMM Conference, Gdańsk, 9th-13th February 2009]. *ZAMM Z. Angew. Math. Mech.* **90**, 259–270 (2010)
69. NIETHAMMER, B., OTTO, F.: Domain coarsening in thin films. *Comm. Pure Appl. Math.* **54**, 361–384 (2001)
70. NIETHAMMER, B., OTTO, F.: Ostwald ripening: the screening length revisited. *Calc. Var. Partial Differ. Equ.* **13**, 33–68 (2001)
71. NIETHAMMER, B., PEGO, R.L.: On the initial-value problem in the Lifshitz–Slyozov–Wagner theory of Ostwald ripening. *SIAM J. Math. Anal.* **31**, 467–485 (2000)
72. NIETHAMMER, B., VELÁZQUEZ, J.J.L.: Homogenization in coarsening systems. I. Deterministic case. *Math. Models Methods Appl. Sci.***14**, 1211–1233 (2004)
73. NOCHETTO, R.H.: A class of nondegenerate two-phase Stefan problems in several space variables. *Comm. Partial Differ. Equ.* **12**, 21–45 (1987)
74. PRÜSS, J.: Maximal regularity for evolution equations in L_p-spacess. *Conf. Sem. Mat. Univ. Bari* **285**, 1–39 (2003)
75. PRÜSS, J., SAAL, J., SIMONETT, G.: Existence of analytic solutions or the classical Stefan problem. *Math. Ann.* **338**(3), 703–755 (2007)
76. PRÜSS, J., SHIBATA, Y., SHIMIZU, S., SIMONETT, G.: On the well-posedness of incompressible two-phase flows with phase transitions: the case of equal densities. *Evol. Eqns. Control Theory* **1**, 171–194 (2012)
77. PRÜSS, J., SHIMIZU, S.: On the well-posedness of incompressible two-phase flows with phase transitions: the case of different densities. *J. Evol. Equ.* (2012)(to appear)
78. PRÜSS, J., SIMONETT, G.: Maximal regularity for evolution equations in weighted L_p-spaces. *Archiv Math.* **82**, 415–431 (2004)
79. PRÜSS, J., SIMONETT, G.: Stability of equilibria for the Stefan problem with surface tension. *SIAM J. Math. Anal.* **40**, 675–698 (2008)
80. PRÜSS, J., SIMONETT, G., WILKE, M.: On thermodynamically consistent Stefan problems with variable surface energy. *arXiv:1109.4542.* (Submitted)
81. PRÜSS, J., SIMONETT, G., ZACHER, R.: Convergence of solutions to equilibria for nonlinear parabolic problems. *J. Differ. Equ.* **246**, 3902–3931 (2009)

82. Prüss, J., Simonett, G., Zacher, R.: On normal stability for nonlinear parabolic problems. Discrete Contin. Dyn. Syst. *Dynamical Systems, Differential Equations and Applications*. 7th AIMS Conference, suppl., pp. 612–621, 2009
83. Radkevitch, E.: The Gibbs-Thompson correction and conditions for the existence of a classical solution of the modified Stefan problem. *Dokl. Akad. Nauk SSSR* **316** (1991), 1311–1315; translation in *Soviet Math. Dokl.* **43**, 274–278 (1991)
84. Radkevitch, E.: Conditions for the existence of a classical solution of a modified Stefan problem (the Gibbs–Thomson law). *Mat. Sb.* **183** (1992), 77–101; translation in *Russian Acad. Sci. Sb. Math.* **75**, 221–246 (1993)
85. Röger, M.: Solutions for the Stefan problem with Gibbs–Thomson law by a local minimisation. *Interfaces Free Bound.* **6**, 105–133 (2004)
86. Röger, M.: Existence of weak solutions for the Mullins–Sekerka flow. *SIAM J. Math. Anal.* **37**, 291–301 (2005)
87. Rubinstein, L.I.: *The Stefan Problem*. Translations of Mathematical Monographs, vol. 27, American Mathematical Society, Providence, 1971
88. Sacks, P.E.: Continuity of solutions of a singular parabolic equation. *Nonlinear Anal.* **7**, 387–409 (1983)
89. Struwe, M.: Curvature flows on surfaces. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **5**(1), 247–274 (2002)
90. Tao, T.: Why are solitons stable? *Bull. Am. Math. Soc. (N.S.)* **46**, 1–33 (2009)
91. Visintin, A.: Supercooling and superheating effects in phase transitions, *IMA J. Appl. Math.* **35**, 233–256 (1985)
92. Visintin, A.: Stefan problem with surface tension. *Istituto di Analisi Numerica del Consiglio Nazionale delle Ricerche, 424*. Istituto di Analisi Numerica del Consiglio Nazionale delle Ricerche, Pavia, 29 pp, 1984
93. Visintin, A.: Stefan problem with surface tension. Mathematical models for phase change problems (Óbidos, 1988), pp. 191–213. *Internat. Ser. Numer. Math.*, vol. 88, Birkhäuser, Basel, 1989
94. Visintin, A.: Remarks on the Stefan problem with surface tension. In: *Boundary value problems for partial differential equations and applications*. (Eds. Lions, J.L. and Baiocchi, C.) RMA: Research Notes in Appl. Math. vol. 29, Masson, Paris, 1993
95. Visintin, A.: Models of Phase Transitions. *Progress in Nonlinear Differential Equations and Their Applications*, vol. 28. Birkhäuser, Boston, 1996
96. Ziemer, W.P.: Interior and boundary continuity of weak solutions of degenerate parabolic equations. *Trans. Am. Math. Soc.* **271**, 733–748 (1982)
97. Yu, W.: A quasisteady Stefan problem with curvature correction and kinetic undercooling. *J. Partial Differ. Equ.* **9**, 55–70 (1996)

Institut für Mathematik
Martin-Luther-Universität Halle-Wittenberg
Theodor-Lieser-Strasse 5, 06120 Halle, Germany.
e-mail: jan.pruess@mathematik.uni-halle.de

and

e-mail: rico.zacher@mathematik.uni-halle.de

and

Department of Mathematics
Vanderbilt University
Nashville, TN 37240, USA.
e-mail: gieri.simonett@vanderbilt.edu

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