On Stable Local Bases for Bivariate Polynomial Spline Spaces

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Abstract. Stable locally supported bases are constructed for the spaces $\mathcal{S}_d^r(\triangle)$ of polynomial splines of degree $d \geq 3r+2$ and smoothness r defined on triangulations $\triangle$, as well as for various superspline subspaces. In addition, we show that for $r \geq 1$, it is impossible to construct bases which are simultaneously stable and locally linearly independent.

§1. Introduction

This paper deals with the classical space of polynomial splines

$$\mathcal{S}_d^r(\triangle) := \{s \in C^r(\Omega) : \; s|_T \in \mathcal{P}_d \; \text{ for all triangles } T \in \triangle\},$$

where $\mathcal{P}_d$ is the space of polynomials of degree d, and $\triangle$ is a regular triangulation of a polygonal set Ω. We also discuss superspline subspaces of the form

$$\mathcal{S}_d^{r,\rho}(\triangle) := \{s \in \mathcal{S}_d^r(\triangle) : \; s \in C^{\rho_v}(v) \; \text{ for all } v \in \mathcal{V}\}, \tag{1.1}$$

with $\rho := \{\rho_v\}_{v \in \mathcal{V}}$, where ρ_v are given integers such that $r \leq \rho_v \leq d$, and $\mathcal{V}$ is the set of all vertices of $\triangle$.

Our aim is to describe algorithms for constructing locally supported bases $\{B_i\}_{\xi \in \mathcal{I}}$ for these spaces which are stable in the following sense:

$$K_1 \|c\|_\infty \; \leq \; \|\sum_{i \in \mathcal{I}} c_i B_i\|_\infty \; \leq \; K_2 \|c\|_\infty \tag{1.2}$$

for all choices of the coefficient vector $c = (c_i)_{i \in \mathcal{I}}$. We are interested in a construction for which (1.2) holds with constants K_1 and K_2 which depend only on d and the smallest angle $\theta_\triangle$ in the triangulation, and not on the number of triangles or any other property of $\triangle$. Stable bases are of critical importance for both theoretical and practical purposes.

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To date, algorithms which produce stable local bases have been described only for certain very special spline spaces, see Remarks 8.2–8.5. In this paper we give algorithms to construct locally supported stable bases for general superspline spaces $\mathcal{S}_d^{r,\rho}(\triangle)$ with $d \ge 3r+2$. Such bases for the full spline spaces $\mathcal{S}_d^r(\triangle)$ (which correspond to the choice $\rho_v = r$ for all $v \in \mathcal{V}$) are of special importance due to the fact that these spaces are nested for nested triangulations, while most superspline spaces are not, see Remark 8.6.

The paper is organized as follows. In Sect. 2 we introduce some notation, and review the minimal determining set approach to constructing bases for spline spaces. In Sect. 3 we describe for later use a special construction of minimal determining sets for superspline spaces defined on near-singular cells. In Sect. 4 we construct stable local bases for the superspline space $\mathcal{S}_d^{r,\mu}(\triangle)$ defined by choosing $\rho_v = \mu := r + \lfloor \frac{r+1}{2} \rfloor$ for all $v \in \mathcal{V}$. In Sect. 5 we consider the superspline spaces on arbitrary cells. The main result for general superspline spaces $\mathcal{S}_d^{r,\rho}(\triangle)$ is established in Sect. 6, and the connection between stability and local linear independence is explored in Sect. 7. We conclude the paper with several remarks in Sect. 8.

§2. Dual bases and minimal determining sets

The key tool for constructing stable spline bases is the well-known Bernstein-Bézier representation for polynomial splines of degree d on a triangulation $\triangle$ used in almost all of the papers cited below. In particular, we use the fact that there is a 1-1 correspondence between the set of splines $\mathcal{S}_d^0(\triangle)$ and the set of B-coefficients $\{c_\xi\}_{\xi \in \mathcal{D}_{d,\triangle}}$, where

$$\mathcal{D}_{d,\triangle} := \{\xi : \xi = \tfrac{(iu+jv+kw)}{d}, \text{ where } i+j+k=d \text{ and } T := \langle u,v,w\rangle \text{ is a triangle in } \triangle\}$$

are the **domain points** associated with d and $\triangle$. Throughout the paper, whenever we write a triangle $T := \langle u,v,w\rangle$ in terms of its vertices, we assume that u, v, w appear in counterclockwise order.

Suppose $\mathcal{S}$ is a linear subspace of $\mathcal{S}_d^0(\triangle)$, and that $\mathcal{M} \subseteq \mathcal{D} \subseteq \mathcal{D}_{d,\triangle}$. Then $\mathcal{M}$ is said to be a **determining set for $\mathcal{S}$ on $\mathcal{D}$** if setting the coefficients of $s \in \mathcal{S}$ associated with the domain points in $\mathcal{M}$ to zero implies that all coefficients of s corresponding to domain points in $\mathcal{D}$ are zero. $\mathcal{M}$ is called a **minimal determining set (MDS) for $\mathcal{S}$ on $\mathcal{D}$** if 1) given any real numbers $\{c_\xi\}_{\xi\in\mathcal{M}}$, there exists at least one spline $s \in \mathcal{S}$ whose B-coefficients in $\mathcal{M}$ are $\{c_\xi\}_{\xi\in\mathcal{M}}$, and 2) all coefficients associated with domain points in $\mathcal{D}\setminus\mathcal{M}$ are *uniquely* determined by the coefficients of s in $\mathcal{M}$. We say that $\mathcal{M}$ is a **stable MDS for $\mathcal{S}$ on $\mathcal{D}$** if all computed coefficients are bounded by $K \max_{\xi\in\mathcal{M}} |c_\xi|$, where K is a constant dependening only on d and the smallest angle $\theta_\triangle$ in $\triangle$. If $\mathcal{M}$ is a MDS for $\mathcal{S}$ on all of $\mathcal{D}_{d,\triangle}$, we simply call it a minimal determining set.

The algorithms presented here for constructing bases is based on the following well-known (cf. [6]) result:

Algorithm 2.1. *Suppose $\mathcal{M}$ is a MDS for $\mathcal{S}$. For each $\xi \in \mathcal{M}$, construct the unique spline $B_\xi \in \mathcal{S}$ satisfying*

$$\lambda_\eta B_\xi = \delta_{\xi,\eta}, \qquad \text{all } \eta \in \mathcal{M}. \tag{2.1}$$

Then the set $\{B_\xi\}_{\xi \in \mathcal{M}}$ is a basis for $\mathcal{S}$. We call it the dual basis *corresponding to $\mathcal{M}$.*

Discussion: To construct the spline B_ξ, choose $c_\xi = 1$, and set all other coefficients corresponding to $\eta \in \mathcal{M}$ to zero. Then since $\mathcal{M}$ is a MDS, all remaining coefficients can be uniquely computed from smoothness conditions. There are two standard ways to use the smoothness conditions – see e.g. Lemmas 6.1 and 6.2 in [24]. We make use of both of them below. □

For a given spline space $\mathcal{S}$, there are generally many different minimal determining sets $\mathcal{M}$. Our aim in this paper is to design algorithms for *choosing minimal determining sets* which produce stable local bases when applied to a space $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$. Recall that given a vertex v of $\triangle$, $\operatorname{star}(v) = \operatorname{star}^1(v)$ is the union of triangles sharing v, and $\operatorname{star}^\ell(v)$, $\ell \geq 2$, is defined recursively as the union of the stars of the vertices in $\mathcal{V} \cap \operatorname{star}^{\ell-1}(v)$.

Definition 2.2. *We call a MDS for a spline space $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$ a* stable local MDS *provided that the corresponding dual basis $\mathcal{B} := \{B_\xi\}_{\xi \in \mathcal{M}}$ satisfies*

$$\operatorname{supp}(B_\xi) \subseteq \operatorname{star}^\ell(v_\xi) \text{ for some vertex } v_\xi, \tag{2.2}$$

$$\|B_\xi\|_\infty \leq K \tag{2.3}$$

for all $\xi \in \mathcal{M}$, where the constants ℓ and K depend only on d and the smallest angle $\theta_\triangle$ in $\triangle$.

Theorem 2.3. *Suppose that $\mathcal{M}$ is a stable local minimal determining set for a spline space $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$. Then the dual basis $\mathcal{B}$ is a stable locally supported basis for $\mathcal{S}$.*

Proof: Let $s \in \mathcal{S}$. Then for any $\xi \in \mathcal{M}$, the corresponding coefficient c_ξ is the B-coefficient of the polynomial $s_T := s|_T$, where T is a triangle containing ξ. But then by the stability of the Bernstein-Bézier basis for polynomials (cf. Lemma 4.1 of [24]), we have $|c_\xi| \leq C\|s_T\|_{\infty,T} \leq C\|s\|_\infty$, where C is a constant which depends only on d. This establishes the left-hand side of (1.2) with the constant $K_1 := 1/C$ which depends only on d. For the right-hand side, we note that by Lemma 3.1 of [24], for any triangle T, the number of basis splines B_ξ whose supports contain T is bounded by a constant depending only on d and $\theta_\triangle$ (cf. the proof of Theorem 9.2 in [24]). Combining this with the boundedness of the basis functions completes the proof. □

With an appropriate renorming, it can be shown that the dual splines also provide a basis which is L_p-stable for $1 \leq p < \infty$, see Remark 8.7.

As we shall see in Sections 3 and 4 below, one of the main problems in creating stable local bases for various spline spaces $\mathcal{S}$ is to make sure that (2.3) holds for triangulations which contain edges which are near-degenerate in the following sense.

Definition 2.4. *Suppose $T = \langle v, v_1, v_2\rangle$ and $\widetilde{T} = \langle v, v_2, v_3\rangle$ are two triangles which share an edge $e = \langle v, v_2\rangle$. Given $\theta > 0$, we say that e is* θ-**near-degenerate at** *v provided that the smaller of the two angles between the edges $\langle v, v_1\rangle$ and $\langle v, v_3\rangle$ is greater than $\pi - \theta$.*

In the case where the edges $\langle v, v_1\rangle$ and $\langle v, v_3\rangle$ are collinear, the edge $e = \langle v, v_2\rangle$ is a classical **degenerate edge**. It is easy to see that if $\triangle$ is a triangulation whose smallest angle is at least θ, then no edge of $\triangle$ can be θ-near-degenerate at both ends. Moreover, in such triangulations, the number of near-degenerate edges attached to any vertex v can only be zero, one, two, or four, and in the case of four near-degenerate edges there are no further edges attached to v, compare Definition 2.6.

Since the Bernstein polynomials on any triangle satisfy $0 \leq B^d_{ijk} \leq 1$, to insure that (2.3) holds, it suffices to make sure that for each dual basis spline B_ξ constructed in Algorithm 2.1, all of its computed coefficients satisfy $|c_\eta| \leq K$, where K depends only on d and the smallest angle $\theta_\triangle$ in $\triangle$. As noted above, these coefficients are typically computed using the smoothness Lemmas 6.1 and 6.2 of [24]. Suppose T and $\widetilde{T}$ are two neighboring triangles as in Definition 2.4, and let $v_3 = b_1 v_1 + b_2 v_2 + b_3 v$. Without going into detail (cf. [24]), we note that Lemma 6.1 is a stable process when applied to triangles whose smallest angle is bounded below by $\theta_\triangle$. In contrast to this, the computation of Lemma 6.2 is not stable if one of $|b_2|, |b_3|$ becomes too small, *i.e.*, if the edge $e_2 := \langle v, v_2\rangle$ is θ-near-degenerate at either end, with $\theta \to 0$. However, Lemma 6.2 certainly can be used in a stable way if the measure of near-degeneracy of e_2 is controlled. The following lemma shows that the computation of Lemma 6.2 is stable when applied to edges which are not $\theta_\triangle$-near-degenerate.

Lemma 2.5. *Suppose the smallest angle in T and $\widetilde{T}$ is at least θ, and that the smaller of the two angles between $e_1 := \langle v, v_1\rangle$ and $e_3 := \langle v, v_3\rangle$ is at most $\pi - \theta$. Then $|b_2| \geq \sin^2\theta$.*

Proof: Let $e_2 := \langle v, v_2\rangle$. Then

$$|b_2| = \frac{|e_1||e_3|\sin a}{|e_1||e_2|\sin\theta_1},$$

where a is the smaller of the two angles between e_1 and e_3, and θ_1 is the angle between the edges e_1 and e_2. It was shown in the proof of Lemma 3.2 of [24] that under the above hypotheses, $|e_3|/|e_2| \geq \sin\theta$. But then the result follows from the fact that $|\sin a| \geq \sin\theta$ while $|\sin\theta_1| \leq 1$. □

Definition 2.6. *Suppose v is a vertex where exactly four θ-near-degenerate edges meet. Then we call v a* θ-**near-singular vertex**.

If v is a near-singular vertex where all four edges are actually degenerate, then v is a classical **singular vertex** formed by the intersection of two lines. It is easy to see that if $\triangle$ is a triangulation whose smallest angle is at least θ and v is not a θ-near-singular vertex, then there must be at least one edge attached to v which is not θ-near-degenerate at either end.

Definition 2.7. *Suppose $\triangle_v$ is a triangulation which consists of four triangles surrounding a singular vertex v. Then we call $\triangle_v$ a* singular cell*. If v is a θ-near-singular vertex v, we call $\triangle_v$ a* θ-near-singular cell.

We conclude this section by recalling some additional standard notation. Given a triangle $T := \langle v, u, w\rangle$, we write ξ^T_{ijk} for the domain points of $\mathcal{S}^0_d(\triangle)$ which lie in T. Given a vertex v, we define the ℓ-th ring around v to be

$$R_\ell(v) := \bigcup \{\xi^T_{d-\ell,j,\ell-j} : \ 0 \le j \le \ell \text{ and } T \text{ is a triangle attached to } v\}, \qquad (2.4)$$

and the ℓ-th disk around v to be

$$D_\ell(v) := \bigcup_{\nu=0}^{\ell} R_\nu(v). \qquad (2.5)$$

For any triangle T attached to v, let

$$R^T_\ell(v) := R_\ell(v) \cap T, \qquad D^T_\ell(v) := D_\ell(v) \cap T. \qquad (2.6)$$

§3. A stable MDS on a near-singular cell

As a first step towards constructing stable local minimal determining sets for general spline spaces, in this section we examine the superspline space

$$\mathcal{S}^{r,\mu}_d(\triangle_v) := \{s \in \mathcal{S}^r_d(\triangle_v) : \ s \in C^\mu(v)\}, \qquad (3.1)$$

where

$$\mu = r + \left\lfloor \frac{r+1}{2} \right\rfloor, \qquad (3.2)$$

and $\triangle_v$ is a near-singular cell. Our aim is to construct a stable MDS for $\mathcal{S}^{r,\mu}_d(\triangle_v)$ on $D_{2r}(v)$.

Suppose $v_1, \ldots, v_4$ are the boundary vertices of $\triangle_v$ in counterclockwise order, where $v_5 = v_1$. Let $T_i := \langle v, v_i, v_{i+1}\rangle$, for $i = 1, \ldots, 4$. For $\mu + 1 \le \ell \le 2r$, we introduce some simplified notation for certain domain points on the ring $R_\ell(v)$. Let

$$\begin{aligned} a^i_{\ell,j} &:= \xi^{T_i}_{d-\ell,\ell-r+j-1,r-j+1}, \quad 1 \le j \le n_\ell, \\ g^i_{\ell,j} &:= \xi^{T_i}_{d-\ell,\ell-r+n_\ell+j-1,r-n_\ell-j+1}, \quad 1 \le j \le n_\ell, \\ d^i_{\ell,j} &:= \xi^{T_i}_{d-\ell,\ell-r+2n_\ell+j-1,r-2n_\ell-j+1}, \quad 1 \le j \le r - 2n_\ell + 1, \end{aligned} \qquad (3.3)$$

where

$$n_\ell := 2r + 1 - \ell. \qquad (3.4)$$

Note that $n_\ell \ge 1$ and $r - 2n_\ell + 1 \ge 1$.

It is not difficult to describe a stable MDS for $\mathcal{S}^{r,\mu}_d(\triangle_v)$ on $D_k(v)$ for $\mu + 1 \le k \le 2r$ if v is actually singular.

Theorem 3.1. *Suppose $\triangle_v$ is a singular cell. For each $\ell = \mu+1,\ldots,2r$, let*

$$\begin{aligned} \mathcal{M}_{v,\ell} &:= \{a^1_{\ell,1},\ldots,a^1_{\ell,n_\ell}\} \;\cup\; \bigcup_{i=1}^{4}\{g^i_{\ell,1},\ldots,g^i_{\ell,n_\ell}\}, \\ O_{v,\ell} &:= \bigcup_{i=1}^{4}\{d^i_{\ell,1},\ldots,d^i_{\ell,r-2n_\ell+1}\}. \end{aligned} \tag{3.5}$$

Then for each $k = \mu,\ldots,2r$,

$$\Gamma_k := D^{T_1}_\mu(v) \;\cup\; \bigcup_{\ell=\mu+1}^{k} \left[\mathcal{M}_{v,\ell} \;\cup\; O_{v,\ell}\right] \tag{3.6}$$

is a stable MDS for the space $\mathcal{S}^{r,\mu}_d(\triangle_v)$ on $D_k(v)$.

Proof: We proceed by induction on k. The result is clear for $k=\mu$. Suppose we set the coefficients c_ξ of $s \in \mathcal{S}^{r,\mu}_d(\triangle_v)$ for $\xi \in \Gamma_k$. Then by the inductive hypothesis, all coefficients in $D_{k-1}(v)$ are uniquely determined by c_ξ, $\xi \in \Gamma_{k-1} \subset \Gamma_k$. We then compute the coefficients on ring $R_k(v)$ using the standard smoothness conditions as in Lemma 6.1 of [24]. Namely, we first use the coefficients in $\Gamma_k \cap R^{T_1}_k(v)$ and $D_{k-1}(v)$ to compute the coefficients in $R^{T_2}_k(v) \setminus \Gamma_k$. This last set includes in particular $\{a^2_{\ell,1},\ldots,a^2_{\ell,n_\ell}\}$, so that we can proceed in the same way and successively compute the coefficients in $R^{T_3}_k(v)\setminus\Gamma_k$, $R^{T_4}_k(v)\setminus\Gamma_k$ and $R^{T_1}_k(v)\setminus\Gamma_k$. Note that here we have not used a portion of the smoothness conditions across the edge $e_1 := \langle v, v_1\rangle$ which involve the coefficients c_ξ for $\xi \in \{a^1_{\ell,1},\ldots,a^1_{\ell,n_\ell}\}$. Nevertheless, these conditions must be satisfied since the number of free parameters c_ξ, $\xi \in \Gamma_k \setminus \Gamma_{k-1}$, used in the above computation on ring $R_k(v)$, is equal to

$$\dim \mathcal{S}^{r,\mu}_k(\triangle_v) - \dim \mathcal{S}^{r,\mu}_{k-1}(\triangle_v) = 4(k-r) + n_k$$

(cf. Theorem 2.2 of [31]). Thus, we are able to compute all coefficients c_ξ, $\xi \in D_k(v)\setminus\Gamma_k$, by applying Lemma 6.1 of [24] several times. By that lemma, the maximum of the computed coefficients is bounded by a constant K times the maximum of the set coefficients, where K depends only on d and the smallest angle in $\triangle_v$. $\square$

For later use in building stable local minimal determining sets for general spline spaces, it is critical that the stable MDS in Theorem 3.1 contains the sets $O_{v,\ell}$. We now extend this result to θ-near-singular cells with $\theta > 0$. In Sect. 5 we construct stable minimal determining sets for supersplines on general cells (which include the near-singular cells considered here). The construction there is simpler, but does not guarantee that the resulting MDS contains the needed sets $O_{v,\ell}$.

Theorem 3.2. *Suppose $\triangle_v$ is a cell associated with a θ-near-singular vertex v, and let $O_{v,\ell}$ be the sets in (3.5). Then for each $\mu+1 \le \ell \le 2r$, there exists a set of domain points*

$$\mathcal{M}_{v,\ell} \subseteq \mathcal{A}_{v,\ell} := \bigcup_{i=1}^{4}\left[\{a^i_{\ell,j}\}_{j=1}^{n_\ell} \cup \{g^i_{\ell,j}\}_{j=1}^{n_\ell}\right].$$

such that for each $k = \mu, \ldots, 2r$,

$$\Gamma_k := D_\mu^{T_1}(v) \cup \bigcup_{\ell=\mu+1}^{k} \left[\mathcal{M}_{v,\ell} \cup O_{v,\ell}\right] \tag{3.7}$$

is a stable MDS for the space $\mathcal{S}_d^{r,\mu}(\triangle_v)$ on $D_k(v)$.

Proof: Since Theorem 3.1 covers the case where $\triangle_v$ is a singular cell, we may assume that $\triangle_v$ is a near-singular cell but not a singular cell. We proceed by induction on k. The statement of the theorem holds for $k = \mu$ since $\Gamma_\mu = D_\mu^{T_1}(v)$ is trivially a stable MDS for $\mathcal{S}_d^{r,\mu}(\triangle_v)$ on $D_\mu(v)$.

Fix $\mu + 1 \leq k \leq 2r$, and suppose that Γ_{k-1} is a stable MDS for $\mathcal{S}_d^{r,\mu}(\triangle_v)$ on $D_{k-1}(v)$. To construct Γ_k which is a stable MDS for $\mathcal{S}_d^{r,\mu}(\triangle_v)$ on $D_k(v)$, we need to supplement Γ_{k-1} with an appropriate subset of the domain points on the ring $R_k(v)$. Using the fact that v is not a singular vertex, it is easy to see that the number of edges attached to v with different slopes is at least three. Then Theorem 2.2 of [31] implies

$$m := \dim \mathcal{S}_k^{r,\mu}(\triangle_v) - \dim \mathcal{S}_{k-1}^{r,\mu}(\triangle_v) = 4(k - r). \tag{3.8}$$

Thus, to get a minimal determining set Γ_k for $\mathcal{S}_d^{r,\mu}(\triangle_v)$ on $D_k(v)$, we need to add to Γ_{k-1} exactly m points on the ring $R_k(v)$.

To simplify the discussion of how to choose these m points, we first reduce the problem to one of considering splines whose coefficients are zero for all points in the disk $D_{k-1}(v)$. Given $s \in \mathcal{S}_d^{r,\mu}(\triangle_v)$, let $\mathcal{T}_{k-1}s$ be the spline in $\mathcal{S}_d^r(\triangle_v)$ constructed in Lemma 3.4 below such that for each triangle attached to v, $g_T := \mathcal{T}_{k-1}s|_T$ interpolates the derivatives up to order $k-1$ of $s|_T$ at v. Note that since $s \in C^\mu(v)$, $\mathcal{T}_{k-1}s$ is also in $C^\mu(v)$. Then the spline $\hat{s} := s - \mathcal{T}_{k-1}s \in \mathcal{S}_d^{r,\mu}(\triangle_v)$ has all zero coefficients in $D_{k-1}(v)$. Computing its coefficients on the ring $R_k(v)$ will stably and uniquely determine the coefficients of s on $R_k(v)$, since by Lemma 3.4 the size of the coefficients of $\mathcal{T}_{k-1}s$ on this ring is bounded by the size of the coefficients of s in $D_{k-1}(v)$.

We now focus on the set $\mathcal{A}_{v,k} \cup O_{v,k}$ of domain points $a_{k,j}^i$, $g_{k,j}^i$, and $d_{k,j}^i$ lying on ring $R_k(v)$. Clearly, if we are given values for the coefficients of $\hat{s}$ in $\mathcal{A}_{v,k} \cup O_{v,k}$, then the remaining coefficients of $\hat{s}$ corresponding to domain points on $R_k(v) \setminus (\mathcal{A}_{v,k} \cup O_{v,k})$ can be computed directly and stably from smoothness conditions using Lemma 6.1 of [24].

Suppose $v_1, \ldots, v_4$ are the boundary vertices of $\triangle_v$ in counterclockwise order, and let $e_i := \langle v, v_i \rangle$ and $T_i := \langle v, v_i, v_{i+1} \rangle$ for $i = 1, \ldots, 4$, where for convenience we identify v_{i+4} with v_i for all i. In addition, suppose the barycentric coordinates of v_{i-1} with respect to the triangle T_i are given by

$$v_{i-1} = r_i v_{i+1} + s_i v + t_i v_i,$$

for $i = 1, 2, 3, 4$. Note that $t_i = 0$ if and only if the edge e_i is degenerate at v. Since v is assumed not to be a singular vertex, at least one t_i is nonzero.

Let $z = (z_1, \ldots, z_{4r+4})$ be the B-coefficients of $\hat{s}$ corresponding to the domain points

$$\bigcup_{i=1}^{4} \{a^i_{k,1}, \ldots, a^i_{k,n}, g^i_{k,1}, \ldots, g^i_{k,n}, d^i_{k,1}, \ldots, d^i_{k,r-2n+1}\}.$$

Here we are writing $n := n_k = 2r+1-k$ for ease of notation. Since the coefficients of $\hat{s}$ corresponding to domain points in $D_{k-1}(v)$ are zero, the smoothness conditions of order $r-n+1, \ldots, r$ across the interior edges of $\triangle_v$ which connect the components of z to each other can be written in the form

$$Hz = 0, \tag{3.9}$$

where

$$H := \begin{pmatrix} H_1^a & H_1^g & H_1^d & & & & & & & -I & & \\ -I & & & H_2^a & H_2^g & H_2^d & & & & & & \\ & & & -I & & & H_3^a & H_3^g & H_3^d & & & \\ & & & & & & -I & & & H_4^a & H_4^g & H_4^d \end{pmatrix},$$

$$H_i^a := \begin{pmatrix} & & & & r_i^{r-n+1} \\ & & & r_i^{r-n+2} & \binom{r-n+2}{r-n+1} r_i^{r-n+1} t_i \\ & & \iddots & & \vdots \\ & r_i^{r-1} & \cdots & & \binom{r-1}{r-n+1} r_i^{r-n+1} t_i^{n-2} \\ r_i^r & \binom{r}{r-1} r_i^{r-1} t_i & \cdots & & \binom{r}{r-n+1} r_i^{r-n+1} t_i^{n-1} \end{pmatrix},$$

$$H_i^g := \begin{pmatrix} \binom{r-n+1}{r-n} r_i^{r-n} t_i & \cdots & \binom{r-n+1}{r-2n+1} r_i^{r-2n+1} t_i^n \\ \vdots & & \vdots \\ \binom{r}{r-n} r_i^{r-n} t_i^n & \cdots & \binom{r}{r-2n+1} r_i^{r-2n+1} t_i^{2n-1} \end{pmatrix},$$

$$H_i^d := \begin{pmatrix} \binom{r-n+1}{r-2n} r_i^{r-2n} t_i^{n+1} & \cdots & \binom{r-n+1}{1} r_i t_i^{r-n} & t_i^{r-n+1} \\ \vdots & & \vdots & \vdots \\ \binom{r}{r-2n} r_i^{r-2n} t_i^{2n} & \cdots & \binom{r}{1} r_i t_i^{r-1} & t_i^r \end{pmatrix},$$

and I is the $n \times n$ identity matrix. We call a column of H a d-**column** when it passes through one of the matrices H_i^d. We define a-**columns** and g-**columns** similarly.

The matrix H has $4n$ rows and $4(r+1)$ columns where $n < r+1$. We claim that it has full rank $4n$. Indeed, the number of independent solutions $4(r+1) - \text{rank}(H)$ of the homogeneous system (3.9) must be equal to m, which implies $\text{rank}(H) = 4n$.

This means that there is a choice of $4n$ indices $1 \le i_1 < \cdots < i_{4n} \le 4r+4$ such that the determinant of the corresponding square submatrix $H(i_1, \cdots, i_{4n})$ is not zero. Moreover, it follows from Lemma 3.3 below that we can choose $i_1, \cdots, i_{4n}$ such that no column of $H(i_1, \cdots, i_{4n})$ is a d-column.

We are ready to describe the set $\mathcal{M}_{v,k}$. Denote by Σ the set of indices of all a- and g-columns of H, and let $\{i_1^*, \cdots, i_{4n}^*\} \subseteq \Sigma$ be such that

$$|\det H(i_1^*, \ldots, i_{4n}^*)| = \max_{i_1, \ldots, i_{4n} \in \Sigma} |\det H(i_1, \ldots, i_{4n})|. \tag{3.10}$$

We take $\mathcal{M}_{v,k}$ to be the set of domain points in $\mathcal{A}_{v,k}$ which correspond to the columns with indices in the set $\Sigma \setminus \{i_1^*, \cdots, i_{4n}^*\}$. Then $\mathcal{M}_{v,k} \cup \mathcal{O}_{v,k}$ is the set of domain points on $R_k(v)$ which correspond to the columns of H with indices in the set $J^* := \{1, \ldots, 4r+4\} \setminus \{i_1^*, \cdots, i_{4n}^*\}$.

Now assuming that the coefficients $\{z_j\}_{j \in J^*}$ of $\hat{s}$ corresponding to points in $\mathcal{M}_{v,k} \cup \mathcal{O}_{v,k}$ have been set, we may compute the remaining coefficients corresponding to points in $\mathcal{A}_{v,k} \cup \mathcal{O}_{v,k}$ from the non-singular system

$$H(i_1^*, \ldots, i_{4n}^*) \begin{pmatrix} z_{i_1^*} \\ \vdots \\ z_{i_{4n}^*} \end{pmatrix} = - \sum_{j \in J^*} z_j H(j),$$

where $H(j)$ is the j-th column of H. Using Cramer's rule and taking account of (3.10) and Lemma 3.3, we conclude that

$$|z_{i_\nu^*}| \le \frac{\sum_{j \in J^*} |z_j| |\det H(i_1^*, \ldots, i_{\nu-1}^*, j, i_{\nu+1}^*, \ldots, i_{4n}^*)|}{|\det H(i_1^*, \ldots, i_{4n}^*)|} \le K \max_{j \in J^*} |z_j|,$$

for $\nu = 1, \ldots, 4n$, where K is a constant depending only on d and the smallest angle in $\triangle_v$. This shows that the computation of $z_{i_1^*}, \ldots, z_{i_{4n}^*}$ is stable. □

We conclude this section with two lemmas which were used in the proof of Theorem 3.2. The first result concerns determinants formed from $4n \times 4n$ submatrices of H. Let

$$R := \begin{pmatrix} \binom{r-n+1}{r-n} & \cdots & \binom{r-n+1}{r-2n+1} \\ \vdots & \ddots & \vdots \\ \binom{r}{r-n} & \cdots & \binom{r}{r-2n+1} \end{pmatrix}.$$

A simple computation shows that

$$\det R = C \det \begin{pmatrix} \frac{1}{1!} & \cdots & \frac{1}{n!} \\ \vdots & \cdots & \vdots \\ \frac{1}{n!} & \cdots & \frac{1}{(2n-1)!} \end{pmatrix},$$

where C is a positive constant depending only on r and n. It is well-known that this determinant is nonzero for all choices of n, and thus the matrix R is nonsingular.

Lemma 3.3. *Let $H(i_1,\dots,i_{4n})$ be a $4n\times 4n$ submatrix of H containing a nontrivial d-column. Then there exists another submatrix $H(j_1,\dots,j_{4n})$ with one less d-column such that*

$$|\det H(i_1,\dots,i_{4n})| \le C\;|\det H(j_1,\dots,j_{4n})|, \tag{3.11}$$

where $C>0$ is a constant depending only on d and the smallest angle of $\triangle_v$.

Proof: Suppose $H(i_1,\dots,i_{4n})$ includes a nontrivial d-column $i_p=(r+1)(i-1)+2n+j$ with $1\le i\le 4$ and $1\le j\le r-2n+1$. Note that the column is nontrivial if and only if the corresponding t_i is nonzero. For any $1\le j\le r-2n+1$, it is not difficult to see that

$$H_i^d(j)=\sum_{\kappa=1}^{n} x_\kappa^{[j]}\left(\frac{t_i}{r_i}\right)^{j+n-\kappa} H_i^g(\kappa),$$

where the numbers $x_\kappa^{[j]}$ are determined from the nonsingular linear system

$$R\begin{pmatrix} x_1^{[j]}\\ \vdots \\ x_n^{[\nu]}\end{pmatrix}=\begin{pmatrix}\binom{r-n+1}{r-2n+1-j}\\ \vdots\\ \binom{r}{r-2n+1-j}\end{pmatrix}. \tag{3.12}$$

Since the κ-th column of H_i^g corresponds to the $(r+1)(i-1)+n+\kappa$-th column of H, this implies that

$$\det H(i_1,\dots,i_{4n})=\sum_{\kappa=1}^{n} x_\kappa^{[j]}\left(\frac{t_i}{r_i}\right)^{j+n-\kappa}\det H_\kappa,$$

where

$$H_\kappa := H(i_1,\dots,i_{p-1},(r+1)(i-1)+n+\kappa,i_{p+1},\dots,i_{4n}).$$

Thus,

$$|\det H(i_1,\dots,i_{4n})| \le K_1|t_i|^j \max_\kappa |\det H_\kappa|, \tag{3.13}$$

where K_1 depends only on d and $\theta_\triangle$. The result follows since $|t_i|\le K_2$ where K_2 is a constant depending only on $\theta_\triangle$. (In fact, $|t_i|$ is quite small since we assume that v is near-singular.) □

The following lemma was used in the proof of Theorem 3.2 above, and will also be useful in Sect. 7 below.

Lemma 3.4. *Let $\triangle_v$ be a cell, and let $0\le r<k\le d$ be integers. Given a spline $s\in\mathcal{S}_d^r(\triangle_v)$, let $\mathcal{T}_{k-1}s$ be such that for each triangle T attached to v,*

$$\mathcal{T}_{k-1}s|_T := \text{the unique polynomial of degree } k-1 \text{ which matches the derivatives of } s|_T \text{ at } v \text{ up to order } k-1.$$

Then $\mathcal{T}_{k-1}s \in \mathcal{S}^r_{k-1}(\triangle_v) \subseteq \mathcal{S}^r_d(\triangle_v)$*. Moreover, if*

$$s|_T = \sum c^T_\xi B^T_\xi, \qquad \mathcal{T}_{k-1}s|_T = \sum \hat{c}^T_\xi B^T_\xi,$$

where B^T_ξ *are the Bernstein polynomials of degree* d *associated with a triangle* T*, then* $\hat{c}^T_\xi = c^T_\xi$ *for all* $\xi \in D^T_{k-1}(v)$*, and*

$$\max_{\xi \in R^T_k(v)} |\hat{c}^T_\xi| \le K \max_{\xi \in D^T_{k-1}(v)} |c^T_\xi|, \tag{3.14}$$

where K *is a constant depending only on* d*.*

Proof: Comparing cross derivatives of neighboring pieces of $\mathcal{T}_{k-1}s$, it is easy to see that it satisfies C^r smoothness conditions across the interior edges of $\triangle_v$, and thus is a spline in $\mathcal{S}^r_{k-1}(\triangle_v) \subseteq \mathcal{S}^r_d(\triangle)$. Now fix a triangle $T := \langle v, v_i, v_{i+1}\rangle$ in $\triangle_v$. Then by the well-known connection between derivatives and coefficients of a polynomial written in Bernstein-Bézier form, it follows that $\hat{c}^T_\xi = c^T_\xi$ for all $\xi \in D^T_{k-1}(v)$. Finally, to establish (3.14), we observe that since $\mathcal{T}_{k-1}s$ is a polynomial of degree $k-1$, its k-th derivatives are identically zero, and thus for all $\nu = 0, \ldots, k$,

$$\begin{aligned} 0 &= D^\nu_{v_i - v} D^{k-\nu}_{v_{i+1}-v} \mathcal{T}_{k-1}s|_T(v) \\ &= \frac{d!}{(d-k)!} \sum_{j_1=0}^{\nu} \sum_{j_2=0}^{k-\nu} \binom{\nu}{j_1} \binom{k-\nu}{j_2} (-1)^{k-j_1-j_2} \hat{c}^T_{d-j_1-j_2, j_1, j_2}. \end{aligned}$$

It follows that

$$\hat{c}^T_{d-k,\nu,k-\nu} = -\frac{d!}{(d-k)!} \sum_{\substack{0 \le j_1 \le \nu,\ 0 \le j_2 \le k-\nu \\ j_1 + j_2 \le k-1}} \binom{\nu}{j_1} \binom{k-\nu}{j_2} (-1)^{k-j_1-j_2} \hat{c}^T_{d-j_1-j_2, j_1, j_2},$$

which immediately implies (3.14). □

§4. Stable local bases for $\mathcal{S}^{r,\mu}_d(\triangle)$

Our ultimate aim is to give stable local bases for the general superspline spaces (1.1) defined on arbitrary triangulations $\triangle$ of a polygonal set Ω. However, in order to illustrate the construction in a somewhat simpler setting, in this section we consider the superspline space

$$\mathcal{S}^{r,\mu}_d(\triangle) := \{s \in \mathcal{S}^r_d(\triangle) : \ s \in C^\mu(v) \ \text{ for all } v \in \mathcal{V}\},$$

for $d \ge 3r + 2$, where μ is defined in (3.2). This is the special case of (1.1) with $\rho_v = \mu$ for all $v \in \mathcal{V}$. The analogous construction for general superspline spaces requires further analysis of cells (see Sect. 5), and is given in Section 6.

In order to describe a minimal determining set for $\mathcal{S}_d^{r,\mu}(\triangle)$ which leads to a stable local basis, we need some additional notation. Given a triangle $T := \langle u, v, w\rangle$, let ξ_{ijk}^T be the domain points of $\mathcal{S}_d^0(\triangle)$ which lie in T. Let

$$C^T := \{\xi_{ijk}^T : \ i > r, \ j > r, \ k > r\}.$$

Associated with the vertex u, let

$$\begin{aligned} D_\mu^T(u) &:= \left\{\xi_{ijk}^T : \ i \geq d - \mu\right\} \\ A^T(u) &:= \bigcup_{i=1}^{\lfloor \frac{r}{2} \rfloor} \bigcup_{j=0}^{i-1} \{\xi_{d-2r+i-1,r-j,r-i+j+1}^T\}, \end{aligned}$$

with similar definitions for the other two vertices of T. Associated with the edge $e := \langle u, v\rangle$, we define

$$\begin{aligned} F^T(e) &:= \left\{\xi_{ijk}^T : \ k \leq r\right\} \\ G_L^T(e) &:= \bigcup_{i=1}^{\lfloor \frac{r}{2} \rfloor} \bigcup_{j=0}^{i-1} \{\xi_{d-2r+i-1,r+1+j,r-i-j}^T\} \\ G_R^T(e) &:= \bigcup_{i=1}^{\lfloor \frac{r}{2} \rfloor} \bigcup_{j=0}^{i-1} \{\xi_{r+1+j,d-2r+i-1,r-i-j}^T\} \\ E^T(e) &:= F^T(e) \setminus \Big[D_\mu^T(u) \cup D_\mu^T(v) \cup A^T(u) \cup A^T(v) \cup G_L^T(e) \cup G_R^T(e)\Big], \end{aligned} \tag{4.1}$$

with similar definitions for the other two edges of T. Note that the definitions of $G_L^T(e)$ and $G_R^T(e)$ depend on the *orientation* of the edge e. Namely, if $\tilde{e} := \langle v, u\rangle$, then $G_L^T(e) = G_R^T(\tilde{e})$ and $G_R^T(e) = G_L^T(\tilde{e})$.

Let $\mathcal{V}_S$ and $\mathcal{V}_{NS}$ be the sets of vertices of $\triangle$ which are singular and $\theta_\triangle$-near-singular, respectively.

Theorem 4.1. *Let $\mathcal{M}$ be the following set of domain points:*

1) *for each triangle T, include the set C^T,*
2) *for each edge e, include the set $E^T(e)$, where T is some triangle sharing e,*
3) *for each edge of a triangle T such that e lies on the boundary of Ω, include the sets $G_L^T(e)$ and $G_R^T(e)$,*
4) *for each vertex $v \in \mathcal{V}$, include $D_\mu^T(v)$ for some triangle T attached to v,*
5) *suppose the vertex $v \notin \mathcal{V}_{NS}$ is connected to $v_1, \ldots, v_n$ in counterclockwise order. Let $T_i := \langle v, v_i, v_{i+1}\rangle$ and set $T_0 := T_n = \langle v, v_n, v_1\rangle$ if v is an interior vertex. Let $1 \leq i_1 < \cdots < i_k < n$ be such that e_{i_j} is $\theta_\triangle$-near-degenerate at either end, where $e_i := \langle v, v_i\rangle$ for $i = 1, \ldots, n$. Let $J_v := \{i_1, \ldots, i_k\}$. Then*
   a) *include $G_L^{T_i}(e_i)$ for all $i \in J_v$,*

b) *include* $A^{T_i}(v)$ *for all* $1 \le i \le n-1$ *such that* $i \notin J_v$,

c) *include* $A^{T_n}(v)$ *if* v *is an interior vertex,*

6) *for each vertex* $v \in \mathcal{V}_S$, *include the sets* $\mathcal{M}_{v,\mu+1}, \ldots, \mathcal{M}_{v,2r}$ *constructed in Theorem 3.1,*

7) *for each* $v \in \mathcal{V}_{NS} \setminus \mathcal{V}_S$ *include the sets* $\mathcal{M}_{v,\mu+1}, \ldots, \mathcal{M}_{v,2r}$ *constructed in Theorem 3.2.*

Then $\mathcal{M}$ *is a stable local minimal determining set for* $\mathcal{S}_d^{r,\mu}(\triangle)$.

Proof: We claim that $\mathcal{M}$ is well-defined. In particular, as observed above, if $v \notin \mathcal{V}_{NS}$, there is always at least one edge attached to v which is not near-degenerate at either end. In the numbering of the edges in step 5) above, we can choose this edge to be $\langle v, v_n \rangle$, and the construction insures that for each interior vertex $v \notin \mathcal{V}_{NS}$ and edge $e_i := \langle v, v_i \rangle$ attached to it, if $v_i \notin \mathcal{V}_{NS}$, then Γ includes exactly one of the two sets $A^{T_i}(v)$ or $G_L^{T_i}(e_i)$.

To see that $\mathcal{M}$ is a MDS for $\mathcal{S}_d^{r,\mu}(\triangle)$, we now show that for each $\xi \in \mathcal{M}$, we can construct a unique dual basis spline B_ξ. Suppose we set the coefficient corresponding to ξ equal to 1, and all other coefficients associated with $\eta \in \mathcal{M}$ to zero. We now show that this uniquely determines all other coefficients of B_ξ.

First, for all vertices v of $\triangle$, we use the coefficients in item 4) and Lemma 6.1 of [24] to uniquely compute all coefficients associated with points in the disks $D_\mu(v)$. Next we compute coefficients on the rings $R_{\mu+1}(v)$ for all v. First we do the vertices v which are not in $\mathcal{V}_{NS}$. Next we use Theorem 3.1 for each vertex $v \in \mathcal{V}_S$, and Theorem 3.2 for each vertex in $\mathcal{V}_{NS} \setminus \mathcal{V}_S$. To do this, we need the coefficients corresponding to the sets $O_{v,\mu+1}$, but these will all have been set or computed at this point. Then we repeat this process one ring at a time until we have completed all of the rings up to $R_{2r}(v)$ for all v. Note that in doing these rings, all computations are done one arc at a time, where an arc consists of points on a ring which lie at a distance up to r on either side of an edge attached to v. As in [24], we process arcs in a *counterclockwise* direction around v. These computations are based on the smoothness conditions of Lemma 6.1 of [24], or (only if the corresponding edge is not $\theta_\triangle$-near-degenerate) those of Lemma 6.2 of [24].

At this point all coefficients of s will have been computed except possibly some points which fall into sets of the form

$$E^T(e) \setminus \Big[D_{2r}(u) \cup D_{2r}(v)\Big],$$

where $e = \langle u, v \rangle$ is an interior edge. These coefficients can be computed from the associated coefficients in the neighboring triangle (which will have been set) using smoothness conditions as in Lemma 6.1 of [24].

We claim that this construction gives dual basis splines whose coefficients satisfy (2.3) with a constant K which depends only on d and the smallest angle in $\triangle$. This follows since the computations in Lemma 6.1 of [24] are always stable, and since we only apply Lemma 6.2 of [24] to edges which are not $\theta_\triangle$-near-degenerate, those computations are also stable, see the discussion proceeding Lemma 2.5 above. The computations in the rings $R_{\mu+1}(v),\ldots,R_{2r}(v)$ for a singular or near-singular vertex v are stable by Theorems 3.1 and 3.2.

Finally, we claim that for each $\xi \in \mathcal{M}$, the support of the dual spline B_ξ is at most $\mathrm{star}^3(v_\xi)$ for some vertex v_ξ. The argument is a modified version of the proof of Theorem 9.1 in [24]. We divide the discussion into cases.

Case 1: Suppose $\xi \in C^T$. Then clearly the support of B_ξ is T.

Case 2: Suppose $\xi \in E^T(e) \setminus \big[D_{2r}(v) \cup D_{2r}(u)\big]$, where $e = \langle v, u\rangle$ is a boundary edge of $\triangle$. Then the support of B_ξ is just the triangle T.

Case 3: Suppose $\xi \in E^T(e) \setminus \big[D_{2r}(v) \cup D_{2r}(u)\big]$, where $e = \langle v, u\rangle$ is an interior edge shared by T and a neighboring triangle $\widetilde{T}$. Then the support of B_ξ is $T \cup \widetilde{T}$.

Case 4: Suppose $\xi \in \mathcal{M} \cap D_{2r}(v)$ with $v \in \mathcal{V}_{NS}$. In addition, suppose that $\xi \notin D_\mu(u_i)$, where $u_1,\ldots,u_4$ are the vertices on the boundary of $\mathrm{star}(v)$. In this case, we claim that the support of B_ξ is at most $\mathrm{star}^2(v)$. Let $e = \langle u, w\rangle$ where u is one of the u_i and w is a vertex on the boundary of $\mathrm{star}^2(v)$. Note that since no edge can be $\theta_\triangle$-near-degenerate at both ends, $u \notin \mathcal{V}_{NS}$. Let $\mathcal{M}_0 := \mathcal{M} \setminus \{\xi\}$. Then all coefficients of B_ξ associated with domain points in $\mathcal{M}_0$ are set to zero and c_ξ is set to 1. After applying the smoothness conditions to compute unset coefficients of B_ξ, we will get several additional nonzero coefficients in $D_{2r}(v)$. Moreover, since $2r$-disks around neighboring vertices overlap whenever $d < 4r+1$, in this case it is also possible to get nonzero coefficients in $D_{2r}(u) \setminus D_\mu(u)$, which in turn could propagate to $D_{2r}(w) \setminus D_\mu(w)$, and possibly even further. We now show that this does not happen. Let $T := \langle u, w, z_2\rangle$ and $\widetilde{T} := \langle z_1, w, u\rangle$ be the two triangles which share the edge e. Note that $\mathcal{M}_0$ contains one of $E^T(e)$ or $E^{\widetilde{T}}(e)$. There are three cases.

a) e is not near-degenerate at either end. Then $\mathcal{M}_0$ contains the set $A^{\widetilde{T}}(w)$, and since coefficients in $D_{2r}(w)$ are computed in counter-clockwise order, nonzero coefficients in $D_{2r}(u)$ do not propagate to nonzero coefficients in $D_{2r}(w)$ which lie outside of $\mathrm{star}^2(v)$.

b) e is near-degenerate at one end and $w \notin \mathcal{V}_{NS}$. Then $\mathcal{M}_0$ contains both $G_L^T(e)$ and $G_R^{\widetilde{T}}(e) = G_L^{\widetilde{T}}(\tilde{e})$, where $\tilde{e} = \langle w, u\rangle$. Lemma 8.2 in [24] then implies that there is no propagation along e into $D_{2r}(w)$.

c) $w \in \mathcal{V}_{NS}$. In this case $\mathcal{M}_0$ contains the set $G_L^T(e)$ and one of the sets $E^T(e)$ or $E^{\widetilde{T}}(e)$. In addition, it also contains sufficient points so that the C^μ supersmoothness at the vertices implies that all coefficients associated with domain points in $D_\mu(u)$ and $D_\mu(w)$ are zero. But then it is easy to see that all coefficients in the $O_{w,\ell}$ sets appearing in Theorems 3.1 and 3.2 are zero. Since all of the sets $\mathcal{M}_{w,\ell}$ appearing in those theorems are also in $\mathcal{M}_0$, we again conclude that there is no propagation along e into $D_{2r}(w)$.

Case 5: Suppose $\xi \in \mathcal{M} \cap D_{2r}(v)$ with $v \notin \mathcal{V}_{NS}$ and that ξ does not fit Case 4. Then we claim that the support of B_ξ is at most $\mathrm{star}^3(v)$. Let u be a vertex on the boundary of $\mathrm{star}(v)$. There are two cases.

a) $u \in \mathcal{V}_{NS}$. Then Case 4 applied to u shows that there is no propagation beyond $\mathrm{star}^2(u)$, and thus none beyond $\mathrm{star}^3(v)$.

b) $u \notin \mathcal{V}_{NS}$. Then the argument of Case 4 shows that there is no propagation beyond $\mathrm{star}^2(v)$.

This completes the proof. □

§5. A stable basis for $\mathcal{S}_\mu^{r,\rho_v}(\triangle_v)$ on a cell $\triangle_v$

Before constructing stable local bases for $\mathcal{S}_d^r(\triangle)$ and for general superspline spaces, we need to examine the superspline space $\mathcal{S}_\mu^{r,\rho_v}(\triangle_v)$ with $r \le \rho_v < \mu$ on an arbitrary cell $\triangle_v$. Suppose that v is a vertex which is connected to the vertices $v_1, \ldots, v_n$ in counterclockwise order, and let $v_{n+1} = v_1$. Let

$$\triangle_v := \{T_i := \langle v, v_i, v_{i+1} \rangle, \quad i = 1, \ldots, n\}$$

form a triangulation of the set

$$\Omega_v := \bigcup_{i=1}^n T_i.$$

In this case $\triangle_v$ is called an interior cell. We now construct a stable basis for $\mathcal{S}_\mu^{r,\rho_v}(\triangle_v)$.

Let e be the number of edges attached to v with different slopes. Then by Theorem 2.2 of [31],

$$m := \dim \mathcal{S}_\mu^{r,\rho_v}(\triangle_v) = \binom{\rho_v + 2}{2} + n\left[\binom{\mu - r + 1}{2} - \binom{\rho_v - r + 1}{2}\right] + \sigma, \quad (5.1)$$

where

$$\sigma := \sum_{j=\rho_v - r + 1}^{\mu - r} (r + j + 1 - je)_+. \quad (5.2)$$

Suppose $\{\xi_i\}_{i=1}^{n_c}$ are the domain points associated with the cell $\triangle_v$. It is easy to see that

$$n_c = n\left[\binom{\mu - 1}{2} + 2\mu - 1\right] + 1 = n\left[\frac{\mu^2 + \mu}{2}\right] + 1. \quad (5.3)$$

Given $s \in \mathcal{S}_\mu^{r,\rho_v}(\triangle_v)$, we denote the B-coefficient associated with ξ_i by c_i for $i = 1, \ldots, n_c$. Associated with each interior edge of $\triangle_v$, there are $\mu - j + 1$ smoothness conditions to insure C^j continuity across that edge, $j = 1, \ldots, r$, and $\rho - r - k + 1$

smoothness conditions to insure C^{ρ_v} continuity at v, $k = 1, \ldots, \rho_v - r$. This gives a total of

$$\begin{aligned} n_s &:= n\Big[\binom{\mu+1}{2} - \binom{\mu-r+1}{2} + \binom{\rho_v-r+1}{2}\Big] \\ &= nr\Big[\frac{2\mu-r+1}{2}\Big] + n\binom{\rho_v-r+1}{2} \end{aligned} \tag{5.4}$$

smoothness conditions to insure that s lies in $\mathcal{S}_\mu^{r,\rho_v}(\triangle_v)$. Note that $n_s < n_c$. These conditions can be written in matrix form

$$Ac = 0, \tag{5.5}$$

where $c = (c_1, \ldots, c_{n_c})^T$, and A is an appropriate $n_s \times n_c$ matrix.

In general, the system (5.5) includes some redundant smoothness conditions, and so $n_r := \operatorname{rank}(A) < n_s$. Indeed, since $\dim \mathcal{S}_\mu^{r,\rho_v}(\triangle_v) = n_c - n_r$, it follows that

$$\begin{aligned} n_r &= n\Big[\frac{\mu^2+\mu}{2}\Big] + 1 - \binom{\rho_v+2}{2} - n\Big[\binom{\mu-r+1}{2} - \binom{\rho_v-r+1}{2}\Big] - \sigma \\ &= nr\Big[\frac{2\mu-r+1}{2}\Big] + 1 - \binom{\rho_v+2}{2} + n\binom{\rho_v-r+1}{2} - \sigma. \end{aligned} \tag{5.6}$$

This implies that the number of redundant equations in (5.5) is

$$n_{red} := \binom{\rho_v+2}{2} - 1 + \sigma. \tag{5.7}$$

Without loss of generality, we may assume that redundant equations have been dropped, and that (5.5) is written in the equivalent form

$$[A_1\, A_2]\, c = 0,$$

where A_1 is an $n_r \times m$ matrix and A_2 is an $n_r \times n_r$ matrix. We may also assume that the columns of A (and the corresponding components of c) have been numbered so that the determinant of A_2 has the maximal absolute value over all $n_r \times n_r$ subdeterminants of A.

Algorithm 5.1. *For each $i = 1, \ldots, m$, let s_i be the spline in $\mathcal{S}_\mu^{r,\rho_v}(\triangle_v)$ with B-coefficients $c = (c_1, \ldots, c_{n_c})^T$ chosen so that $c_i = 1$, $c_j = 0$ for $j = 1, \ldots, m$ with $j \neq i$, and $c_{m+1}, \ldots, c_{n_c}$ are determined from the linear system*

$$A_2 \begin{pmatrix} c_{m+1} \\ \vdots \\ c_{n_c} \end{pmatrix} = -A_1(i), \tag{5.8}$$

where $A_1(i)$ is the i-th column of the matrix A_1.

The splines $\{s_i\}_{i=1}^m$ are clearly linearly independent since

$$\lambda_j s_i = \delta_{i,j}, \qquad j = 1, \ldots, m, \tag{5.9}$$

where λ_j is the linear functional which picks off the j-th B-coefficient. It follows that they form a basis for $\mathcal{S}_\mu^{r,\rho_v}(\triangle_v)$. We now show that their construction is a stable process, *i.e.*, for each i, all of the coefficients of s_i are uniformly bounded.

Theorem 5.2. *Suppose s_i is one of the basis splines constructed by Algorithm 5.1. Then its B-coefficients satisfy*

$$|c_j| \le 1, \qquad j = 1, \ldots, n_c. \tag{5.10}$$

Proof: Fix $1 \le i \le m$, and let $c = (c_1, \ldots, c_{n_c})$ be the vector of coefficients of s_i as computed from Algorithm 5.1. Then (5.10) clearly holds for $j = 1, \ldots, m$. Let $m + 1 \le j \le n_c$. Then by Cramer's rule,

$$c_j = \frac{\det(\tilde{A}_2)}{\det(A_2)},$$

where $\tilde{A}_2$ is the matrix obtained from A_2 by replacing the j-th column by $-A_1(i)$. But then $|c_j| \le 1$ follows by the choice of A_2. □

Note that this is a constructive algorithm for building dual basis splines. Indeed, if we take $\mathcal{M}_v$ to be the set of domain points corresponding to the m coefficients $c_1, \ldots, c_m$ which are set (as opposed to calculated) in Algorithm 5.1, then obviously $\mathcal{M}_v$ is a minimal determining set for $\mathcal{S}_\mu^{r,\rho_v}(\triangle_v)$.

A completely analogous algorithm can be used to create stable dual basis splines for $\mathcal{S}_\mu^{r,\rho_v}(\triangle_v)$ in the case where $\triangle_v$ is a boundary cell.

§6. A stable basis for $\mathcal{S}_d^{r,\rho}(\triangle)$

In this section we combine the constructions of the two previous sections to create stable local bases for the spaces of supersplines $\mathcal{S}_d^{r,\rho}(\triangle)$ defined in (1.1) for all $d \ge 3r + 2$. As in [23], we assume that

$$k_v + k_u < d \quad \text{for each pair of neighboring vertices } v, u \in \mathcal{V},$$

where

$$k_v := \max\{\rho_v, \mu\}, \qquad v \in \mathcal{V},$$

with μ as in (3.2).

Given a triangle $T = \langle u, v, w \rangle$, let

$$\tilde{C}^T := C^T \setminus [D_{k_u}^T(u) \cup D_{k_v}^T(v) \cup D_{k_w}^T(w)].$$

Associated with u, let

$$\tilde{A}^T(u) := A^T(u) \setminus D_{k_u}^T(u),$$

with similar definitions for the other two vertices of T. Associated with the edge $e := \langle u, v \rangle$, we define

$$\begin{aligned}
\tilde{G}_L^T(e) &:= G_L^T(e) \setminus D_{k_u}^T(u)\\
\tilde{G}_R^T(e) &:= G_R^T(e) \setminus D_{k_v}^T(v),\\
\tilde{E}^T(e) &:= E^T(e) \setminus [D_{k_u}^T(u) \cup D_{k_v}^T(v)],
\end{aligned}$$

with similar definitions for the other edges of T.

Theorem 6.1. *Let $\mathcal{M}$ be the following set of domain points:*

1) *for each triangle T, include the set $\tilde{C}^T$,*
2) *for each edge e, include the set $\tilde{E}^T(e)$, where T is some triangle sharing e,*
3) *for each edge of a triangle T such that e lies on the boundary of Ω, include the sets $\tilde{G}_L^T(e)$ and $\tilde{G}_R^T(e)$,*
4) *for each vertex $v \in \mathcal{V}$,*
   a) *include the set $D_{\rho_v}^T(v)$ for some triangle attached to v if $\rho_v \geq \mu$,*
   b) *include the domain points in $D_\mu(v)$ corresponding to the stable minimal determining set $\mathcal{M}_v$ of Section 5 for $S_\mu^{r,\rho_v}(\triangle_v)$ if $\rho_v < \mu$,*
5) *suppose the vertex $v \notin \mathcal{V}_{NS}$ is connected to $v_1, \ldots, v_n$ in counterclockwise order. Let $T_i := \langle v, v_i, v_{i+1}\rangle$ and set $T_0 := T_n = \langle v, v_n, v_1\rangle$ if v is an interior vertex. Let $1 \leq i_1 < \cdots < i_k < n$ be such that e_{i_j} is $\theta_\triangle$-near-degenerate at either end, where $e_i := \langle v, v_i\rangle$ for $i = 1, \ldots, n$. Let $J_v := \{i_1, \ldots, i_k\}$. Then*
   a) *include $\tilde{G}_L^{T_i}(e_i)$ for all $i \in J_v$,*
   b) *include $\tilde{A}^{T_i}(v)$ for all $1 \leq i \leq n-1$ such that $i \notin J_v$,*
   c) *include $\tilde{A}^{T_n}(v)$ if v is an interior vertex,*
6) *for each vertex $v \in \mathcal{V}_S$, include the sets $\mathcal{M}_{v,k_v+1}, \ldots, \mathcal{M}_{v,2r}$ constructed in Theorem 3.1,*
7) *for each $v \in \mathcal{V}_{NS} \setminus \mathcal{V}_S$ include the sets $\mathcal{M}_{v,k_v+1}, \ldots, \mathcal{M}_{v,2r}$ constructed in Theorem 3.2.*

Then $\mathcal{M}$ is a stable local minimal determining set for $\mathcal{S}_d^{r,\rho}(\triangle)$.

Proof: It is straightforward to check that $\mathcal{M}$ is a determining set for $\mathcal{S}_d^{r,\rho}(\triangle)$. To see that $\mathcal{M}$ is a minimal determining set, we construct splines $B_\xi \in \mathcal{S}_d^r(\triangle)$ corresponding to each $\xi \in \mathcal{M}$. The support properties of these basis splines are the same as in Theorem 4.1, and the boundedness of their coefficients follows by the same argument as before. □

§7. Stability and local linear independence

We recall (cf. [10,15,18,19,20]) that a set $\mathcal{B} = \{B_\nu\}_{\nu\in\mathcal{I}}$ of basis splines for a spline space $\mathcal{S}$ is called locally linearly independent (LLI) provided that for every $T \in \triangle$, the splines $\{B_\nu\}_{\nu\in\Sigma_T}$ are linearly independent on T, where

$$\Sigma_T := \{\nu : \ T \subseteq \text{supp } B_\nu\}. \tag{7.1}$$

Since the classical univariate B-splines are both stable and locally linearly independent (cf. Theorems 4.18 and 4.41 in [30]), it seems natural to expect that there also exist bases for bivariate spline spaces which possess both of these properties simultaneously. Here we have constructed stable local bases for the spline spaces $\mathcal{S}_d^r(\triangle)$ and their superspline subspaces, while star-supported LLI bases for the same spaces were recently constructed in [18]. But these bases are different, and in fact we have the following surprising result.

Theorem 7.1. *Given $r \geq 1$ and $d \geq 3r+2$, there is no construction which will lead to bases for $\mathcal{S}_d^r(\triangle)$ on general triangulations which are simultaneously stable and locally linearly independent.*

Proof: Suppose $\mathcal{B} := \{B_\nu\}_{\nu \in \mathcal{I}}$ is a stable LLI basis for $\mathcal{S}_d^r(\triangle)$ on a triangulation which contains an interior near-singular vertex v. Suppose v is connected to v_1, v_2, v_3, v_4 in counterclockwise order. For each $1 \leq i \leq 4$, let $e_i := \langle v, v_i \rangle$, and $T_i := \langle v, v_i, v_{i+1} \rangle$. Let

$$v_{i-1} = r_i v_{i+1} + s_i v + t_i v_i,$$

and suppose that none of the e_i is degenerate at v, *i.e.*, $t_i \neq 0$. For convenience, we define $\alpha_i, \beta_i, \gamma_i, \mu_i$ to be the linear functionals picking off B-coefficients corresponding to the domain points $\xi^{T_i}_{d-2r,r,r}, \xi^{T_i}_{d-2r,r-1,r+1}, \xi^{T_i}_{d-2r,r+1,r-1}, \xi^{T_i}_{d-2r-1,r,r+1}$, respectively.

For each $1 \leq j \leq 4$, we claim that there is a unique spline $S_j \in \mathcal{S}_d^r(\triangle)$ whose only nonzero coefficients are

$$\alpha_j S_j := 1, \quad \gamma_j S_j := -r_j/(r t_j), \quad \gamma_{j+1} S_j := r_{j+1}^{1-r}/(r t_{j+1}),$$
$$\beta_{j-1} S_j := r_j^{r-1} \gamma_j S_j, \quad \beta_j S_j := r_{j+1}^{r-1} \gamma_{j+1} S_j,$$
$$\mu_{j-1} S_j := r r_j^{r-1} s_j \gamma_j S_j, \quad \mu_j S_j := r r_{j+1}^{r-1} s_{j+1} \gamma_{j+1} S_j.$$

It can be verified directly that S_j satisfies all C^r smoothness conditions, and thus belongs to $\mathcal{S}_d^r(\triangle)$. It is also easy to see that

$$\text{supp } S_j = T_{j-1} \cup T_j \cup T_{j+1},$$

and by a property of LLI bases (see [10,20]),

$$S_j = \sum_{\nu \in I_j} c_\nu^{[j]} B_\nu, \tag{7.2}$$

where $I_j := \{\nu : \text{ supp } B_\nu \subseteq T_{j-1} \cup T_j \cup T_{j+1}\}$ for $j = 1, 2, 3, 4$. We now define

$$\hat{S} = r_2^r S_1 + S_2 + r_3^{-r} S_3 + (r_3 r_4)^{-r} S_4.$$

Using the fact that $r_1 r_2 r_3 r_4 = 1$, it is easy to check that all of the coefficients of $\hat{S}$ are zero except for

$$\alpha_1 \hat{S} := r_2^r, \quad \alpha_2 \hat{S} := 1, \quad \alpha_3 \hat{S} := r_3^{-r}, \quad \alpha_4 \hat{S} := (r_3 r_4)^{-r}.$$

This immediately implies

$$\|\hat{S}\|_\infty \leq K_3,$$

where K_3 depends only on d and $\theta_\triangle$.

In view of (7.2), we can write

$$\hat{S} = \sum_{\nu \in I_1 \cup I_2 \cup I_3 \cup I_4} a_\nu B_\nu.$$

By the assumption that the basis $\mathcal{B}$ is stable, we have

$$\|a\|_\infty \le K_1^{-1} \|\hat{S}\|_\infty \le K_3/K_1,$$

where K_1 is the constant in (1.2).

For each ν, let $\tilde{B}_\nu = B_\nu - \mathcal{T}_{2r-1} B_\nu$, where $\mathcal{T}_{2r-1} B_\nu \in \mathcal{S}^r_{2r-1}(\triangle_v) \subseteq \mathcal{S}^r_d(\triangle_v)$ is the spline constructed in Lemma 3.4 which interpolates the derivatives of B_ν at v up to order $2r-1$. Then the B-coefficients of $\tilde{B}_\nu$ corresponding to domain points in the disk $D_{2r-1}(v)$ are zero. Moreover, since the basis $\mathcal{B}$ is stable, it follows from Lemma 3.4 that the B-coefficients of $\tilde{B}_\nu$ corresponding to domain points on the ring $R_{2r}(v)$ are bounded in absolute value by a constant K_4 depending only on d and $\theta_\triangle$.

Since all of the derivatives of $\hat{S}$ up to order $2r-1$ at v are zero, $\mathcal{T}_{2r-1}\hat{S} = 0$, and we have (on $\triangle_v$)

$$\hat{S} = \sum_{\nu \in I_1 \cup I_2 \cup I_3 \cup I_4} a_\nu \tilde{B}_\nu.$$

Since the support of $\tilde{B}_\nu$ is a subset of the support of B_ν (on $\triangle_v$), it follows that $\alpha_2 \tilde{B}_\nu \ne 0$ only if ν lies in the set

$$\tilde{I}_2 := \{\nu : \ \text{supp } B_\nu = T_1 \cup T_2 \cup T_3\}.$$

This implies

$$1 = \alpha_2 \hat{S} = \sum_{\nu \in \tilde{I}_2} a_\nu \, \alpha_2 \tilde{B}_\nu \le \# \tilde{I}_2 \|a\|_\infty \max_{\nu \in \tilde{I}_2} |\alpha_2 \tilde{B}_\nu|,$$

Now clearly $\#\tilde{I}_2 \le 3\binom{d+2}{2}$, and hence there exists $\nu_0 \in \tilde{I}_2$ such that

$$|\alpha_2 \tilde{B}_{\nu_0}| \ge K_5 > 0,$$

where K_5 depends only on d and $\theta_\triangle$.

Now consider the following C^r smoothness condition across the edge e_2:

$$\alpha_1 \tilde{B}_{\nu_0} = r_2^r \alpha_2 \tilde{B}_{\nu_0} + r r_2^{r-1} t_2 \gamma_2 \tilde{B}_{\nu_0} + \sum_{k=1}^{r-1} \binom{r}{r-k-1} r_2^{r-k-1} t_2^{k+1} \eta_{2,k} \tilde{B}_{\nu_0},$$

where $\eta_{2,k}$ is the linear functional which picks off the B-coefficient corresponding to $\xi^{T_i}_{d-2r,r+k+1,r-k-1}$ for $k = 1, \dots, r-1$. Since $\alpha_1 \tilde{B}_{\nu_0} = 0$, this implies

$$|\gamma_2 \tilde{B}_{\nu_0}| + \frac{1}{r} \sum_{k=1}^{r-1} \binom{r}{r-k-1} \left| \frac{t_2}{r_2} \right|^k |\eta_{2,k} \tilde{B}_{\nu_0}| \ge \left| \frac{r_2}{t_2} \right| \frac{K_5}{r},$$

which is unbounded as $t_2 \to 0$. On the other hand, since the B-coefficients $\gamma_2 \tilde{B}_{\nu_0}$, $\eta_{2,k} \tilde{B}_{\nu_0}$, $k = 1, \ldots, r-1$, correspond to domain points on the ring $R_{2r}(v)$, they cannot exceed K_4 in absolute value, which leads to a contradiction and completes the proof. □

Note that the above proof also applies to the superspline spaces $\mathcal{S}_d^{r,\rho}(\triangle)$ whenever $\rho_v < 2r$ for some near-singular vertex v. On the other hand, if $d \geq 4r+1$ and $\rho_v \geq 2r$ for *all* vertices, then the basis constructed in [23] is both stable and LLI.

§8. Remarks

Remark 8.1. It is well known [5–8,22,23] that the dimension of spline spaces and superspline spaces (when $\rho_v < 2r$) generally depends on the exact geometry of the triangulation, and in particular may change as certain near-singular vertices become singular. Thus, it may seem surprising that it is possible to construct stable bases even though the dimension is not stable. This fact was realized already in [1] for $\mathcal{S}_2^1(\triangle_v)$, where $\triangle_v$ is a near-singular cell. The spaces considered in [11] are also examples where stable bases were constructed even though the dimensions were not stable.

Remark 8.2. For $d \geq 3r+2$, algorithms for constructing star-supported bases for $\mathcal{S}_d^r(\triangle)$ were presented in [22], and for general superspline subspaces in [23]. The constructions there produce stable bases for $r = 0$, and for $d \geq 4r+1$, $\rho_v \geq 2r$ for $r > 0$. However, they are not generally stable if $\rho_v < 2r$, since some of the basis functions do not remain bounded for sequences of triangulations containing vertices which become singular, even if the smallest angle in the triangulations is bounded away from zero. If $\rho_v < \mu$, then many other sequences of triangulations lead to unbounded basis functions when two edges attached to the same vertex become collinear.

Remark 8.3. Stable bases were constructed in [11] for the superspline space $\mathcal{S}_d^{r,\mu}(\triangle)$, and in [24] for a certain special subspace $\mathcal{SS}$ of $\mathcal{S}_d^{r,\mu}(\triangle)$, as a first step in constructing quasi-interpolation operators with optimal approximation order. Note that these constructions differ from our algorithm for $\mathcal{S}_d^{r,\mu}(\triangle)$ described in Section 4. Compared to the construction in [11], our basis splines have substantially smaller supports in general (see also Remark 8.10). For the space $\mathcal{SS}$ the algorithm in [24] produces basis splines with similar small supports, but does not appear to extend to the full spaces $\mathcal{S}_d^r(\triangle)$ and $\mathcal{S}_d^{r,\mu}(\triangle)$.

Remark 8.4. Well-known finite element results, see e.g. [32], imply that the classical superspline subspaces of $\mathcal{S}_d^1(\triangle)$ have stable local bases. In [17] we have recently extended this construction to the full spline spaces $\mathcal{S}_d^1(\triangle)$ with $d \geq 5$. The construction there uses nodal functionals (point evaluation of certain derivatives). Here we have used the linear functionals λ_ξ which pick off the coefficients of the Bernstein-Bézier form.

Remark 8.5. No constructions of stable bases for spline spaces with $d < 3r+2$ are known for general triangulations. However, it is possible to construct stable bases for some values of $d < 3r + 2$ for classes of splines defined on *special* triangulations using macro-element techniques. These include Clough-Tocher and Powell-Sabin refinements, for example. See [25,26].

Remark 8.6. For multiresolution applications, it is important to work with sequences of triangulations which are **nested**. In such cases, the corresponding spline spaces $\mathcal{S}_d^r(\triangle)$ are also nested, but in general the various superspline subspaces are not. (See the discussion of this "super-spline effect" in [14,27].)

Remark 8.7. Following the proof of Theorem 2.3 (see also the proof of Theorem 9.2 in [24]), it is not hard to establish

Theorem 8.8. *Suppose $\mathcal{M}$ is a stable local MDS for a spline space $\mathcal{S}$, and let $\{B_\xi\}_{\xi\in\mathcal{M}}$ be the corresponding dual basis splines. Given $1 \le p < \infty$, let $B_{\xi,p} := A_\xi^{-1/p} B_\xi$, where A_ξ denotes the area of the support of B_ξ. Then $\{B_{\xi,p}\}_{\xi\in\mathcal{M}}$ is an L_p-stable basis for $\mathcal{S}$, i.e., there exist constants K_1 and K_2 depending only on d and $\theta_\triangle$ such that*

$$K_1\|c\|_p \;\le\; \|\sum_{\xi\in\mathcal{M}} c_\xi B_{\xi,p}\|_{p,\Omega} \;\le\; K_2\|c\|_p \tag{8.1}$$

for all choices of the coefficient vector $c = (c_\xi)_{\xi\in\mathcal{M}}$.

Remark 8.9. The definitions of near-degenerate edges and near-singular vertices given here are not exactly the same as in [24], but are essentially equivalent.

Remark 8.10. Our construction guarantees that the supports of the basis splines are at most $\mathrm{star}^3(v)$ for some vertex v in general. In some cases the supports can be made smaller. An explicit construction of stable star-supported bases for C^1 splines can be found in [17]. A careful examination of the construction here shows that for $r = 2$, we also get star-supported stable bases. Moreover, the same holds for general $r > 2$ provided $d \ge 3r+\lfloor (r+1)/2\rfloor+1$, since in this case $D_{2r}(u)\cap D_\mu(w) = \emptyset$ for any two vertices u, w connected by an edge. For $d = 3r + \lfloor (r+1)/2\rfloor$ we can construct star^2-supported stable bases. Indeed, using the fact that $D_{2r-1}(u) \cap D_\mu(w) = \emptyset$, we simply modify Theorem 3.2 to choose $\mathcal{M}_{v,2r} = \{a_{2r,1}^1, g_{2r,1}^2, g_{2r,1}^3, g_{2r,1}^4\}$ where it is assumed without loss of generality that e_1 is the "best edge" attached to the near-singular vertex v in the sense that e_1 is θ-near-degenerate with the greatest θ among all edges attached to v.

Remark 8.11. The proofs of Theorems 3.2 and 5.2 are based on Cramer's rule. This idea of choosing a submatrix with the greatest determinant was used already in [11].

Remark 8.12. The fact that stability and local linear independence are mutually exclusive for spline spaces $\mathcal{S}_d^r(\triangle)$ with $r \ge 1$ was first established for $r = 1$ in [17]. The proof was based on nodal determining sets. Here we have used the Bernstein-Bézier form to establish the same result for general $r \ge 1$.

Remark 8.13. Our construction of a stable local basis can be easily adapted to the spaces of splines and supersplines on a triangulation on the sphere or a sphere-like surface introduced in [2]. Indeed, since we are using exclusively Bernstein-Bézier techniques, our construction and the entire argument can be carried over in the same way as was done in [3] for the standard local bases of $\mathcal{S}_d^{r,\rho}(\triangle)$.

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