On the Approximation Order of Splines on Spherical Triangulations

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Abstract. Bounds are provided on how well functions in Sobolev spaces on the sphere can be approximated by spherical splines, where a spherical spline of degree d is a C^r function whose pieces are the restrictions of homogoneous polynomials of degree d to the sphere. The bounds are expressed in terms of appropriate seminorms defined with the help of radial projection, and are obtained using appropriate quasi-interpolation operators.

§1. Introduction

In a series of papers [2–4], together with P. Alfeld we have developed a theory of spherical splines on triangulations of the sphere in $\mathbb{R}^3$. Such splines closely resemble the classical piecewise polynomials on planar triangulations. Whereas splines in the plane are ordinary piecewise polynomials, spherical splines are formed by piecing together so-called spherical polynomials which are defined as the restrictions of homogeneous trivariate polynomials to the sphere. Since spherical polynomials are combinations of spherical harmonics, they are good candidates for constructing spaces of splines on the sphere.

In [4] we presented a variety of methods for fitting scattered data on the sphere using spherical splines. These included analogs of several classical bivariate methods, such as interpolation, least squares, and minimum-energy methods. That paper also presented the results of extensive numerical experimentation, and confirmed our expectations that spherical splines perform very well numerically. However, in contrast to the planar case, to date no theory has been developed to justify these experimental findings.

In this paper we fill this gap by providing bounds on the error of approximation of smooth functions by spherical splines. As in the planar case, the error bounds will be expressed in terms of the mesh size (the diameter of the largest triangle in a given triangulation), and also in terms of appropriate seminorms measuring the smoothness of the approximated functions. The error estimates will show that for spherical splines of sufficiently high degree d, namely $d \geq 3r + 2$, where r is the

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degree of global smoothness of the spline, the approximation order of these splines is the same as that of their planar counterparts.

To measure the norm of a **spherical function** (a function defined on the unit sphere S) and its smoothness, appropriate analogs of Sobolev smoothness classes on the sphere will be needed. Such Sobolev classes have been known for many years. However, it has not been clear, to us at least, how to define appropriate spherical analogs of Sobolev seminorms. In fact, this has been the main obstacle in our efforts to prove approximation rates for spherical splines. While it is not difficult to define seminorms that annihilate spherical harmonics (spherical polynomials) of a given degree on the whole sphere, for example written in terms of the Laplace-Beltrami operator, such seminorms are not adequate when working on a subdomain of the sphere. Sobolev-type seminorms are needed, among other things, to be able to establish Whitney-type theorems on the sphere, expressing how well a function can be approximated locally by polynomials (see Sect. 4 below). A theorem of this type was established recently in [13], but the error bound failed to provide an explicit dependence on the smoothness of the function being approximated. In our approach, we utilize the intimate connection between spherical polynomials and homogeneous polynomials, along with the fact that the spherical Sobolev spaces can be obtained as restrictions to the sphere of Sobolev spaces of homogeneous functions.

Given a nonnegative integer d, we write Π_d for the space of trivariate homogeneous polynomials of degree d. We write $\Pi_d(\Omega)$ for the restriction of Π_d to any subset Ω of the unit sphere S, and refer to it as the space of **spherical polynomials** of degree d. Similarly, we write $\Pi_d(H)$ for the restriction of Π_d to any hyperplane H in $\mathbb{R}^3$. This is just the usual space of **bivariate polynomials**. All of these polynomial spaces have the same dimension, namely $\binom{d+2}{2}$. Let $\triangle$ be a **spherical triangulation** consisting of a finite collection of spherical triangles (*i.e.*, triangles whose edges are segments of great circles) whose union is S, and such that each pair of triangles in $\triangle$ are either disjoint or share a common vertex or an edge. Given integers d and r, we define the space of **spherical splines** of degree d and smoothness r associated with $\triangle$ to be

$$S_d^r(\triangle) := \{s \in C^r(S) : s|_T \in \Pi_d(T),\ T \in \triangle\}, \tag{1.1}$$

where $s|_T$ denotes the restriction of s to T.

Given a set $\Omega \subset S$, we define its **diameter** to be

$$\operatorname{diam}(\Omega) := \sup\{\arccos(u \cdot v),\ u, v \in \Omega\}.$$

By $|\triangle|$, we denote the **mesh size** of $\triangle$, *i.e.*, the diameter of the largest triangle in $\triangle$. We can now state the main result of the paper (to be proved in Sect. 5 below).

Theorem 1.1. *Let $d \ge 3r+2$ and $1 \le p \le \infty$. Then there exists a spline $s \in S_d^r(\triangle)$ and a constant C, depending only on d, p, and the smallest angle in $\triangle$, such that*

$$|f - s|_{k,p,S} \le C|\triangle|^{d+1-k}|f|_{d+1,p,S},$$

for all $f \in W^{d+1,p}(S)$ *and all* $0 \le k \le d$ *such that* $s \in W^{k,p}(S)$.

Here $W^{\ell,p}(S)$ denote Sobolev spaces of spherical functions and $|\cdot|_{\ell,p,S}$ are associated Sobolev-type seminorms to be introduced in Sect. 3.

§2. Radial Projection

To get approximation results for spherical splines, we shall make use of known results for bivariate splines, along with a natural radial projection operator that we now define. Suppose Ω is a subset of S with $\mathrm{diam}(\Omega) \le 1$. Then we define r_Ω to be the center of a spherical cap of smallest possible radius containing Ω, and let T_Ω be the tangent plane touching S at r_Ω.

We now define the radial projection from T_Ω into S by

$$R_\Omega\,\bar\omega := \omega := \frac{\bar\omega}{|\bar\omega|} \in S, \quad \bar\omega \in T_\Omega.$$

Clearly, R_Ω is one-to-one, and hence R_Ω^{-1} is well defined. Let $\bar\Omega$ be the image of Ω under R_Ω^{-1}.

For later use we note that the diameters of Ω and $\bar\Omega$ are comparable. In particular, it is not hard to see that

$$A_1^{-1}\mathrm{diam}(\Omega) \le \mathrm{diam}(\bar\Omega) \le A_1\,\mathrm{diam}(\Omega), \tag{2.1}$$

where $A_1 := 2\tan(1/2)$. Similarly, it is easy to see that there exists a constant $1 \le A_2 < \infty$ such that

$$A_2^{-1}\rho_\Omega \le \rho_{\bar\Omega} \le A_2\,\rho_\Omega, \tag{2.2}$$

where ρ_Ω is the diameter of the largest spherical cap contained in Ω and $\rho_{\bar\Omega}$ is the diameter of the largest disk contained in $\bar\Omega$.

It is also important to observe that if T is a spherical triangle, then since great circles are mapped into straight lines under R_Ω^{-1}, any spherical triangulation lying in Ω will be mapped into a planar triangulation in $\bar\Omega$.

§3. Spherical Sobolev Spaces and Seminorms

Throughout the remainder of the paper we fix $1 \le p \le \infty$. Suppose B is an open set in $\mathbb{R}^2$ and that k is a nonnegative integer. Then the corresponding classical Sobolev space $W^{k,p}(B)$ is just the space of functions on B whose derivatives up to order k belong to $L_p(B)$, see *e.g.* [1]. A norm on $W^{k,p}(B)$ can be defined as

$$\|g\|_{W^{k,p}(B)} := \sum_{\gamma_1+\gamma_2\le k} \|D_\xi^{\gamma_1} D_\eta^{\gamma_2} g\|_{L_p(B)}, \tag{3.1}$$

where $D_\xi^{\gamma_1} D_\eta^{\gamma_2} = \partial^{\gamma_1+\gamma_2}/\partial\xi^{\gamma_1}\partial\eta^{\gamma_2}$.

Our aim in this section is to define analogous spherical Sobolev spaces defined on open sets $\Omega \subset S$, and to construct corresponding seminorms which annihilate spherical polynomials. To get started, suppose that $\{(\Gamma_j, \phi_j)\}$ is an atlas for Ω, *i.e.*, a finite collection of charts (Γ_j, ϕ_j), where Γ_j are open subsets of Ω, covering Ω, and where ϕ_j are infinitely differentiable mappings $\phi_j : \Gamma_j \to B_j$, B_j an open subset of $\mathbb{R}^2$, whose inverses ϕ_j^{-1} are also infinitely differentiable. Also, let $\{\alpha_j\}$ be a partition of unity subordinated to the atlas $\{(\Gamma_j, \phi_j)\}$, *i.e.*, a set of infinitely differentiable functions α_j on Ω vanishing outside the sets Γ_j, such that $\sum_j \alpha_j = 1$ on Ω.

We now define spherical Sobolev spaces $W^{k,p}(\Omega)$ as follows:

$$W^{k,p}(\Omega) := \{f : (\alpha_j f) \circ \phi_j^{-1} \in W^{k,p}(B_j), \text{ for all } j\}. \tag{3.2}$$

A norm on $W^{k,p}(\Omega)$ can be defined as

$$\|f\|_{W^{k,p}(\Omega)} := \sum_j \|(\alpha_j f) \circ \phi_j^{-1}\|_{W^{k,p}(B_j)}. \tag{3.3}$$

Then the Sobolev space $W^{k,p}(\Omega)$ is just the space of all functions f defined on Ω for which $\|f\|_{W^{k,p}(\Omega)}$ is finite. It is well known [5,14] that this definition does not depend on the choice of the atlas and the partition of unity, in the sense that other choices will give rise to the same space with a norm that is equivalent to (3.3). For other ways to define a norm on $W^{k,p}(\Omega)$, see Remark 6.1 in Sect. 6.

We now turn to the problem of defining seminorms for the spaces $W^{k,p}(\Omega)$. In analogy with the Euclidean case where the Sobolev seminorms annihilate algebraic polynomials, we want to construct seminorms that annihilate spherical polynomials. When considering functions on the entire sphere S, *i.e.* when $\Omega = S$, then one possibility is to define a seminorm that annihilates all spherical polynomials of a given degree d in terms of the well-known Laplace-Beltrami operator $\triangle^*$. Namely, the following operator, used extensively in [10,16], has $\Pi_d(S)$ as its null space:

$$\triangle_d := \begin{cases} (\triangle^* + \lambda_0)(\triangle^* + \lambda_2)\cdots(\triangle^* + \lambda_d), & d \text{ even}, \\ (\triangle^* + \lambda_1)(\triangle^* + \lambda_3)\cdots(\triangle^* + \lambda_d), & d \text{ odd}, \end{cases} \tag{3.4}$$

where $\lambda_k := k(k+1)$, $k \in \mathbb{Z}_+$. Unfortunately, $\triangle_d$ is not suitable for working with functions on domains Ω smaller than S since in general its null space could be strictly larger than $\Pi_d(\Omega)$, *i.e.*, $\triangle_d$ could annihilate other functions beside polynomials. For an example, see Remark 6.2.

Thus, in this paper we need to construct seminorms in a different way. Our approach will be to view each spherical function as a restriction of a trivariate homogeneous function to the sphere S. In particular, given any spherical function f and any integer n, let f_n be its homogeneous extension of degree n, defined by

$$f_n(u) := |u|^n f\left(\frac{u}{|u|}\right), \quad u \in \mathbb{R}^3 \backslash \{0\}, \tag{3.5}$$

where $|u|$ denotes the Euclidean norm of u. If Ω is such that $\operatorname{diam}(\Omega) \le 1$ and $\bar{\Omega}$ is the planar domain defined in Sect. 2, we will write

$$\bar{f}_n := f_n|_{\bar{\Omega}}, \tag{3.6}$$

which is a bivariate function defined on $\bar{\Omega}$.

We need several elementary facts concerning homogeneous extensions.

Lemma 3.1. *Let Ω be an open subset of S with $\operatorname{diam}(\Omega) \le 1$. Let $g \in L_p(\Omega)$. Then for any integer n, the function $\bar{g}_n$ defined as in (3.6) satisfies*

$$M_1 \|g\|_{L_p(\Omega)} \le \|\bar{g}_n\|_{L_p(\bar{\Omega})} \le M_2 \|g\|_{L_p(\Omega)}, \tag{3.7}$$

where

$$M_1 := \begin{cases} m_\Omega^{n+3/p}, & n+3/p \ge 0, \\ M_\Omega^{n+3/p}, & n+3/p < 0, \end{cases} \qquad M_2 := \begin{cases} M_\Omega^{n+3/p}, & n+3/p \ge 0, \\ m_\Omega^{n+3/p}, & n+3/p < 0, \end{cases} \tag{3.8}$$

with

$$m_\Omega := \inf\left\{|\bar{\omega}|,\ \bar{\omega} \in \bar{\Omega}\right\} \ge 1, \quad M_\Omega := \sup\left\{|\bar{\omega}|,\ \bar{\omega} \in \bar{\Omega}\right\} \le (\cos 1/2)^{-1}.$$

Here the exponents in (3.8) are understood to be equal to n for $p = \infty$.

Proof: We prove (3.7) for $p < \infty$. The case $p = \infty$ is similar and simpler. Let σ and $\bar{\sigma}$ denote the Lebesgue measures on S and T_Ω, respectively. Using the substitution $\omega \to \bar{\omega} := R_\Omega^{-1}\omega \in \bar{\Omega}$, it follows by an elementary calculation that

$$\int_\Omega |g(\omega)|^p d\sigma(\omega) = \int_{\bar{\Omega}} |g(R_\Omega \bar{\omega})|^p |\bar{\omega}|^{-3} d\bar{\sigma}(\bar{\omega}).$$

By the homogeneity of g_n and the identity $|R_\Omega \bar{\omega}| = |\omega| = 1$, $\bar{\omega} \in \bar{\Omega}$, we can write

$$\bar{g}_n(\bar{\omega}) = g_n(\bar{\omega}) = g_n\left(|\bar{\omega}| R_\Omega \bar{\omega}\right) = |\bar{\omega}|^n g_n(R_\Omega \bar{\omega}) = |\bar{\omega}|^n g(R_\Omega \bar{\omega}),$$

and therefore

$$\int_\Omega |g(\omega)|^p d\sigma(\omega) = \int_{\bar{\Omega}} |\bar{\omega}|^{-(np+3)} |\bar{g}_n(\bar{\omega})|^p d\bar{\sigma}(\bar{\omega}).$$

Now (3.7) follows immediately using $m_\Omega \le |\bar{\omega}| \le M_\Omega$. The bound $m_\Omega \ge 1$ is trivial, while $M_\Omega \le (\cos 1/2)^{-1}$ is a consequence of $\operatorname{diam}(\Omega) \le 1$. $\square$

Lemma 3.1 asserts that for any n, $g \in L_p(\Omega)$ if and only if $\bar{g}_n \in L_p(\bar{\Omega})$. The following lemma makes an analogous assertion for Sobolev spaces.

Lemma 3.2. *Let $k, n \in \mathbb{Z}_+$, and suppose f is a function defined on Ω, with* $\operatorname{diam}(\Omega) \le 1$. *Then $f \in W^{k,p}(\Omega)$ if and only if $\bar{f}_n \in W^{k,p}(\bar{\Omega})$.*

Proof: Let $\hat{f} : \bar{\Omega} \to \mathbb{R}$ be defined as $\hat{f}(\bar{\omega}) := f(R_\Omega \bar{\omega})$, $\bar{\omega} \in \bar{\Omega}$. It is well known that $f \in W^{k,p}(\Omega)$ if and only if $\hat{f} \in W^{k,p}(\bar{\Omega})$. This is because in the definition (3.2) of $W^{k,p}(\Omega)$, we can choose an atlas consisting of a single chart (Γ, ϕ), where $\Gamma = \Omega$ and $\phi : \Gamma \to B := \bar{\Omega}$, where $\phi(\omega) := R_\Omega^{-1} \omega = \bar{\omega} \in \bar{\Omega}$, for $\omega \in \Omega$. Clearly, since $\operatorname{diam}(\Omega) \le 1$, the mapping ϕ is a C^∞-diffeomorphism of Ω onto $\bar{\Omega}$. Thus, in this case (3.2) expresses the fact that $f \in W^{k,p}(\Omega)$ if and only if $f \circ \phi^{-1} = \hat{f} \in W^{k,p}(\bar{\Omega})$.

Now note that $\bar{f}_n(\bar{\omega}) = |\bar{\omega}|^n \hat{f}(\bar{\omega})$, $\bar{\omega} \in \bar{\Omega}$. Since the functions $|\bar{\omega}|^n$ and $|\bar{\omega}|^{-n}$ are bounded infinitely differentiable functions whose derivatives are also bounded, using the Leibnitz rule, we see that multiplying any function in $W^{k,p}(\bar{\Omega})$ by $|\bar{\omega}|^n$ and $|\bar{\omega}|^{-n}$ results in a new function in the same Sobolev space. We conclude that $\hat{f} \in W^{k,p}(\bar{\Omega})$ if and only if $\bar{f}_n \in W^{k,p}(\bar{\Omega})$, which combined with the above completes the proof. $\square$

The following lemma shows that the trivariate functions obtained as homogeneous extensions of spherical functions (belonging to a Sobolev space) are differentiable in some sense. As usual, for $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}_+^3$, we write $D^\alpha := D_x^{\alpha_1} D_y^{\alpha_2} D_z^{\alpha_3}$ and $|\alpha| := \alpha_1 + \alpha_2 + \alpha_3$.

Lemma 3.3. *Let $f \in W^{k,p}(\Omega)$ for some $k \ge 1$ with* $\operatorname{diam}(\Omega) \le 1$. *Then $(D^\alpha f_{k-1})|_\Omega \in L_p(\Omega)$ for all multi-indices α with $|\alpha| = k$.*

Proof: Let $g := (D^\alpha f_{k-1})|_\Omega$. Note that whenever a trivariate homogeneous function is differentiated, the derivative is also homogeneous. In fact, $D^\alpha f_{k-1}$ is homogeneous of degree -1 and hence, $g_{-1} = D^\alpha f_{k-1}$. It will be sufficient to show that $\bar{g}_{-1} := g_{-1}|_{\bar{\Omega}} \in L_p(\bar{\Omega})$ since then, by Lemma 3.1, also $g = g_{-1}|_\Omega \in L_p(\Omega)$.

To do this, we assume without loss of generality that r_Ω is the north pole (*i.e.*, T_Ω is the plane $z = 1$) so that we can choose a Cartesian coordinate system (ξ, η) in T_Ω with $D_\xi = D_x$ and $D_\eta = D_y$. Let D_r denote differentiation in the radial direction, *i.e.*, for $|r| = 1$ and a trivariate function h, we have

$$D_r h(r) = \nabla h(r) \cdot r.$$

Since with $|\beta| = k-1$ the function $D^\beta f_{k-1}$ is homogeneous of degree zero, it follows that

$$D_r D^\beta f_{k-1} = 0. \tag{3.9}$$

Using $D_z = z^{-1}(D_r - xD_x - yD_y)$, $z \ne 0$, and (3.9), we obtain

$$D_z D^\beta f_{k-1} = z^{-1}(D_r - xD_x - yD_y) D^\beta f_{k-1} = -z^{-1}(x D_x D^\beta f_{k-1} + y D_y D^\beta f_{k-1}).$$

Iterating this identity, we obtain the more general formula

$$D_z^{\alpha_3} D_x^{\alpha_1} D_y^{\alpha_2} f_{k-1} = (-z)^{-\alpha_3} \sum_{\ell=0}^{\alpha_3} \binom{\alpha_3}{\ell} x^\ell y^{\alpha_3 - \ell} D_x^{\alpha_1 + \ell} D_y^{\alpha_2 + \alpha_3 - \ell} f_{k-1}, \tag{3.10}$$

which holds whenever $\alpha_1+\alpha_2+\alpha_3=k$.

Let $(x,y,z)\in\bar\Omega$. By our assumption on T_Ω, we have $z=1$. Moreover,

$$|x|\le|(x,y,z)|\le\sup\left\{|\bar\omega|:\bar\omega\in\bar\Omega\right\}=M_\Omega,$$

and similarly, $|y|\le M_\Omega$. Hence, it follows that $|\binom{\alpha_3}{\ell}x^\ell y^{\alpha_3-\ell}|\le(|x|+|y|)^{\alpha_3}\le(2M_\Omega)^{\alpha_3}$. We can now bound (3.10) as

$$\begin{aligned}
\|\bar g_{-1}\|_{L_p(\bar\Omega)} &= \|D_x^{\alpha_1}D_y^{\alpha_2}D_z^{\alpha_3}f_{k-1}\|_{L_p(\bar\Omega)}\\
&= \left\|\sum_{\ell=0}^{\alpha_3}\binom{\alpha_3}{\ell}x^\ell y^{\alpha_3-\ell}D_x^{\alpha_1+\ell}D_y^{\alpha_2+\alpha_3-\ell}f_{k-1}\right\|_{L_p(\bar\Omega)}\\
&\le (2M_\Omega)^{\alpha_3}\sum_{\gamma_1+\gamma_2=k}\|D_x^{\gamma_1}D_y^{\gamma_2}f_{k-1}\|_{L_p(\bar\Omega)} \qquad (3.11)\\
&= (2M_\Omega)^{\alpha_3}\sum_{\gamma_1+\gamma_2=k}\|D_x^{\gamma_1}D_y^{\gamma_2}\bar f_{k-1}\|_{L_p(\bar\Omega)}\\
&\le (2M_\Omega)^{\alpha_3}\|\bar f_{k-1}\|_{W^{k,p}(\bar\Omega)}<\infty,
\end{aligned}$$

where it is understood that the trivariate homogeneous functions involved in the above inequalities are first restricted to $\bar\Omega$ before we take their L_p norm. The last inequality follows from Lemma 3.2 with $n=k-1$ since $f\in W^{k,p}(\Omega)$. $\square$

The above lemmas motivate the following definition of a Sobolev-type seminorm on the sphere or any open subset Ω of S. For $k\ge0$ and $f\in W^{k,p}(\Omega)$, let

$$|f|_{k,p,\Omega}:=\sum_{|\alpha|=k}\|D^\alpha f_{k-1}\|_{L^p(\Omega)}, \tag{3.12}$$

where $\|D^\alpha f_{k-1}\|_{L^p(\Omega)}$ should be understood as the L_p-norm of the restriction of the trivariate function $D^\alpha f_{k-1}$ to Ω. For $k=0$, the above seminorm reduces to the usual L_p-norm

$$|f|_{0,p,\Omega}=\|f\|_{L^p(\Omega)}.$$

One reason why the above seminorms make sense is that they are locally equivalent to the usual Sobolev seminorms of functions defined in a plane. The precise statement is as follows. Let Ω be an open subset of S with $\mathrm{diam}(\Omega)\le1$. Let (ξ,η) be a local Cartesian system in the tangent plane T_Ω, *i.e.*, $|\xi|=|\eta|=1$, $\xi\cdot\eta=0$, and $\xi\cdot r_\Omega=\eta\cdot r_\Omega=0$. Let $|\cdot|_{k,p,\bar\Omega}$ be the usual Sobolev seminorm on $\bar\Omega$, *i.e.*,

$$|g|_{k,p,\bar\Omega}:=\sum_{\gamma_1+\gamma_2=k}\|D_\xi^{\gamma_1}D_\eta^{\gamma_2}g\|_{L^p(\bar\Omega)},\quad g\in W^{k,p}(\bar\Omega). \tag{3.13}$$

Proposition 3.4. *Let $\Omega\subset S$ with $\mathrm{diam}(\Omega)\le1$. Then the seminorms $|\cdot|_{k,p,\Omega}$ and $|\cdot|_{k,p,\bar\Omega}$ are equivalent in the sense that there exist positive constants C_1 and C_2 depending only on k and p such that for every $f\in W^{k,p}(\Omega)$,*

$$C_1\,|f|_{k,p,\Omega}\le|\bar f_{k-1}|_{k,p,\bar\Omega}\le C_2\,|f|_{k,p,\Omega}. \tag{3.14}$$

Proof: Suppose $f \in W^{k,p}(\Omega)$. Then

$$\begin{aligned}
|f|_{k,p,\Omega} &= \sum_{\alpha_1+\alpha_2+\alpha_3=k} \|D_x^{\alpha_1} D_y^{\alpha_2} D_z^{\alpha_3} f_{k-1}\|_{L^p(\Omega)} \\
&\le M_1^{-1} \sum_{\alpha_1+\alpha_2+\alpha_3=k} \|D_x^{\alpha_1} D_y^{\alpha_2} D_z^{\alpha_3} f_{k-1}\|_{L^p(\bar\Omega)} \\
&\le M_1^{-1} \sum_{\alpha_1+\alpha_2+\alpha_3=k} (2M_\Omega)^{\alpha_3} \sum_{\gamma_1+\gamma_2=k} \|D_x^{\gamma_1} D_y^{\gamma_2} \bar f_{k-1}\|_{L^p(\bar\Omega)} \\
&\le M_1^{-1}(2M_\Omega)^k \binom{k+2}{2} |\bar f_{k-1}|_{k,p,\bar\Omega},
\end{aligned}$$

where above, in the first inequality, we used Lemma 3.1 with $n = -1$, and in the second inequality, we employed (3.11). This proves the left-hand inequality in (3.14) with $C_1 = M_1(2M_\Omega)^{-k}\binom{k+2}{2}^{-1} > 0$. On the other hand,

$$\begin{aligned}
|f|_{k,p,\Omega} &= \sum_{\alpha_1+\alpha_2+\alpha_3=k} \|D_x^{\alpha_1} D_y^{\alpha_2} D_z^{\alpha_3} f_{k-1}\|_{L^p(\Omega)} \\
&\ge M_2^{-1} \sum_{\alpha_1+\alpha_2+\alpha_3=k} \|D_x^{\alpha_1} D_y^{\alpha_2} D_z^{\alpha_3} f_{k-1}\|_{L^p(\bar\Omega)} \\
&\ge M_2^{-1} \sum_{\gamma_1+\gamma_2=k} \|D_x^{\gamma_1} D_y^{\gamma_2} f_{k-1}\|_{L^p(\bar\Omega)} \\
&= M_2^{-1} |\bar f_{k-1}|_{k,p,\bar\Omega},
\end{aligned}$$

where in the first inequality, we used (3.7). This gives the right-hand inequality in (3.14) with $C_2 = M_2$. □

The main motivation behind definition (3.12) of Sobolev seminorms for spherical functions is the requirement that these seminorms annihilate spherical polynomials. We are now ready to prove this fact.

Proposition 3.5. *Suppose Ω is an open connected subset of S. For all $f \in W^{k,p}(\Omega)$ with $k \ge 1$, $|f|_{k,p,\Omega} = 0$ if and only if $f \in \Pi_{k-1}(\Omega)$.*

Proof: Clearly, $|f|_{k,p,\Omega} = 0$ if and only if $|f|_{k,p,\Omega'} = 0$, for all $\Omega' \subset \Omega$ such that $\operatorname{diam}(\Omega') \le 1$. By Proposition 3.4 (applied to Ω'), $|f|_{k,p,\Omega'} = 0$ if and only if $|\bar f_{k-1}|_{k,p,\bar\Omega'} = 0$. Since $\bar\Omega'$ is a planar region, it is well known that $|\bar f_{k-1}|_{k,p,\bar\Omega'} = 0$ if and only if $\bar f_{k-1} \in \Pi_{k-1}(\bar\Omega')$, *i.e.*, $\bar f_{k-1}$ is an ordinary bivariate polynomial of total degree at most $k-1$ on every open subset of $\bar\Omega$. Since $\bar\Omega$ is connected, this is equivalent to f_{k-1} being a trivariate homogeneous polynomial of degree $k-1$ on $\bar\Omega$. This in turn is equivalent to $f \in \Pi_{k-1}(\Omega)$ since $\Pi_{k-1}(\Omega)$ is precisely the space of trivariate homogeneous polynomials of degree $k-1$ restricted to Ω, see *e.g.*, [15]. □

Next, we show that the Sobolev norm of $\bar f_n = f_n|_{\bar\Omega}$ does not depend in an essential way on the degree n of the homogeneous extension of f that is used to define f_n.

Lemma 3.6. *Let $\Omega \subset S$ with* $\operatorname{diam}(\Omega) \le 1$. *Suppose $f \in W^{k,p}(\Omega)$ and let $\bar f_m$ and $\bar f_n$ be two homogeneous extensions of f restricted to $\bar\Omega$. Then*

$$\|\bar f_m\|_{k,p,\bar\Omega} \le C_3 \|\bar f_n\|_{k,p,\bar\Omega}, \tag{3.15}$$

for some constant C_3 depending only on k, m, and n.

Proof: Note that $\bar f_m = g \bar f_n$, where $g(u) := \|u\|^{m-n}, u \in T_\Omega$, *i.e.*, g is the restriction of the trivariate function $\|\cdot\|^{m-n}$ to T_Ω. It is not difficult to see that g is infinitely differentiable, and hence all of its partial derivatives are bounded on $\bar\Omega$, since $\bar\Omega$ is bounded. Let (ξ, η) be a Cartesian coordinate system in T_Ω, and let

$$K := \sup\{\|D^\gamma g\|_{\infty,\bar\Omega}\,,\ |\gamma| \le k\} < \infty,$$

where $D^\gamma = D_\xi^{\gamma_1} D_\eta^{\gamma_2}$. By the Leibnitz rule, we obtain

$$\begin{aligned}
\|\bar f_m\|_{k,p,\bar\Omega} &= \sum_{|\alpha|\le k} \|D^\alpha (g \bar f_n)\|_{L_p(\bar\Omega)} \le K \sum_{|\alpha|\le k} \sum_{\beta \le \alpha} \|D^\beta \bar f_n\|_{L_p(\bar\Omega)} \\
&= K \sum_{|\beta|\le k} \#\{\alpha : |\alpha| \le k,\ \alpha \ge \beta\}\, \|D^\beta \bar f_n\|_{L_p(\bar\Omega)} \\
&\le K \binom{k+2}{2} \|\bar f_n\|_{k,p,\bar\Omega},
\end{aligned}$$

which shows (3.15) with $C_3 = K\binom{k+2}{2}$. Now since $\operatorname{diam}(\Omega) \le 1$, by (2.1) the diameter of a smallest disk B in T_Ω containing $\bar\Omega$ and centered at r_Ω is at most A_1 which implies that the constant K is bounded by

$$\sup\{\|D^\gamma g\|_{\infty,B}\,,\ |\gamma| \le k\} < \infty,$$

which thus depends only on $m - n$ and k. $\square$

§4. Local Approximation by Spherical Polynomials

As in the planar case, cf. [7,12], the key to getting error bounds for approximation of smooth functions by spherical splines is to first investigate how well such functions can be approximated locally by polynomials. As both a guide to the type of result to be expected and as a tool in proving that result, we begin by recalling what is known in the bivariate setting.

Let $\bar\Omega \subset \mathbb{R}^2$ be a set whose interior is a connected set and such that Ω is the union of a finite number of non-degenerate planar triangles, which are either disjoint or share a common edge or a vertex.

Proposition 4.1. *Let $g \in W^{d+1,p}(\bar{\Omega})$ with $d \geq 0$. Then there exists a polynomial $q \in \Pi_d(\bar{\Omega})$ such that for every $0 \leq k \leq d$,*

$$|g - q|_{k,p,\bar{\Omega}} \leq C_4 \operatorname{diam}(\bar{\Omega})^{d+1-k} |g|_{d+1,p,\bar{\Omega}}, \tag{4.1}$$

where C_4 is a constant depending only on d, p, the minimum angle of the triangulation of $\bar{\Omega}$, and the Lipschitz constant of the boundary $\partial\bar{\Omega}$ of $\bar{\Omega}$. If $\bar{\Omega}$ is convex, then C_4 can be chosen to be independent of the Lipschitz constant of $\partial\bar{\Omega}$.

Proposition 4.1 has been proved in various degrees of generality by several authors, see *e.g.*, Lemma (4.3.8) in [6] and Theorem 3.1.5 in [8]. In the above form, the result was essentially proved in [12], Lemma 4.6, the only difference being that here the Sobolev seminorm $|\cdot|_{k,p,\bar{\Omega}}$ differs from that in [12], but is equivalent to it. The proof in [12] provides an explicit construction of an optimal polynomial q.

We now prove a spherical analog of Proposition 4.1. Suppose that $\Omega \subset S$ is a connected set that is the union of a finite number of non-degenerate spherical triangles which are either disjoint or share a common edge or a vertex. We define the Lipschitz constant of the boundary $\partial\Omega$ of Ω to be the minimum exterior angle of $\partial\Omega$. Also, from now on we shall assume that Ω is such that $\operatorname{diam}(\Omega) \leq 1$. In addition to (2.1) and (2.2), this condition implies that the minimum angles of the triangulations of Ω and $\bar{\Omega}$ are of comparable size, and the same is true of the Lipschitz constants of $\partial\Omega$ and $\partial\bar{\Omega}$. A rigorous proof of these facts is elementary.

Theorem 4.2. *Suppose Ω is an open subset of S with $\operatorname{diam}(\Omega) \leq 1$. Let $f \in W^{d+1,p}(\Omega)$ with $d \geq 0$. Then there exists a spherical polynomial $s \in \Pi_d(S)$ such that for every $0 \leq k \leq d$,*

$$|f - s|_{k,p,\Omega} \leq C_5 \operatorname{diam}(\Omega)^{d+1-k} |f|_{d+1,p,\Omega}. \tag{4.2}$$

Here C_5 is a constant that depends only on d, p, the smallest angle in the triangulation of Ω, and the Lipschitz constant of $\partial\Omega$. If Ω is such that the cone $\hat{\Omega} := \{u \in \mathbb{R}^3 \backslash \{0\}, u/|u| \in \Omega\}$ is convex, then C_5 can be taken to be independent of the Lipschitz constant of $\partial\Omega$.

Proof: As before, let $\bar{\Omega}$ be the planar region which is the radial projection of Ω onto the tangent plane T_Ω. Since $f \in W^{d+1,p}(\Omega)$, Lemma 3.2 implies $\bar{f}_d \in W^{d+1,p}(\bar{\Omega})$. But then Proposition 4.1 guarantees the existence of a bivariate polynomial $\bar{s} \in \Pi_d(\bar{\Omega})$ satisfying

$$\begin{aligned}|\bar{f}_d - \bar{s}|_{\ell,p,\bar{\Omega}} &\leq C_4 \operatorname{diam}(\bar{\Omega})^{d+1-\ell} |\bar{f}_d|_{d+1,p,\bar{\Omega}} \\ &\leq C_4 A_1^{d+1} C_2 \operatorname{diam}(\Omega)^{d+1-\ell} |f|_{d+1,p,\Omega},\end{aligned}$$

for every $\ell = 0, \ldots, k$, where C_4, A_1, and C_2 are the constants in (4.1), (2.1), and (3.14), respectively.

We are now able to define a polynomial $s \in \Pi_d(\Omega)$ which we claim will satisfy (4.2). Indeed, since the polynomial $\bar{s}$ can be thought of as the restriction to $\bar{\Omega}$ of a unique trivariate polynomial that is homogeneous of degree d, it follows that there exists a unique spherical polynomial $s \in \Pi_d(\Omega)$ whose homogeneous extension of degree d restricted to $\bar{\Omega}$ is equal to $\bar{s}$, *i.e.*, such that $\bar{s} = s_d|_{\bar{\Omega}}$. This is the desired polynomial.

Using Proposition 3.4 and Lemma 3.6, in conjunction with the previous inequality, it follows that

$$
\begin{aligned}
|f - s|_{k,p,\Omega} &\leq C_1^{-1} |\bar{f}_{k-1} - \bar{s}_{k-1}|_{k,p,\bar{\Omega}} \leq C_1^{-1} \|\bar{f}_{k-1} - \bar{s}_{k-1}\|_{k,p,\bar{\Omega}} \\
&\leq C_1^{-1} C_3 \|\bar{f}_d - \bar{s}_d\|_{k,p,\bar{\Omega}} = C_1^{-1} C_3 \sum_{\ell=0}^{k} |\bar{f}_d - \bar{s}|_{\ell,p,\bar{\Omega}} \\
&\leq C_1^{-1} C_3 C_4 A_1^{d+1} C_2 \sum_{\ell=0}^{k} \operatorname{diam}(\Omega)^{d+1-\ell} |f|_{d+1,p,\Omega} \\
&= C_1^{-1} C_3 C_4 A_1^{d+1} C_2 \left(\sum_{\ell=0}^{k} \operatorname{diam}(\Omega)^{\ell} \right) \operatorname{diam}(\Omega)^{d+1-k} |f|_{d+1,p,\Omega} \\
&\leq C_5 \operatorname{diam}(\Omega)^{d+1-k} |f|_{d+1,p,\Omega},
\end{aligned}
$$

where C_1 and C_3 are the constants in (3.14) and (3.15), respectively, and $C_5 := C_1^{-1} C_3 C_4 A_1^{d+1} C_2 (d+1)$. Here we have used the fact $\operatorname{diam}(\Omega) \leq 1$ implies $\sum_{\ell=0}^{k} \operatorname{diam}(\Omega)^{\ell} \leq k+1 \leq d+1$.

The dependence of C_5 on the Lipschitz constant of $\partial\Omega$ enters via the constant C_4, *i.e.*, via the Lipschitz constant of $\partial\bar{\Omega}$. However, the convexity of $\hat{\Omega}$ is equivalent with the convexity of $\bar{\Omega}$, in which case C_4 can be chosen to be independent of the Lipschitz constant of $\partial\bar{\Omega}$ and hence also independent of the Lipschitz constant of $\partial\Omega$. $\square$

We close this section with several results on spherical polynomials. Our first result is a spherical analog of a Markov-type inequality for spherical polynomials.

Proposition 4.3. *Let $\Omega \subset S$ with $\operatorname{diam}(\Omega) \leq 1$. Then there exists a constant M depending on d such that for all $s \in \Pi_d(\Omega)$ and all $0 \leq k \leq d$,*

$$|s|_{k,p,\Omega} \leq M \rho_{\Omega}^{-k} \|s\|_{L_p(\Omega)}, \tag{4.3}$$

where ρ_Ω is the diameter of the largest spherical cap contained in Ω.

Proof: As in the previous theorem, we will utilize the connection between spherical polynomials and bivariate polynomials defined on T_Ω. The following Markov inequality for bivariate polynomials is well known (see *e.g.*, [12], Lemma 4.2):

$$|\bar{s}_d|_{k,p,\bar{\Omega}} \leq \bar{M} \rho_{\bar{\Omega}}^{-k} \|\bar{s}_d\|_{L_p(\bar{\Omega})}, \tag{4.4}$$

for some constant $\bar{M}$ depending on d. Thus, as in the proof of Theorem 4.2, we conclude that

$$\begin{aligned}
|s|_{k,p,\Omega} &\le C_1^{-1}|\bar{s}_{k-1}|_{k,p,\bar{\Omega}} \le C_1^{-1}\|\bar{s}_{k-1}\|_{k,p,\bar{\Omega}} \le C_1^{-1}C_3\|\bar{s}_d\|_{k,p,\bar{\Omega}} \\
&\le C_1^{-1}C_3\bar{M}\sum_{\ell=0}^{k}\rho_{\bar{\Omega}}^{-\ell}\|\bar{s}_d\|_{p,\bar{\Omega}} \\
&= C_1^{-1}C_3\bar{M}\left(\sum_{\ell=0}^{k}\rho_{\bar{\Omega}}^{\ell}\right)\rho_{\bar{\Omega}}^{-k}\|\bar{s}_d\|_{p,\bar{\Omega}} \\
&\le C_1^{-1}C_3\bar{M}A_2^{d+1}(d+1)\rho_{\Omega}^{-k}\|s\|_{p,\Omega},
\end{aligned}$$

where in the second line we used (4.4). For the last step we used (2.2) to get $\rho_{\bar{\Omega}} \le A_2\rho_\Omega \le A_2\mathrm{diam}(\Omega) \le A_2$ which in turn implies $\sum_{\ell=0}^{k}\rho_{\bar{\Omega}}^{\ell} \le A_2^{k+1}(k+1) \le A_2^{d+1}(d+1)$. We have shown that (4.3) holds with $M := C_1^{-1}C_3\bar{M}A_2^{d+1}(d+1)$. $\square$

Our next result compares the sizes of the L_p and L_∞ norms of a polynomial on a triangle T.

Lemma 4.4. *Let T be a spherical triangle such that* $\mathrm{diam}(T) \le 1$. *Then there exists a constant M_3 such that*

$$A_T^{-1/p}\|s\|_{L_p(T)} \le \|s\|_{L_\infty(T)} \le M_3\, A_T^{-1/p}\|s\|_{L_p(T)}, \quad \forall s \in \Pi_d(T), \tag{4.5}$$

where A_T is the area of T. The constant M_3 depends only on d, p, and the minimum angle of T.

Proof: The first inequality is an elementary consequence of Hölder's inequality. As for the second inequality, let $\bar{s}_d := s_d|_{\bar{T}}$. We already noted that $\bar{s}_d$ is an ordinary bivariate polynomial, *i.e.*, $\bar{s}_d \in \Pi_d(\bar{T})$. The planar analog of the second inequality in (4.5) is (see the proof of Lemma 4.1 in [12])

$$\|\bar{s}_d\|_{L_\infty(\bar{T})} \le \bar{M}_3 A_{\bar{T}}^{-1/p}\|\bar{s}_d\|_{L_p(\bar{T})}, \quad \forall \bar{s}_d \in \Pi_d(\bar{T}), \tag{4.6}$$

where $\bar{M}_3$ depends only on d, p, and the minimum angle in $\bar{T}$. Using this fact and Lemma 3.1 (with $n = d$), we obtain

$$\begin{aligned}
\|s\|_{L_\infty(T)} &\le M_1^{-1}\|\bar{s}_d\|_{L_\infty(\bar{T})} \\
&\le M_1^{-1}\bar{M}_3 A_{\bar{T}}^{-1/p}\|\bar{s}_d\|_{L_p(\bar{T})} \\
&\le M_1^{-1}M_2\bar{M}_3 A_T^{-1/p}\|s\|_{L_p(T)},
\end{aligned}$$

where M_1 and M_2 depend only on d and p. Here we used the elementary fact that $A_{\bar{T}} \ge A_T$. To finish the proof, it is enough to note that since $\mathrm{diam}(T) \le 1$, the minimum angle of $\bar{T}$ is bounded from below by an absolute constant multiplied by the minimum angle of T. $\square$

§5. Approximation Order of Spherical Splines

In this section we prove Theorem 1.1 by giving bounds on how well smooth functions can be approximated by spherical splines. Our results are direct analogs of similar results for bivariate splines on planar triangulations, see *e.g.*, [7,12] and references therein.

Let $\triangle$ be a spherical triangulation of S. Since we are interested in approximation order for small values of the mesh size $|\triangle|$. without loss of generality we assume throughout the remainder of the paper that $|\triangle| \le 1$. Let d, r be integers such that $d \ge 3r+2$, and let $S_d^r(\triangle)$ be a space of spherical splines defined in (1.1). We write N for the dimension of $S_d^r(\triangle)$. Given any vertex v of $\triangle$, we write $\mathrm{star}^0(v) := \{v\}$, and for all $\ell \ge 1$ inductively define $\mathrm{star}^\ell(v)$ to be the union of all triangles in $\triangle$ sharing a vertex with some vertex of $\mathrm{star}^{\ell-1}(v)$. In [3] we gave explicit formulae for N, along with a construction of a set of basis functions $\{B_i\}_{i=1}^N$ which are locally supported in the sense that for each i, there is a vertex v_i of the triangulation $\triangle$ such that $\mathrm{supp}(B_i) \subseteq \mathrm{star}(v_i)$. This basis is, however, not adequate for our purposes here since it is not stable in the presence of near-singular vertices (cf. the discussion in [9,12]).

The construction of stable bases is a delicate process, even in the case of bivariate splines, see [9] and references therein. Fortunately, as pointed out in [9], Remark 13.13, the construction presented there for the bivariate analog of $S_d^r(\triangle)$ also carries over to the spherical spline space $S_d^r(\triangle)$. We now briefly outline this construction. The key is to work with the Bernstein-Bézier representation of splines in $\mathcal{S}_d^0(\triangle)$, see [2–4]. First we introduce the set of domain points

$$\mathcal{D} := \bigcup_{T := \langle u,v,w \rangle \in \triangle} \left\{ \xi^T_{\nu\mu\kappa} = \frac{\nu u + \mu v + \kappa w}{d} \right\}_{\nu+\mu+\kappa=d}.$$

It is well known [4] that each spline in $\mathcal{S}_d^0(\triangle)$ is uniquely determined by associating one Bézier coefficient with each domain point.

A set $\mathcal{M} := \{\xi_i\}_{i=1}^N \subset \mathcal{D}$ is called a minimal determining set for $S_d^r(\triangle)$ if the values of the coefficients of $s \in S_d^r(\triangle)$ associated with domain points in $\mathcal{M}$ uniquely determine all of the coefficients of s. Given a minimal determining set, we can construct a basis $\{B_i\}_{i=1}^N$ for $S_d^r(\triangle)$ by requiring

$$\mu_j B_i = \delta_{ij}, \quad 1 \le i, j \le N, \tag{5.1}$$

where for all $1 \le j \le N$, μ_j is the linear functional which picks the coefficient associated with the domain point ξ_j. In particular, B_i has the property that the coefficient associated with ξ_i is 1 while the coefficients associated with all other ξ_j in $\mathcal{M}$ are zero. The remaining coefficients of B_i are computed using smoothness conditions.

For any given spline space $S_d^r(\triangle)$, there are many possible choices for a minimal determining set $\mathcal{M}$. To get a stable local basis requires choosing $\mathcal{M}$ carefully to insure the local support, and to insure that for each B_i all of the calculated coefficients remain bounded. The choice of $\mathcal{M}$ presented in [9] leads to a basis with the following properties, where for each i, $\Omega_i := \mathrm{supp}(B_i)$ and T_i is the triangle in which ξ_i lies.

Proposition 5.1. *Let $\{B_i\}_{i=1}^N$ be the basis for $S_d^r(\triangle)$ corresponding to the minimal determining set $\mathcal{M}$ described in [9]. Then there exist constants $K_1, \ldots, K_7$ depending only on d, p, and the minimal angle in $\triangle$ such that for each $1 \le i \le N$,*

1) *there exists a vertex $v_i \in \triangle$ such that $\Omega_i \subseteq \mathrm{star}^3(v_i)$,*
2) $\|B_i\|_{L_\infty(S)} \le K_1$,
3) $|\mu_i s| \le K_2 \|s\|_{L_\infty(T_i)}$, *for all $s \in S_d^r(\triangle)$,*
4) $|\mu_i s| \le K_3 A_{T_i}^{-1/p} \|s\|_{L_p(T_i)}$, *for all $s \in S_d^r(\triangle)$,*

and for every $T \in \triangle$,

5) $\|B_i\|_{L_p(T)} \le K_4 A_T^{1/p}$,
6) $\#I_T \le K_5$, where $I_T := \{i : T \subset \Omega_i\}$,
7) $|B_i|_{k,\infty,T} \le K_6 \rho_T^{-k}$, for all $0 \le k \le d$,
8) $|B_i|_{k,p,T} \le K_7 \rho_T^{-k} A_T^{1/p}$, for all $0 \le k \le d$.

Proof: Properties 1) and 2) follow directly from the construction of the basis. Property 3) follows from the stability of the spherical Bernstein basis (see Remark 6.3) since restricted to T_i, s is just a spherical polynomial. Combining 3) and (4.5), we get 4) with $K_3 := K_2 M_3$. Combining 2) with (4.5) leads to 5) with $K_4 := K_1$. Property 6) follows from 1) as in the bivariate case (cf. Lemma 3.1 of [12]). Combining 2) with the Markov inequality (4.3) gives 7) with $K_6 := MK_1$. Combining 5) with the Markov inequality (4.3) gives 8) with $K_7 := MK_4$. $\square$

We are now ready to construct a quasi-interpolation operator $Q : L_p(S) \to S_d^r(\triangle)$ which will produce the desired approximation results. To this end, we first use the Hahn-Banach theorem to extend the linear functionals μ_i appearing in (5.1) to all of $L_p(S)$. We will continue to use the same symbol for these extensions. By the theorem and by 4) of Proposition 5.1, we know that for every $f \in L_p(T_i)$,

$$|\mu_i f| \le K_3 A_{T_i}^{-1/p} \|f\|_{L_p(T_i)}, \quad 1 \le i \le N. \tag{5.2}$$

This inequality immediately implies that for each i, the carrier of the extended functional μ_i is contained in T_i, *i.e.*, if $f \equiv 0$ on T_i, then $\mu_i f = 0$.

Proposition 5.2. *For each $f \in L_p(\Omega)$, let*

$$Qf := \sum_{i=1}^N (\mu_i f) B_i. \tag{5.3}$$

Then $Qg = g$ for all $g \in \Pi_d(S)$. Moreover, there exists a constant K_8 depending only on d, p, and the smallest angle in $\triangle$ such that for each triangle $T \in \triangle$,

$$|Qf|_{k,p,T} \le K_8\, \rho_T^{-k} \|f\|_{L_p(\Omega_T)}, \tag{5.4}$$

where $I_T := \{i : T \subset \Omega_i\}$ and $\Omega_T := \cup_{i \in I_T} \Omega_i$.

Proof: The fact that Q reproduces polynomials in $\Pi_d(S)$ is clear from (5.1). To prove (5.4), by (5.2) we have

$$\begin{aligned}
|Qf|_{k,p,T} &= \left| \sum_{i \in I_T} (\mu_i f) B_i \right|_{k,p,T} \\
&\le \sum_{i \in I_T} |\mu_i f| |B_i|_{k,p,T} \le K_3 \sum_{i \in I_T} A_{T_i}^{-1/p} \|f\|_{L_p(T_i)} |B_i|_{k,p,T} \\
&\le K_3 A^{-1/p} \|f\|_{L_p(\Omega_T)} \sum_{i \in I_T} |B_i|_{k,p,T} \\
&\le K_3 A^{-1/p} \|f\|_{L_p(\Omega_T)} \# I_T K_7 \rho_T^{-k} A_T^{1/p},
\end{aligned}$$

where $A := \min\{A_{T_i} : i \in I_T\}$ and where we have inserted 8) from Proposition 5.1 in the last inequality. Using the fact that A_T/A is bounded by a constant K_9 depending only on the smallest angle in $\triangle$ (cf. (3.8) in [12] in the bivariate case), we get (5.4) with $K_8 := K_3 K_7 K_9^{1/p} \# I_T$. $\square$

We are now ready to prove the main result of this paper. Instead of Theorem 1.1, we prove a slightly stronger result.

Theorem 5.3. *Let $1 \le p \le \infty$, $d \ge 3r + 2$, and $0 \le k \le d$. Then there exists a constant C, depending only on d, p, and the smallest angle in $\triangle$, such that*

$$|f - Qf|_{k,p,T} \le C \operatorname{diam}(\Omega_T)^{d+1-k} |f|_{d+1,p,\Omega_T}, \tag{5.5}$$

for all $f \in W^{d+1,p}(S)$ and all $T \in \triangle$. Moreover, there exists a constant C', depending only on d, p, and the smallest angle in $\triangle$, such that

$$|f - Qf|_{k,p,S} \le C' \, |\triangle|^{d+1-k} |f|_{d+1,p,S}, \tag{5.6}$$

for all $f \in W^{d+1,p}(S)$ and all $0 \le k \le d$ such that $Qf \in W^{k,p}(S)$.

Proof: For $T \in \triangle$, let $s \in \Pi_d(\Omega)$ be such that

$$|f - s|_{k,p,T} \le |f - s|_{k,p,\Omega_T} \le C'_T \operatorname{diam}(\Omega_T)^{d+1-k} |f|_{d+1,p,\Omega_T} \tag{5.7}$$

and

$$\|f - s\|_{L_p(\Omega_T)} \le C''_T \operatorname{diam}(\Omega_T)^{d+1} |f|_{d+1,p,\Omega_T}, \tag{5.8}$$

where C'_T and C''_T depend only on d, p, the smallest angle in the triangulation of Ω_T, and the Lipschitz constant of $\partial\Omega_T$, which in turn depends only on the smallest angle in the triangulation $\triangle$. The existence of such a polynomial s is guaranteed by Theorem 4.2. By the linearity of Q and the fact that Q reproduces polynomials of degree d, we can write

$$|f - Qf|_{k,p,T} \leq |f - s|_{k,p,T} + |Q(f - s)|_{k,p,T}. \tag{5.9}$$

We now estimate the second term on the right-hand side of (5.9). It follows from 1) of Proposition 5.1 that there exists a constant $D \geq 1$, depending only on the minimum angle in the triangulation of Ω_T, such that $\operatorname{diam}(\Omega_T) \leq D\,\rho_T$, or equivalently, $\rho_T^{-1} \leq D \operatorname{diam}(\Omega_T)^{-1}$. By (5.4) and (5.8), we thus have

$$\begin{aligned} |Q(f - s)|_{k,p,T} &\leq K_8 \rho_T^{-k} \|f - s\|_{L_p(\Omega_T)} \\ &\leq K_8 C''_T \rho_T^{-k} \operatorname{diam}(\Omega_T)^{d+1} |f|_{d+1,p,\Omega_T} \\ &\leq K_8 C''_T D^k \operatorname{diam}(\Omega_T)^{d+1-k} |f|_{d+1,p,\Omega_T}. \end{aligned} \tag{5.10}$$

Combining (5.9), (5.7), and (5.10), we get the first assertion of the theorem with $C := C'_T + K_8 C''_T D^d$.

We now consider the second assertion. To prove (5.6), we sum (5.5) over all triangles in $\triangle$. To do this, we first note that by the local support property of the basis functions B_i, there exists a constant $L \geq 1$ which depends only on the smallest angle in $\triangle$ such that $\operatorname{diam}(\Omega_T) \leq L \operatorname{diam}(T)$, for every $T \in \triangle$. By (5.5), we obtain

$$\begin{aligned} |f - Qf|_{k,p,S} &= \sum_{T \in \triangle} |f - Qf|_{k,p,T} \\ &\leq C \sum_{T \in \triangle} \operatorname{diam}(\Omega_T)^{d+1-k} |f|_{d+1,p,\Omega_T} \\ &\leq C L^{d+1-k} \sum_{T \in \triangle} \operatorname{diam}(T)^{d+1-k} |f|_{d+1,p,\Omega_T} \\ &\leq C L^{d+1} |\triangle|^{d+1-k} \sum_{T \in \triangle} \sum_{\substack{T' \in \triangle \\ T' \subset \Omega_T}} |f|_{d+1,p,T'} \\ &= C L^{d+1} |\triangle|^{d+1-k} \sum_{T' \in \triangle} \#\{T : T' \subset \Omega_T\} |f|_{d+1,p,T'} \\ &\leq C L^{d+1} M' |\triangle|^{d+1-k} \sum_{T' \in \triangle} |f|_{d+1,p,T'} \\ &= C L^{d+1} M' |\triangle|^{d+1-k} |f|_{d+1,p,S}, \end{aligned}$$

where $M' := \max\{\#\{T : T' \subset \Omega_T\},\ T' \in \triangle\}$, which is a constant depending only on the minimum angle in $\triangle$. $\square$

§6. Remarks

Remark 6.1. The seminorms defined in (3.12) can be used to define a norm on the space $W^{k,p}(\Omega)$ as

$$\|f\|'_{W^{k,p}(\Omega)} := \sum_{\ell=0}^{k} |f|_{\ell,p,\Omega}.$$

Using the results of Sect. 3 and a partition of unity $\{\alpha_j\}$ associated with an atlas $\{(\Gamma_j, \phi_j)\}$ for Ω, it can be shown that this norm is equivalent to the norm $\|f\|_{W^{k,p}(\Omega)}$ defined in (3.3). Another alternative is to define Sobolev norms using covariant derivatives, see *e.g.* [5,11].

Remark 6.2. In Section 3, we mentioned that the operators defined by (3.4) were not well suited for defining Sobolev seminorms for domains Ω that are proper subsets of S. The following example illustrates this fact. Consider the function

$$f(x, y, z) := \log\left(\frac{1+z}{1-z}\right), \quad (x, y, z) \in S.$$

Clearly, f is a spherical function that is infinitely differentiable except at the south and north poles ($z = \pm 1$). Let Ω be any open subset of S not containing the poles. Then one can easily check that

$$\triangle_0 f = \triangle^* f = 0 \quad \text{on } \Omega,$$

i.e., in addition to constants, which are spherical harmonics of degree $d = 0$, the operator $\triangle_0$ also annihilates other smooth functions defined on Ω.

Remark 6.3. Let $s \in \Pi_d(T)$ be a spherical polynomial defined on a spherical triangle T with $\operatorname{diam}(T) \le 1$, and let c be the vector of Bézier coefficients of s (see [2]). Then the Bernstein-Bézier representation of s is stable in the sense that there exist positive constants D_1 and D_2, depending only on d, such that

$$D_1 \|c\|_\infty \le \|s\|_{L_\infty(T)} \le D_2 \|c\|_\infty.$$

This is a standard result for bivariate polynomials, see *e.g.*, [12], Lemma 4.1. It is easy to see that a similar proof works in the spherical case. The inequalities can also be proved using radial projection.

Remark 6.4. Since $Qf \in C^r(S)$, the largest integer k in (5.6) for which $Qf \in W^{k,p}(S)$ is always at least $r + 1$.

Remark 6.5. Theorem 5.3 can easily be generalized to provide bounds on $|f - Qf|_{k,p,\Omega}$, where Ω is a set formed from a union of triangles of $\triangle$. The only change in this case is that the constant C' on the right-hand side of (5.6) then depends on the Lipschitz constant of the boundary of Ω, and the semi-norm on the right-hand side of (5.6) has to be taken over the set $\tilde{\Omega} := \cup\{\Omega_i : \Omega_i \cap \Omega \neq \emptyset\}$.

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