Scattered Data Interpolation Using C^2 Supersplines of Degree Six

by

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Abstract. We show how C^2 supersplines of degree 6 can be used to interpolate Hermite data at the vertices of a quadrangulation. We also present error bounds which show that our method has full approximation order 7, and compare its efficiency with other C^2 interpolation methods in the literature.

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1. Introduction

Suppose $\mathcal{V} = \{v_i = (x_i, y_i)\}_{i=1}^n$ is a set of points lying in a domain $\Omega \subset \mathbb{R}^2$, and suppose $\{z_i^{\nu,\mu}\}_{0\le\nu+\mu\le 2}$, are corresponding real numbers for $i = 1, \ldots, n$. Our aim in this paper is to construct a function $s \in C^2(\Omega)$ such that

$$D_x^\nu D_y^\mu s(v_i) = z_i^{\nu,\mu}, \qquad 0 \le \nu + \mu \le 2, \quad i = 1, \ldots, n. \tag{1.1}$$

This is the classical *Hermite scattered data interpolation problem.* In practice we may be given only function values (or function values and first derivatives) at each data point. In this case we can estimate the missing derivative information by

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standard local quadrature rules, and then proceed as if the full Hermite data were available. There are several methods in the literature for solving this interpolation problem using polynomial splines of the form

$$\mathcal{S}_d^r(\triangle) := \{s \in C^r(\Omega) \ : \ s|_T \in \mathcal{P}_d \text{ for all triangles } T \in \triangle\},$$

where $\triangle$ is a triangulation with vertices at the points of $\mathcal{V}$, $\mathcal{P}_d$ is the space of polynomials of total degree d, and Ω is the union of the triangles. For references, see Remarks 1 – 5. Most of these methods involve refining the initial triangulation which results in many triangles and spline spaces of fairly high dimension. Following ideas introduced in [11], in this paper we show that we can solve the Hermite interpolation problem using splines of degree 6 which

1) produces an interpolant with optimal approximation order $\mathcal{O}(h^7)$, where h is the mesh size,
2) uses locally supported basis functions,
3) is very efficient compared with other C^2 methods in that it uses a less complicated triangulation, and generally involves fewer parameters,
4) can be displayed using quadrilaterals rather than triangles.

2. The Main Result

Before stating our main result, we need some additional notation. Following [11], we define a quadrangulation.

Definition 2.1. *Let $\mathcal{V} = \{v_i\}_{i=1}^n$ be a set of points in $\mathbb{R}^2$. A set $\diamondsuit$ of quadrangles (quadrilaterals) with vertices $\mathcal{V}$ is called a quadrangulation if the intersection of any two quadrangles $q_i, q_j \in \diamondsuit$ is either empty, a common vertex, or a common edge.*

Throughout this paper we assume that starting with the set of data sites $\mathcal{V}$, there is a quadrangulation whose vertices fall at the points of $\mathcal{V}$. For a reference to algorithms for constructing quadrangulations, see Remark 10. We denote the union of the quadrangles in $\diamondsuit$ by Ω. Note that we do not require either Ω or the quadrangles in $\diamondsuit$ to be convex. Given a quadrangulation $\diamondsuit$, we now introduce a unique natural triangulation associated with it.

Definition 2.2. *Given a quadrangulation $\diamondsuit$, we define a corresponding triangulation $\oplus$ as follows:*

1) *for each convex quadrangle, draw in both diagonals,*
2) *for each nonconvex quadrangle, draw in the diagonal which lies inside, and then connect its center point to the remaining two vertices.*

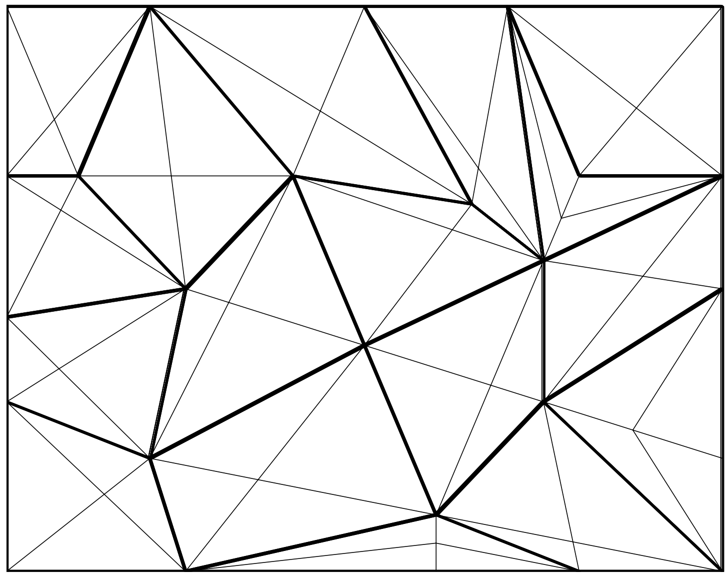

Fig. 1. A quadrangulation and associated triangulation.

Fig. 1 shows a typical quadrangulation and its associated triangulation. As discussed in Remark 11, it may be better to divide some convex quadrilaterals using method 2) rather than method 1).

The following is the main result of this paper. It follows from the slightly stronger Theorem 5.4 to be proved in Sect. 5 below.

Theorem 2.3. *Let* $\oplus$ *be the triangulation associated with a quadrangulation* $\Diamond$ *of a set of scattered data points* $\mathcal{V}$. *Then for any given data* $\{z_i^{\nu,\mu}\}$, *there exists a spline* $s \in \mathcal{S}_6^2(\oplus)$ *satisfying (1.1). Moreover, if* $z_i^{\nu,\mu} = D_x^\nu D_y^\mu f(v_i)$ *for* $0 \leq \nu + \mu \leq 2$ *and* $i = 1, \ldots, n$ *for some function* $f \in C^7(\Omega)$, *then*

$$\|f - s\| \leq K h^7 |D^7 f|, \tag{2.1}$$

where $\|\cdot\|$ *measures the uniform norm on* Ω, h *is the diameter of the largest triangle in* $\oplus$, *and*

$$|D^i f| = \sum_{\nu+\mu=i} \|D_x^\nu D_y^\mu f\|. \tag{2.2}$$

Here K *is a constant which depends only on the smallest angle in* $\oplus$.

3. Preliminaries

To prove Theorem 2.3, we are going to work with the following *superspline subspace* of $\mathcal{S}_6^2(\oplus)$:

$$\mathcal{SS}_6^2(\oplus) := \{s \in \mathcal{S}_6^2(\oplus) \;:\; s \in C^3(v) \text{ for all } v \in \mathcal{V}_3\}, \tag{3.1}$$

where $s \in C^3(v)$ means that s has three continuous derivatives at v, and where $\mathcal{V}_3$ is a certain subset of the vertices of $\diamondsuit$ to be defined in Theorem 4.2 below.

To describe our spline interpolant, we first need to identify the dimension of $\mathcal{SS}_6^2(\diamondsuit)$ and construct a local basis for it. Our main tool is the Bernstein-Bézier representation for the polynomial pieces of a spline, see e.g., Farin [8] or de Boor [3]. Given a spline function $s \in \mathcal{S}_6^0(\diamondsuit)$, then its restriction to the triangle $T := \langle v_1, v_2, v_3 \rangle \in \diamondsuit$ can be expressed as

$$s|_T(v) = \sum_{i+j+k=6} c_{ijk}^T \frac{6!}{i!j!k!} b_1^i b_2^j b_3^k, \tag{3.2}$$

where b_1, b_2, b_3 are the barycentric coordinates of v with respect to the triangle T defined by

$$v = b_1 v_1 + b_2 v_2 + b_3 v_3, \qquad b_1 + b_2 + b_3 = 1.$$

As usual, we associate each coefficient c_{ijk}^T with a corresponding *domain point* $\xi_{ijk}^T := (iv_1 + jv_2 + kv_3)/6$, where v_1, v_2, v_3 are the vertices of T.

Following Alfeld and Schumaker [2], we say that the domain points ξ_{ijk}^T with $i = 6 - \nu$ are on the *ν-th ring around* v_1, with similar definitions for the rings around v_2 and v_3. The set of domain points ξ_{ijk}^T with $6 - \nu \le i \le 6$ are in the *ν-th disk around* v_1, etc. For later use, we note that the coefficients of s are directly related to values of s and its derivatives at the vertices of T. In particular, $c_{600}^T = s(v_1), c_{060}^T = s(v_2)$, and $c_{006}^T = s(v_3)$, and in general (cf Lai [10]), for all $i + j + k = 6$,

$$c_{ijk}^T = \sum_{\substack{\nu_1 \le j \\ \nu_2 \le k}} \binom{j}{\nu_1} \binom{k}{\nu_2} \frac{(6 - \nu_1 - \nu_2)!}{6!} D_{v_2 - v_1}^{\nu_1} D_{v_3 - v_1}^{\nu_2} s(v_1), \tag{3.3}$$

where in general, D_{a-b} is the directional derivative

$$D_{a-b} f(x) = \lim_{t \to 0} \frac{f(x + t(a - b)) - f(x)}{t}$$

in the direction $a - b$.

Every spline $s \in \mathcal{S}_6^0(\diamondsuit)$ is uniquely associated with a vector $\boldsymbol{c} = (c_1, \ldots, c_M)$ with

$$M = \dim \mathcal{S}_6^0(\diamondsuit) = n + 5(E_I + E_B) + 10N, \tag{3.4}$$

where E_I and E_B denote the number of interior and boundary edges of $\diamondsuit$ and N denotes the number of triangles in $\diamondsuit$.

We can think of $\boldsymbol{c}$ as consisting of an ordered list of the Bézier coefficients of the polynomial pieces of s, using the convention that when two polynomial pieces join along an edge, then the corresponding Bézier coefficients associated with that edge are identified with each other and included just once in the list.

The space of splines $\mathcal{SS}_6^2(\diamondplus)$ is the linear subspace of $\mathcal{S}_6^0(\diamondplus)$ which satisfies the C^3 smoothness conditions at each vertex $v \in \mathcal{V}_3$ along with C^2 smoothness conditions across each of the edges of $\diamondplus$. As is well known [3,8], these can all be expressed as simple linear conditions on the Bézier coefficients. Following [2], to find the dimension of $\mathcal{SS}_6^2(\diamondplus)$ and to construct a basis for it, it suffices to find a so-called *minimal determining set* of coefficients, i.e., a set $\mathcal{C} = \{c_{i_1}, \ldots, c_{i_m}\} \subset \{c_1, \ldots, c_M\}$ with m as small as possible so that setting the coefficients c_{i_j}, $j = 1, \ldots, m$, uniquely defines s. Then as shown in [2], the dimension of $\mathcal{SS}_6^2(\diamondplus)$ is m, and using $\mathcal{C}$ it is easy to construct a basis for the spline space.

As an aid to analyzing $\mathcal{SS}_6^2(\diamondplus)$, we now present several lemmas concerning minimal determining sets for splines in the set $\mathcal{S}_3^2(\mathrm{star}(v))$, where $\mathrm{star}(v)$ is a triangulation consisting of the set of triangles in $\diamondplus$ surrounding a vertex v. Later we shall apply these lemmas to deduce which coefficients of a spline s in $\mathcal{SS}_6^2(\diamondplus)$ have to be fixed in order to determine s on a 3-disk around each vertex. Since we eventually want to construct *locally* supported basis splines, we want to include the coefficients associated with domain points located at the centers of edges of $\Diamond$.

First we consider the case where v is a boundary vertex of $\diamondplus$. By the construction, every boundary vertex in $\diamondplus$ has an odd number of edges attached to it.

Lemma 3.1. *Suppose that v is a boundary vertex of $\diamondplus$ with $2n-1$ edges attached, see Fig. 2. Let the boundary vertices of star(v) be $v, v_1, w_1, v_2, w_2, \cdots, w_{n-1}, v_n$ in counterclockwise order. Then the following set of $2n+7$ coefficients form a minimal determining set for $\mathcal{S}_3^2$(star(v)):*

1) $c_{ijk}^{\langle v,v_1,w_1\rangle}, \quad i+j+k=3,$

2) $c_{030}^{\langle v,v_i,w_i\rangle}, c_{003}^{\langle v,v_i,w_i\rangle}, \quad i=2,\ldots,n-1$

3) $c_{003}^{\langle v,w_{n-1},v_n\rangle}$.

These coefficients are marked with $\circ$ in the figure.

Discussion: This lemma was first established in Alfeld and Schumaker[2]. The marked coefficients in Fig. 2 can be fixed, and the rest can be found from the smoothness conditions. ■

We now consider an interior vertex v. By the construction of $\diamondplus$, each of its interior vertices has an even number $2n$ of attached edges. For convenience, we number the boundary vertices of $\mathrm{star}(v)$ in counterclockwise order as $v_1, w_1, v_2, w_2, \cdots, v_n, w_n$. Fig. 3 shows the case where $n = 4$. By a basic result in [14],

$$\dim \mathcal{S}_3^2(\mathrm{star}(v)) = 6 + 2n + (4-e)_+, \tag{3.5}$$

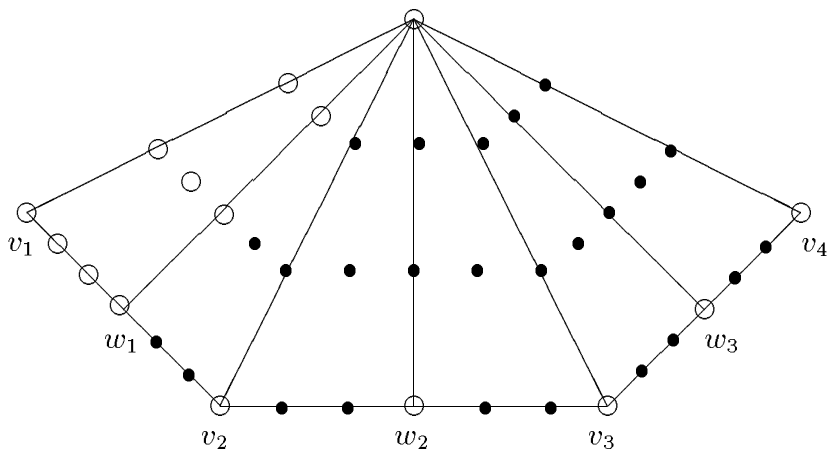

Fig. 2. The minimal determining set for $\mathcal{S}_3^2(\mathrm{star}(v))$ in Lemma 3.1 ($n = 4$).

where e is the number of edges attached to v with different slopes.

Our first result deals with the case where $n \geq 3$ and for some numbering of the vertices,

$$\angle v_1 v v_3 \leq 180°. \tag{3.6}$$

It is easy to see that this condition automatically holds whenever $n \geq 4$.

Lemma 3.2. *Suppose that v is an interior vertex with $2n$ attached edges such that (3.6) holds with $n \geq 3$. Then the following set of $2n+6$ coefficients form a minimal determining set for $\mathcal{S}_3^2(star(v))$:*

1) $c_{ijk}^{\langle v,w_n,v_1\rangle}$, $\quad i+j+k=3$,

2) $c_{030}^{\langle v,v_i,w_i\rangle}$, $\quad i=2,\ldots,n$,

3) $c_{003}^{\langle v,v_i,w_i\rangle}$, $\quad i=3,\ldots,n-1$.

These coefficients are marked with $\circ$ in the Fig. 3.

Proof: This lemma follows from Theorem 3.3 in Schumaker [16]. Since we later make use of certain systems of equations which arise in the proof, we give full details here. We suppose that the coefficients listed in 1) – 3) of an $s \in \mathcal{S}_3^2(\mathrm{star}(v))$ are set to zero, and show that $s \equiv 0$. Using the C^1 and C^2 smoothness conditions, the only possible nonzero coefficients of s are those labeled $a_1, \ldots, a_6$ in Fig. 3. The C^1 and C^2 smoothness conditions imply that these coefficients must satisfy the linear

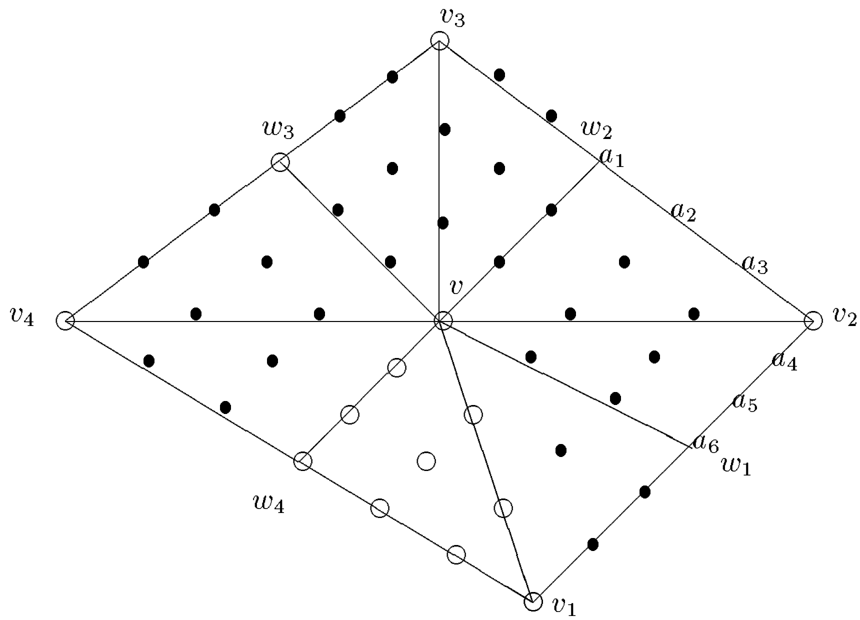

Fig. 3. The minimal determining set for $\mathcal{S}_3^2(\text{star}(v))$ in Lemma 3.2 ($n = 4$).

system of equations

$$\begin{pmatrix} \gamma_1 & -1 & 0 & 0 & 0 & 0 \\ \gamma_1^2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & \gamma_2 \\ 0 & 0 & 0 & -1 & 0 & \gamma_2^2 \\ 0 & 0 & -1 & \beta_3 & 0 & 0 \\ 0 & -1 & 0 & 2\beta_3\gamma_3 & \beta_3^2 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = 0,$$

where $v_2 = \beta_1 v_3 + \gamma_1 w_2$, $v_2 = \beta_2 v_1 + \gamma_2 w_1$, and $w_2 = \alpha_3 v + \beta_3 w_1 + \gamma_3 v_2$.

The determinant of this linear system is

$$-\gamma_1\gamma_2\beta_3(2\gamma_1\gamma_2\gamma_3 - \gamma_2 + \gamma_1\beta_3).$$

By the geometric meaning of the α_i, β_i, and γ_i, along with the assumption (3.6), we have $\beta_i < 0$ and $\gamma_i > 0$. Also,

$$\begin{aligned} \beta_1 v_3 &= v_2 - \gamma_1 w_2 \\ &= v_2 - \gamma_1(\alpha_3 v + \beta_3 w_1 + \gamma_3 v_2) \\ &= (1 - \gamma_1\gamma_3)(\beta_2 v_1 + \gamma_2 w_1) - \gamma_1\alpha_3 v - \gamma_1\beta_3 w_1 \\ &= -\gamma_1\alpha_3 v + \beta_2(1 - \gamma_1\gamma_3)v_1 + (\gamma_2 - \gamma_1\beta_3 - \gamma_1\gamma_2\gamma_3)w_1. \end{aligned}$$

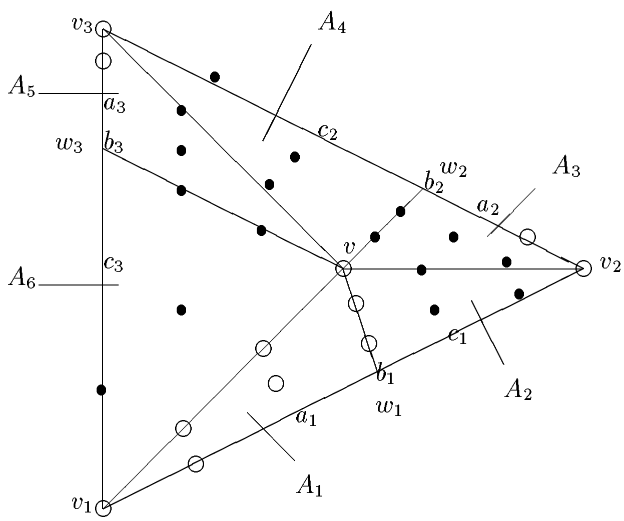

Fig. 4. The minimal determining set for $\mathcal{P}_3$ in Lemma 3.3.

Thus

$$\frac{\gamma_2 - \gamma_1\beta_3 - \gamma_1\gamma_2\gamma_3}{\beta_1} \geq 0.$$

Since $\beta_1 < 0$, we have $-\gamma_2 + \gamma_1\beta_3 + \gamma_1\gamma_2\gamma_3 \geq 0$, and we conclude that the determinant is nonzero. This implies $s \equiv 0$, and the proof is complete. ■

This lemma says that in constructing a spline $s \in \mathcal{S}_3^2(\text{star}(v))$, we can set the coefficients described in 1) – 3) to arbitrary values, and then find the remaining coefficients in the 3-disk around v by solving the above system of equations.

In our next lemma we deal with the case where $n = 3$ but (3.6) is not satisfied. To make sure that certain locally supported basis splines to be constructed in Theorem 4.3 are bounded, in this case we consider the space $\mathcal{S}_3^2(\text{star}(v)) \cap C^3(v)$.

Lemma 3.3. *Let v be an interior vertex of $\diamondsuit$ of degree 3 as in Fig. 4, and suppose (3.6) does not hold. Then the following set of 10 coefficients form a minimal determining set for $\mathcal{P}_3 = \mathcal{S}_3^2(star(v)) \cap C^3(v)$:*

1) $c_{ijk}^{\langle v,v_1,w_1\rangle}$, $\quad i+j+k=3$ *and* $i \geq 1$,

2) $c_{030}^{\langle v,v_i,w_i\rangle}$, $\quad i = 1,2,3$,

3) $c_{021}^{\langle v,v_1,w_1\rangle}$.

These coefficients are marked with $\circ$ *in the figure.*

Proof: Since (3.6) does not hold, $\angle v_1 v v_3 > 180°$. We may assume that the location of w_1 on the edge $e = \langle v_1, v_2\rangle$ is between v_1 and the intersection $\bar{v}$ of e with the line containing $\langle v_3, v\rangle$. The proof is similar in the case where w_1 lies between $\bar{v}$ and v_2. Then we can choose $c_{ijk}^{\langle v,w_1,v_2\rangle}$, $i+j+k=3$, $i \geq 1$, in place of 1), and $c_{012}^{\langle v,w_1,v_2\rangle}$ in place of 3).

The space $\mathcal{P}_3 = \mathcal{S}_3^2(\mathrm{star}(v)) \cap C^3(v)$ has dimension 10, and we now show that the coefficients in 1) – 3) are a determining set for it. Suppose that the coefficients in item 1) along with $c_{030}^{\langle v,v_1,w_1\rangle}$ and $c_{021}^{\langle v,v_1,w_1\rangle}$ are set to zero. Then assuming $s \in \mathcal{P}_3$ is written in Bernstein-Bézier form relative to the triangle $\langle v, v_1, w_1\rangle$, it follows that $s = a_1 B_{012}^3 + a_2 B_{003}^3$, where $a_1 := c_{012}^{\langle v,v_1,w_1\rangle}$ and $a_2 := c_{003}^{\langle v,v_1,w_1\rangle}$. Now setting the remaining coefficients in 2) and 3) to zero is equivalent to requiring that $s(v_2) = s(v_3) = 0$. Thus a_1 and a_2 must satisfy the system

$$\begin{pmatrix} 3b_2 b_3^2 & b_3^3 \\ 3\tilde{b}_2 \tilde{b}_3^2 & \tilde{b}_3^3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where (b_1, b_2, b_3) and $(\tilde{b}_1, \tilde{b}_2, \tilde{b}_3)$ are the barycentric coordinates of v_2 and v_3 relative to the triangle $\langle v, v_1, w_1\rangle$, respectively. By the geometry, $b_2 < 0$, $b_3 > 1$, $\tilde{b}_2 \leq 0$, and $\tilde{b}_3 < 0$. Thus the determinant of the above linear system satisfies

$$3b_3^2\tilde{b}_3^2(b_2\tilde{b}_3 - \tilde{b}_2 b_3) \geq 3b_3^2\tilde{b}_3^2 b_2\tilde{b}_3 \neq 0,$$

and we conclude that $a_1 = a_2 = 0$. This completes the proof. ■

This lemma says that in constructing a spline $s \in \mathcal{S}_3^2(\mathrm{star}(v)) \cap C^3(v)$, we can set the coefficients described in 1) – 3) to arbitrary values, and then find the coefficients a_1 and a_2 by solving the above 2×2 linear system. Then we use the C^3 smoothness conditions to determine the remaining coefficients in the 3-disk around v.

We turn now the case where $n = 2$. This case occurs when an interior vertex v of $\Diamond$ is shared by two quadrilaterals (at least one of which must be nonconvex). There are three cases depending on the number e of edges in $\Diamond\!\!\!\!\!+$ attached to v with different slopes. Our first lemma deals with the case where v is a singular vertex. (Recall that a singular vertex of a triangulation is one formed by two crossing lines so that $e = 2$).

Lemma 3.4. *Let v be an interior vertex of $\Diamond$ of degree 2 such that v is a singular vertex of $\Diamond\!\!\!\!\!+$. Then the following set of 12 coefficients form a minimal determining set for $\mathcal{S}_3^2$(star(v)):*

1) $c_{ijk}^{\langle v,v_1,w_1\rangle}$, $\quad i+j+k=3$,

2) $c_{030}^{\langle v,v_2,w_2\rangle}$, $c_{003}^{\langle v,v_2,w_2\rangle}$.

Proof: By the smoothness conditions, it is clear that these coefficients form a determining set. Since the dimension of $\mathcal{S}_3^2(\mathrm{star}(v))$ is 12 in this case, they form a minimal determining set. ■

Lemma 3.5. *Let v be an interior vertex of $\diamondsuit$ of degree 2 such that $e = 3$. Then the following set of 11 coefficients form a minimal determining set for $\mathcal{S}_3^2(star(v))$:*

1) $c_{ijk}^{\langle v,v_1,w_1\rangle}$, $\quad i+j+k=3$,

2) $c_{030}^{\langle v,v_2,w_2\rangle}$.

Proof: Without loss of generality we may assume that the edges $\langle v_1, v\rangle$ and $\langle v, v_2\rangle$ are not collinear (otherwise we could change the quadrangulation $\diamondsuit$ by connecting v to the other two vertices of the pair of quadrangles which share these two edges). By the smoothness conditions, it is clear that these coefficients form a determining set. Since here the dimension of $\mathcal{S}_3^2(\text{star}(v))$ is 11, they form a minimal determining set. ■

Lemma 3.6. *Let v be an interior vertex of $\diamondsuit$ of degree 2 such that $e = 4$. Then the following set of 10 coefficients form a minimal determining set for $\mathcal{S}_3^2(star(v))$:*

1) $c_{ijk}^{\langle v,v_1,w_1\rangle}$, $\quad i+j+k=3$ and $i \geq 1$,

2) $c_{030}^{\langle v,v_1,w_1\rangle}$, $c_{021}^{\langle v,v_1,w_1\rangle}$, $c_{012}^{\langle v,v_1,w_1\rangle}$,

3) $c_{030}^{\langle v,v_2,w_2\rangle}$.

Proof: By (3.5), the dimension of $\mathcal{S}_3^2(\text{star}(v))$ in this case is 10, and so it reduces to the space $\mathcal{P}_3$ of cubic polynomials, and we actually have C^3 continuity at v. Suppose the coefficients in items 1) – 2) are set to zero for a spline s. Then $s = aB_{003}^3$ on $\langle v, v_1, w_1\rangle$, where B_{ijk}^3 are the Bernstein polynomials associated with this triangle. Now setting the coefficient in 3) to zero is equivalent to setting $s(v_2) = 0$ which immediately implies $a = 0$. Thus, these coefficients form a determining set, which in view of the dimension of $\mathcal{S}_3^2(\text{star}(v))$ must be minimal. ■

We conclude this section with two lemmas concerning minimal determining sets for $\mathcal{S}_6^2(\text{star}(v))$ on a single quadrangle. The first deals with the case where v is a singular vertex.

Lemma 3.7. *Let $star(v)$ consist of 4 triangles surrounding a vertex v formed by two crossing lines as in Fig. 5. We denote the boundary vertices of $star(v)$ by $v_1, \ldots, v_4$. Let $T_\ell = \langle v, v_\ell, v_{\ell+1}\rangle$ for $\ell = 1, \ldots, 4$, where we identify $v_5 = v_1$. Then the following set of 49 coefficients form a minimal determining set for $\mathcal{S}_6^2(star(v))$:*

1) $c_{ijk}^{T_\ell}$, $\quad i+j+k=6$, $\ j \geq 3$, $\ \ell = 1, \ldots 4$,

2) $c_{222}^{T_\ell}$, $\quad \ell = 1, \ldots, 4$,

3) $c_{420}^{T_\ell}$, $\quad \ell = 1, \ldots, 4$,

4) $c_{600}^{T_1}$.

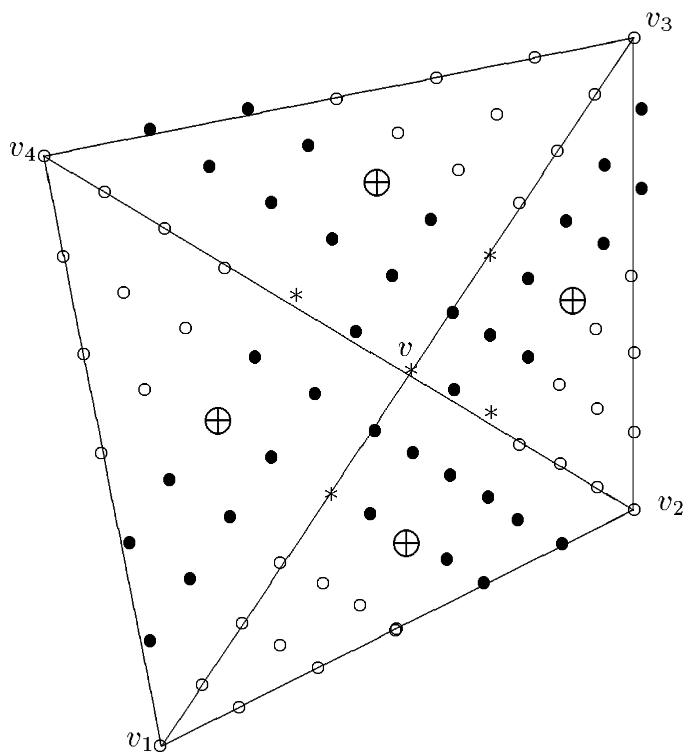

Fig. 5. The determining set for $\mathcal{S}_6^2(\mathrm{star}(v))$ in Lemma 3.7.

The coefficients in 1) are marked with $\circ$ *in the figure. Those in 2) are marked with* $\oplus$*, and the 5 coefficients in 3) – 4) are marked with* $*$.

Proof: By Theorem 2.1 in Schumaker [14], the dimension of $\mathcal{S}_6^2(\mathrm{star}(v))$ is 49. To prove the lemma, we have to show that if $s \in \mathcal{S}_6^2(\mathrm{star}(v))$ with the above 49 coefficients equal to zero, then $s \equiv 0$. But this follows immediately from the C^1 and C^2 smoothness conditions. This lemma shows that in constructing a spline in $\mathcal{S}_6^2(\mathrm{star}(v))$, we can set the coefficients described above to arbitrary values, and then solve for the remaining coefficients in $\mathrm{star}(v)$ using the smoothness conditions. ■

The following is the analog of Lemma 3.7 when v is nonsingular.

Lemma 3.8. *Let star(v) consist of 4 triangles surrounding a vertex v, where two of the edges at v are collinear as in Fig. 6. We denote the boundary vertices of the star by* $v_1, \ldots, v_4$*, and assume that the edges* $\langle v_2, v\rangle$ *and* $\langle v_4, v\rangle$ *are not collinear. Let* $T_\ell = \langle v, v_\ell, v_{\ell+1}\rangle$ *for* $\ell = 1, \ldots, 4$*, where we identify* $v_5 = v_1$*. Then the following*

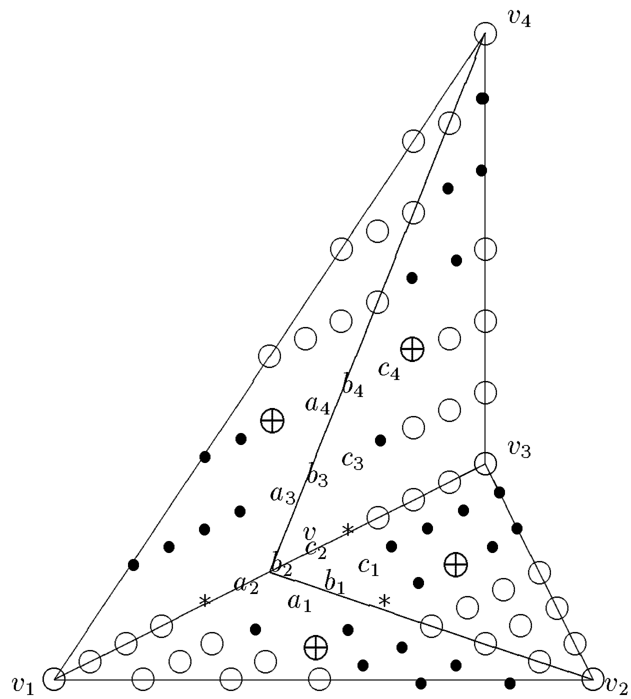

Fig. 6. The determining set for $\mathcal{S}_6^2(\text{star}(v))$ in Lemma 3.8.

set of 47 coefficients form a minimal determining set for $\mathcal{S}_6^2(star(v))$*:*

1) $c_{ijk}^{T_\ell}$, $\quad i+j+k=6, \quad j \geq 3, \quad \ell = 1, \ldots 4$
2) $c_{222}^{T_\ell}$, $\quad \ell = 1, \ldots, 4$
3) $c_{420}^{T_\ell}$, $\quad \ell = 1, \ldots, 3.$

The coefficients in 1) are marked with $\circ$ *in the figure. Those in 2) are marked with* $\oplus$*, and the three coefficients in 3) are marked with* $*$.

Proof: By Theorem 2.1 in Schumaker [14], the dimension of $\mathcal{S}_6^2(\text{star}(v))$ is 47. To prove the lemma, we have to show that if $s \in \mathcal{S}_6^2(\text{star}(v))$ has the above 47 coefficients equal to zero, then $s \equiv 0$. Using the C^1 and C^2 continuity conditions, it is easy to see that the only possible nonzero coefficients of s are those labeled a_i, b_i, c_i, $i = 1, \ldots, 4$, in Fig. 6. Suppose

$$\begin{aligned} v_4 =& \alpha v_1 + \beta v_2 + \gamma v, \\ v_4 =& \hat{\alpha} v_3 + \hat{\beta} v_2 + \hat{\gamma} v, \\ v_3 =& \eta v + \theta v_1. \end{aligned}$$

The assumption that $\langle v, v_4 \rangle$ and $\langle v, v_2 \rangle$ are not collinear implies $\beta = \hat{\beta}$ and $\gamma \neq \hat{\gamma}$.

By the C^1 and C^2 smoothness conditions,

$$0 = \eta^2 b_i + 2\eta\theta a_i \quad \text{and} \quad c_i = \eta b_i + \theta a_i,$$

for $i = 1, \ldots, 4$. Thus,

$$b_i = -2\theta a_i/\eta \quad \text{and} \quad c_i = -\theta a_i,$$

for $i = 1, \ldots, 4$. On the other hand, we also have

$$\begin{aligned} a_4 =& \gamma^2 a_2 + 2\gamma\beta a_1 \\ b_4 =& \gamma^2 b_2 + 2\gamma\beta b_1 + 2\alpha\gamma a_2 + 2\alpha\beta a_1 \\ c_4 =& \hat{\gamma}^2 c_2 + 2\hat{\gamma}\hat{\beta} c_1. \end{aligned} \tag{3.7}$$

Substituting a_i's for the c_i's in the last equation, we get

$$a_4 = \hat{\gamma}^2 a_2 + 2\hat{\gamma}\hat{\beta} a_1.$$

We now have two equations in a_1, a_2 and a_4. Combining them, we get

$$0 = (\hat{\gamma} + \gamma) a_2 + 2\beta a_1.$$

Next we substitute b_i's for the a_i's in the first equation in (3.7) to get $b_4 = \gamma^2 b_2 + 2\gamma\beta b_1$, which combined with the second equation in (3.7) gives

$$0 = \gamma a_2 + \beta a_1.$$

We now have a system of two equations for the a_1, a_2 whose determinant is $(\hat{\gamma} - \gamma)\beta \neq 0$. We conclude that $a_1 = a_4 = 0$. This in turn implies that the other coefficients are all zero. This lemma shows that in constructing a spline in $\mathcal{S}_6^2(\text{star}(v))$, we can set the coefficients described in 1) – 3) to arbitrary values and then solve for the remaining ones in star(v) using the smoothness conditions. ■

4. A Locally Supported Basis for $\mathcal{SS}_6^2(\diamondplus)$

Let $\diamondsuit$ be a quadrangulation of the given data set $\mathcal{V}$, and let $\diamondplus$ be the associated triangulation described in Sect. 1. We write

$$\begin{aligned} \mathcal{V}_I =& \text{number of interior vertices of } \diamondsuit \\ \mathcal{V}_B =& \text{number of boundary vertices of } \diamondsuit. \end{aligned}$$