A Trivariate Box Macro-element

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Abstract. Given a rectangular box which has been split into twenty-four tetrahedra, we show how to construct a C^1 macro-element using polynomial pieces of degree six.

§1. Introduction

Suppose $\triangle$ is a partition of a rectangular box B into tetrahedra, and let

$$\mathcal{S}_d^1(\triangle) := \{s \in C^1(B) : s|_T \in \mathcal{P}_d, \text{ all } T \in \triangle\}$$

be the associated space of trivariate polynomial splines of degree d, where as usual, $\mathcal{P}_d$ denotes the space of trivariate polynomials of degree d. Suppose $\mathcal{S}$ is a superspline subspace of $\mathcal{S}_d^1(\triangle)$ (defined by enforcing some appropriate set of additional smoothness conditions) and that Λ is a set of linear functionals (consisting only of point evaluation of s or its derivatives at points on the faces of B) such that

1) each spline $s \in \mathcal{S}$ is uniquely determined by the values $\{\lambda s\}_{\lambda \in \Lambda}$,
2) if $\mathcal{B}$ is a collection of rectangular boxes forming a partition of some set Ω and if $s|_B$ is defined on each box by the above construction, then the corresponding piecewise polynomial s belongs to $C^1(\Omega)$.

In this case we refer to $(\triangle, \mathcal{S}, \Lambda)$ as a C^1 macro-element of degree d. The linear functionals in Λ are usually referred to as the degrees of freedom of the element.

The construction of trivariate macro-elements goes back to the early finite-element literature, where polynomial elements were constructed on tetrahedra, see [19]. Elements based on subdivided tetrahedra can be found in [1,11,17,18].

The aim of this paper is to construct a C^1 macro-element on a box. Our construction will be based on a partition of B into 24 tetrahedra as in Definition 3.1 below, and will involve polynomials of degree six (see Remark 7.1, where we show that this is the lowest degree which will work on this partition).

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The paper is organized as follows. In Sect. 2 we recall some standard Bernstein-Bézier notation, while in Sect. 3 we discuss the partition of B of interest here. The main result is contained in Sect. 4, where we introduce and study our macro-element space and its associated degrees of freedom set. Several useful lemmas for bivariate splines are stated and proved Sect. 5. In Sect. 6 we show how to assemble our macro-elements into spline spaces defined on box partitions, and discuss their use for Hermite interpolation. The approximation power of the spaces is also treated there. Finally, we conclude with several remarks in Sect. 7.

§2. The Bernstein–Bézier Representation

We will make essential use of the standard Bernstein-Bézier representation for both bivariate and trivariate polynomials, see e.g. [3,11,15] and references therein. We recall that the space of trivariate polynomials $\mathcal{P}_d$ has dimension $\binom{d+3}{3}$, and that given a tetrahedron $T := \langle u_1, u_2, u_3, u_4 \rangle$, each $p \in \mathcal{P}_d$ has a unique representation

$$p = \sum_{i+j+k+l=d} c^T_{ijkl} B^d_{ijkl},$$

where for each $v \in \mathbb{R}^3$ with barycentric coordinates $(\beta_1, \beta_2, \beta_3, \beta_4)$ with respect to T,

$$B^d_{ijkl}(v) := \frac{d!}{i!\,j!\,k!\,l!} \beta_1^i \beta_2^j \beta_3^k \beta_4^l, \qquad i+j+k+l = d,$$

are the classical Bernstein basis polynomials of degree d. Each coefficient c^T_{ijkl} can be identified with a corresponding domain point

$$\xi^T_{ijkl} := \frac{iu_1 + ju_2 + ku_3 + lu_4}{d}.$$

If $\triangle$ is a tetrahedral partition of a set Ω, we write $\mathcal{D}_{d,\triangle}$ for the collection of all domain points associated with tetrahedra in $\triangle$, where ξ^T_{ijkl} associated with different tetrahedra but located at the same point in $\mathbb{R}^3$ are not repeated. It is easy to check that the cardinality of $\mathcal{D}_{d,\triangle}$ is equal to

$$n := V + (d-1)E + \binom{d-1}{2} F + \binom{d-1}{3} N,$$

where V, E, F, and N are the number of vertices, edges, faces, and tetrahedra in $\triangle$, respectively. It is well known that n is precisely the dimension of the space $\mathcal{S}^0_d(\triangle)$ of continuous piecewise polynomials of degree d on $\triangle$.

As usual, we say that the domain points $\{\xi^T_{d-m,j,k,l}\}_{j+k+l=m}$ lie on the ring $R_m(u_1)$ of radius m around u_1 with similar definitions for the other vertices of T. We refer to the set

$$D_m(u_1) := \bigcup_{i=0}^{m} R_i(u_1)$$

as the disk of radius m around u_1.

As is well known, a spline $s \in \mathcal{S}_d^0(\triangle)$ will belong to $C^1(\Omega)$ if and only if certain smoothness conditions across faces between adjoining tetrahedra are satisfied. To describe these in more detail, suppose that $T := \langle v_1, v_2, v_3, v_4 \rangle$ and $\widetilde{T} := \langle v_5, v_2, v_3, v_4 \rangle$ are two adjoining tetrahedra sharing the face $F := \langle v_2, v_3, v_4 \rangle$. Suppose

$$s|_T = \sum_{i+j+k+l=d} c_{ijkl}^T B_{ijkl}^d,$$

$$s|_{\widetilde{T}} = \sum_{i+j+k+l=d} c_{ijkl}^{\widetilde{T}} \widetilde{B}_{ijkl}^d,$$

where $\{\widetilde{B}_{ijkl}^d\}_{i+j+k+l=d}$ are the Bernstein polynomials of degree d associated with $\widetilde{T}$.

Given $0 \le i \le d$ and a spline $s \in \mathcal{S}_d^0(\triangle)$, let

$$\tau_{jkl}^i s := c_{ijkl}^T - \sum_{\nu+\mu+\kappa+\ell=i} c_{\nu,j+\mu,k+\kappa,l+\ell}^{\widetilde{T}} \widetilde{B}_{\nu\mu\kappa\ell}^i(v_1) \tag{2.1}$$

for all $j+k+l = d-i$. Following [4], we call τ_{jkl}^i a smoothness functional of order i. Note that for a given pair of adjoining tetrahedra, this functional is uniquely associated with the domain point $\xi_{ijkl}^T \in \mathcal{D}_T$, which we call the tip of the smoothness functional. It is well known that a spline $s \in \mathcal{S}_d^0(\triangle)$ is C^1 continuous across the face F if and only if

$$\tau_{jkl}^1 s = 0, \qquad \text{for all } j+k+l = d-1.$$

We will also need some notation for bivariate smoothness conditions. Suppose $T := \langle u, v, w \rangle$ and $\widetilde{T} := \langle z, w, v \rangle$ are two adjoining triangles lying in the same plane which share the oriented edge $e := \langle v, w \rangle$. Let

$$\begin{aligned} s_T &= \sum_{i+j+k=d} c_{ijk} B_{ijk}^d, \\ s_{\widetilde{T}} &= \sum_{i+j+k=d} \tilde{c}_{ijk} \widetilde{B}_{ijk}^d, \end{aligned} \tag{2.2}$$

where B_{ijk}^d and $\widetilde{B}_{ijk}^d$ are the Bernstein polynomials of degree d on the triangles T and $\widetilde{T}$, respectively. Then following [4], we write $\tau_{j,e}^n$ for the bivariate smoothness functional of order n defined by

$$\tau_{j,e}^n s := c_{n,d-j,j-n} - \sum_{\nu+\mu+\kappa=n} \tilde{c}_{\nu,\mu+j-n,\kappa+d-j} \widetilde{B}_{\nu\mu\kappa}^n(u). \tag{2.3}$$

We emphasize that $\tau_{j,e}^n$ depends on the orientation of the edge e, and involves coefficients lying on the ring of order j around v.

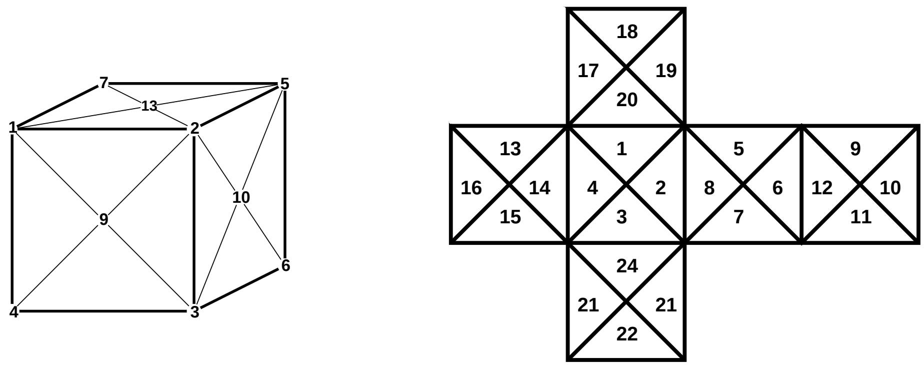

Fig. 1. The numbering of the vertices and exterior faces of $\triangle_B$.

Suppose $\mathcal{S}$ is a linear space of splines contained in $\mathcal{S}_d^0(\triangle)$. Then we recall that $\mathcal{M} \subset \mathcal{D}_{d,\triangle}$ is called a determining set for $\mathcal{S}$ provided that if we set to zero the B-coefficients of $s \in \mathcal{S}$ corresponding to all points $\xi \in \mathcal{M}$, then s must vanish identically. The set $\mathcal{M}$ is called a minimal determining set (MDS) for $\mathcal{S}$ provided that it is the smallest determining set, or equivalently, if prescribing the B-coefficients corresponding to $\xi \in \mathcal{M}$ *uniquely* determines a spline $s \in \mathcal{S}$.

§3. The Partition

We begin by definining the partition of B to be used throughout the paper.

Definition 3.1. *Given a rectangular box B in $\mathbb{R}^3$ with vertices $v_1, \ldots, v_8$, suppose we split each of its six faces into four triangles by drawing in both diagonals. This creates six new vertices which we denote by $v_9, \ldots, v_{14}$. Now let v_0 be the center of the box. Then connecting v_0 to each of the v_i, $i = 1, \ldots, 14$, produces a tetrahedral decomposition of B which we denote by $\triangle_B$.*

It is easy to see that the partition $\triangle_B$ consists of 24 tetrahedra, and that it has 50 edges and 60 triangular faces. Each tetrahedron T has one vertex at v_0, and three vertices on B. We call the face of T lying on B the outer face of T. The face of T which contains v_0 and one edge of B will be called the principal face of B. We call the other two faces of T the side faces.

For convenience, we number the tetrahedra $T_1, \ldots, T_{24}$ of $\triangle_B$ (and their outer faces) as shown in Fig. 1. For a list of the vertices of the tetrahedra, see the table in Remark 7.2. In accordance with the figure, we call the faces containing the vertices v_9, v_{13}, and v_{10} the front, top, and right faces of B, respectively.

The set of domain points $\mathcal{D}_B$ associated with $\triangle_B$ consists of $n_\triangle = 15 + 5 \cdot 50 + 10 \cdot 60 + 10 \cdot 24 = 1105$ points. For each face F of B, the domain points on that face can be regarded as the domain points of the bivariate spline $s|_F$, which belongs to $\mathcal{S}_6^1(\triangle_F)$, where $\triangle_F$ is the triangulation of F obtained by drawing in both diagonals of F.

For each $1 \le m \le 6$, the domain points on the ring $R_m(v_0)$ can be regarded as lying on the surface of a box B_m that is split into tetrahedra in the same way as B.

We shall number the corners of B_m in the same way as those of B. We refer to the set of 12 edges of B_m as the frame of B_m. Although for $m < 6$ we cannot speak of s as being restricted to B_m, setting the coefficients of s for domain points on any face F of B_m is equivalent to setting the corresponding coefficients of a spline in the bivariate space $\mathcal{S}_m^1(\triangle_F)$. We denote this spline by $s_{m,F}$, and will make extensive use of this association below. Given such a face $F := \langle u_1, u_2, u_3, u_4 \rangle$, we write u_F for the point of intersection of the diagonals of F, and write $e_i := \langle u_i, u_F \rangle$ for $i = 1, \ldots, 4$. Finally, if u is a vertex of $\triangle_F$, we say that $s \in C_F^r(u)$ provided that the bivariate spline $s_{m,F}$ is C^r continuous at u, and $s \in \mathcal{S}_m^r(\triangle_F)$ if the bivariate spline $s_{m,F}$ is C^r continuous on F.

§4. A Macro-element

To define a C^1 macro-element of degree 6 on the partition $\triangle_B$, we need to construct an appropriate superspline subspace of $\mathcal{S}_6^1(\triangle_B)$ and an associated set of degrees of freedom. We recall that if v is a vertex of $\triangle_B$, then s is said to be in $C^r(v)$ provided for any two tetrahedra T and $\widetilde{T}$ of $\triangle$ sharing the vertex v, all derivatives of $s|_T$ and $s|_{\widetilde{T}}$ up to order r agree at v. Let $B_1, \ldots, B_6$ be the subboxes of B described in the previous section.

Definition 4.1. *Suppose $\triangle_B$ is the partition of a rectangular box B described in Definition 3.1. Let $\mathcal{S}_B$ be the space of all splines in $\mathcal{S}_6^1(\triangle_B)$ satisfying the following additional smoothness conditions:*

1) *For each corner v of box B, $s \in C^2(v)$;*
2) *For each face F of box B_6,*
   a) $s_{6,F} \in \mathcal{S}_6^2(\triangle_F) \cap C_F^4(u_F)$,
   b) $\tau_{4,e_i}^3 s_{6,F} = 0$, *for* $i = 1, \ldots, 4$,
3) *For each face F of box B_5, $s_{5,F} \in \mathcal{S}_5^3(\triangle_F) \cap C_F^4(u_F)$;*
4) *For each face $F := \langle u_1, u_2, u_3, u_4 \rangle$ of box B_4,*
   a) $s_{4,F} \in C_F^2(u_F)$,
   b) $\tau_{4,e_2}^3 s_{4,F} = 0$,
   c) $\tau_{3,e_2}^2 s_{4,F} = 0$;
5) *On the front face of B_4, $\tau_{4,e_1}^3 s_{4,F} = 0$;*
6) *On the front face of B_3, $\tau_{3,e_1}^2 s_{3,F} = \tau_{3,e_1}^3 s_{3,F} = 0$;*
7) *On the right face of B_3, $\tau_{3,e_1}^2 s_{3,F} = 0$;*
8) *For each $i \in \mathcal{I} := \{2, 3, 4, 5, 7, 9, 10, 11, 12, 13, 15\}$, s should satisfy the following smoothness conditions across the principal face of the tetrahedron T_i:*
   a) *the C^2 smoothness condition corresponding to the domain points $\xi_{1212}^{T_i}$ and $\xi_{1221}^{T_i}$,*
   b) *the C^3 smoothness condition corresponding to the domain points $\xi_{0312}^{T_i}$ and $\xi_{0321}^{T_i}$;*

9) *For the principal face of the tetrahedron T_1, s should satisfy the following smoothness conditions across that face:*
   a) *the C^2 smoothness condition corresponding to the domain point $\xi^{T_1}_{1212}$,*
   b) *the C^3 smoothness condition corresponding to the domain point $\xi^{T_1}_{0312}$;*
10) *For each $i \in \mathcal{J} := \{3, 4, 9, 10, 11, 12, 13, 15\}$, s should satisfy the following smoothness conditions across the principal face of the tetrahedron T_i:*
   a) *the C^2 smoothness condition corresponding to the domain point $\xi^{T_i}_{2211}$,*
   b) *the C^3 smoothness condition corresponding to the domain points $\xi^{T_i}_{1311}$.*

Our choice of the space $\mathcal{S}_B$ in Definition 4.1 was guided by several requirements. First, we wanted a C^1 macro-element using polynomials of the lowest possible degree, which in this case is 6, see Remark 7.1. Secondly, we wanted a space where the faces of the boxes B_5 and B_6 are all treated in a symmetric way. The remaining smoothness conditions were chosen to eliminate unnecessary degrees of freedom, see Remark 7.6.

As an aid to understanding these various special smoothness conditions, we have drawn several figures illustrating the conditions geometrically. Thus, for example, the smoothness condition corresponding to τ^3_{4,e_1} in 2b) involves the coefficients corresponding to the domain points lying on ring $R_4(u_1)$ between the pair of arrows in Fig. 2 (left). Similarly, the two smoothness conditions in 4b)–4c) are shown in Fig. 3 (left), and the additional condition in 5) for the front face is illustrated in Fig. 3 (right). The numbering of domain points in the figures will be used later in analyzing the macro-element. Similarly, the conditions in 6) and 7) are shown in Fig. 4 (right) and Fig. 5 (left), respectively.

The special smoothness conditions listed in 8)–10) are smoothness conditions across the principal faces of neighboring tetrahedra. They are 3D conditions which could be depicted geometrically by marking the domain points corresponding to their tips. However, due to the large number of domain points lying on boxes B_6 and B_5, we only indicate schematically where these tips are located. In particular, in Fig. 6 (left) the black dots indicate which faces contain tips of the smoothness conditions in 8a) and 9a). The same figure works for the smoothness conditions in 8b) and 9b), although the actual domain points are on the faces of different boxes (B_5 and B_6, respectively). Fig. 6 (right) indicates the locations of the tips of the smoothness conditions for 10a) and also for 10b).

Definition 4.2. *Associated with the space $\mathcal{S}_B$ described in Definition 4.1, let $\mathcal{M}$ be the following set of domain points:*

1) *for each of the eight corners u of the box B, the domain points in $D^T_2(u)$, where T is some tetrahedron attached to u;*
2) *for each of the twelve edges e of the box B, the point ξ^T_{0033} for some tetrahedron T attached to e;*
3) *for each face F of B_6, the point ξ^T_{0600} for some tetrahedron T whose face lies on F;*

4) *for each tetrahedron T, the points $\xi^T_{0123}, \xi^T_{0132}, \xi^T_{0222}, \xi^T_{1122}$;*
5) *for each face F of B_5, the point ξ^T_{1500} for some tetrahedron T whose face lies on F.*

Theorem 4.3. *The dimension of the space $\mathcal{S}_B$ is 200, and the set $\mathcal{M}$ forms a minimal determining set for $\mathcal{S}_B$.*

Proof: First we show that $\mathcal{M}$ is a minimal determining set. To this end, suppose we set the coefficients of $s \in \mathcal{S}_B$ corresponding to the domain points in $\mathcal{M}$. We now show that for each $1 \leq i \leq 6$ and each face F of B_i, the coefficients associated with the domain points lying on F are *uniquely* determined.

The box B_6. Lemma 5.1 below shows that on each face F of B_6, the 41 coefficients of s associated with $\mathcal{M} \cap F$ (shown as black dots in Fig. 2 (left)) uniquely determine the coefficients of s associated with the remaining points on F.

The box B_5. Let F be a face of B_5 as in Fig. 2 (right), where the coefficients of s associated with points in $\mathcal{M} \cap F$ are marked with black dots. Clearly, the four coefficients corresponding to domain points marked with $\otimes$ are determined from C^1 smoothness conditions on F. The eight coefficients corresponding to domain points marked with $\oplus$ in the figure can be uniquely determined from the coefficients on B_6 by using C^1 conditions across principal faces of tetrahedra. Fig. 7 shows how a coefficient x corresponding to a point on the frame of B_5 (which need not be at a corner) can be computed from coefficients A, C, f, e corresponding to points on B_6 via the formula $x = e + f - (A + C)/2$. Now Lemma 5.2 below implies that the coefficients of s associated with the remaining points on F are also uniquely determined.

The frame of B_4. Let F be a face of B_4 as shown in Fig. 3. The four points marked with black dots are in $\mathcal{M}$. Now the coefficients corresponding to points marked with $\oplus$ in the figure can be uniquely determined from known coefficients corresponding to points on B_5 by using the C^1 smoothness conditions across principal faces of tetrahedra. Note that at this point, we do not have enough information to compute the coefficients of s corresponding to the remaining points on F.

The frame of B_3. For each face F of B_3, we can uniquely compute the coefficients corresponding to the corners of F from C^1 smoothness conditions across principal faces of neighboring tetrahedra. In particular, the coefficient x corresponding to a corner point as shown in Fig. 7 can be computed from the formula $x = (B + C + D - A)/2$ which involves coefficients on B_4. These points are marked with $\oplus$ in Fig. 4. Since we do not yet know all coefficients on B_4, we cannot get the other two points on each edge of F directly. Instead, we make use of the special smoothness conditions listed in 8) and 9) of the definition of $\mathcal{S}_B$. Let $\mathcal{I}$ be in the index set given in 8). Then for each tetrahedron $T := T_i$ with $i \in \mathcal{I}$, we combine C^1 smoothness with the first C^2 condition in 8a) and the first C^3 condition in 8b) to get a system of three linear equations for the coefficients c^T_{2121}, c^T_{3021} and $c^{\widetilde{T}}_{2112}$,

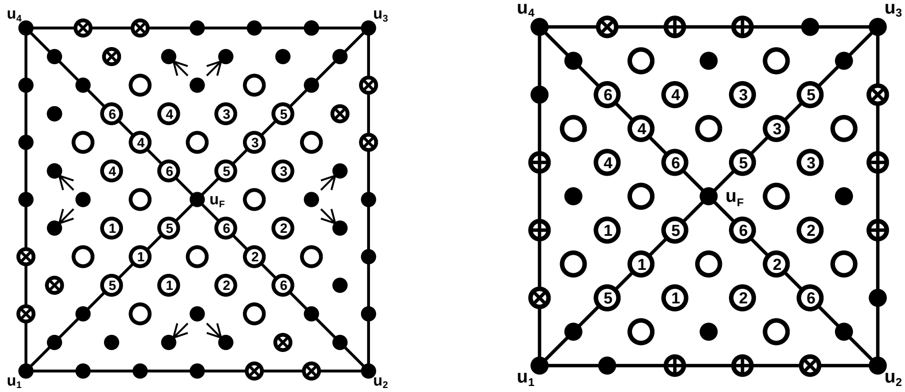

Fig. 2. Computing coefficients on faces of B_6 and B_5.

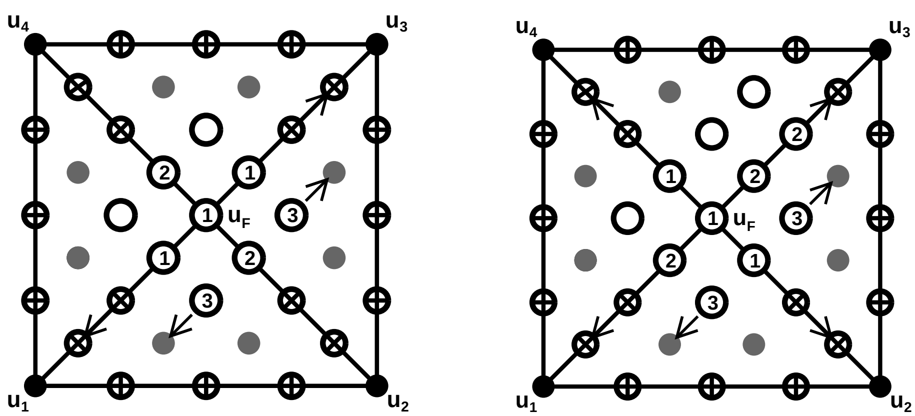

Fig. 3. Computing coefficients on faces of B_4.

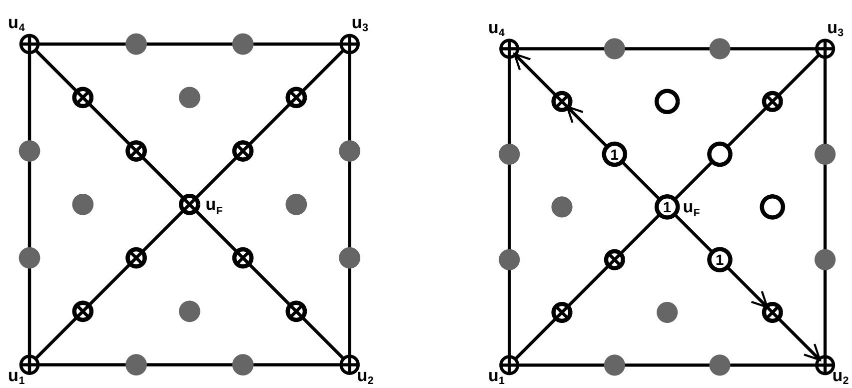

Fig. 4. Computing coefficients on faces of B_3.

where $\widetilde{T}$ is tetrahedron which shares a principal face with T. The matrix of this

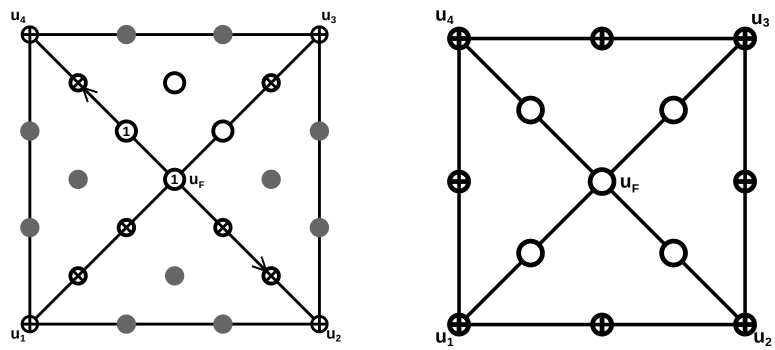

Fig. 5. Computing coefficients on faces of B_3 and B_2.

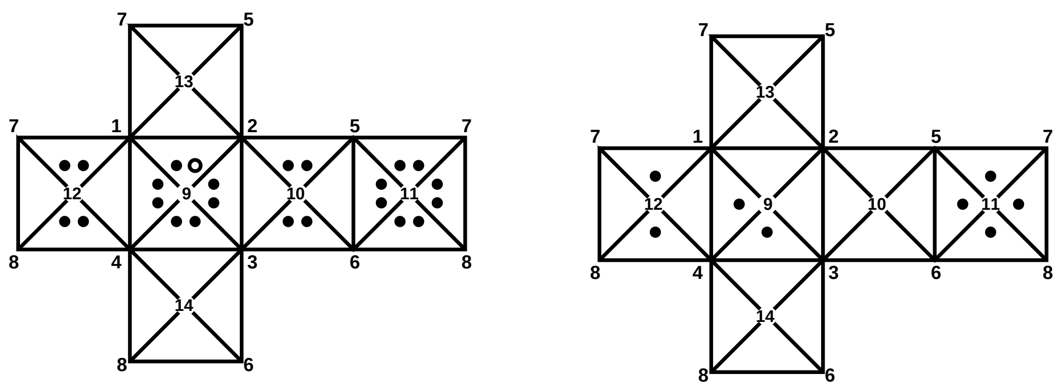

Fig. 6. Tips of the smoothness conditions in 8)–10).

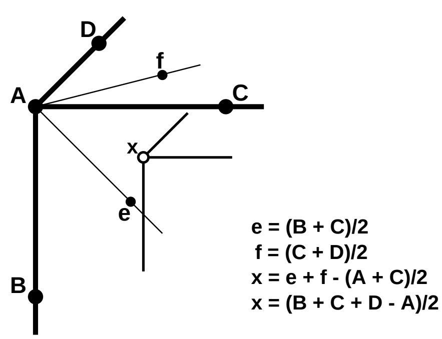

Fig. 7. Use of C^1 smoothness conditions across principal faces.

system is

$$\begin{pmatrix} -1 & 1 & -1 \\ -2 & 1 & 0 \\ -3 & 1 & 0 \end{pmatrix}. \tag{4.1}$$

The points ξ^T_{2121} and $\xi^{\widetilde{T}}_{2112}$ lie on B_4, and are shown as grey dots in Fig. 3 (left). The point ξ^T_{3021} lies on B_3, and is shown as a grey dot in Fig. 4. Similarly, we can solve

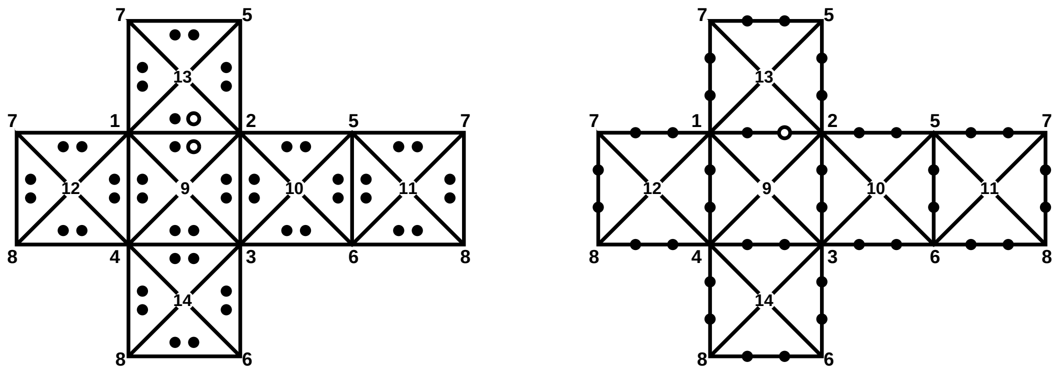

Fig. 8. The coefficients on B_4 and B_3 determined by smoothness conditions 8)–9).

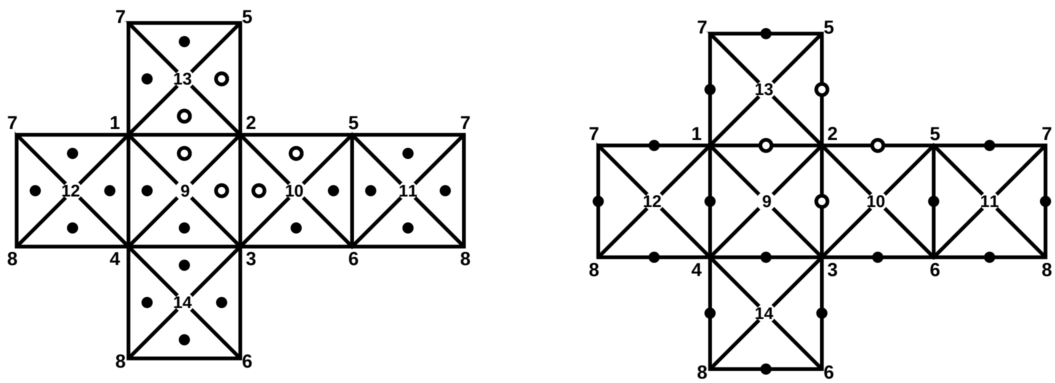

Fig. 9. The coefficients B_3 and B_2 determined by smoothness conditions 10).

for the coefficients $c^{T}_{2112}, c^{T}_{3012}$ and $c^{\widetilde{T}}_{2121}$. Using the smoothness conditions 9), we can now compute $c^{T_{20}}_{2112}, c^{T_1}_{3012}$ and $c^{T_1}_{2121}$ in the same way. We have now computed a total of 46 coefficients associated with points on B_4 (whose locations are indicated by the grey dots in Fig. 8 (left)), and 23 coefficients associated with points on the interiors of the edges of the frame of B_3 (whose locations are indicated by the grey dots in Fig. 8 (right)). Note that at this point the three coefficients corresponding to domain points marked with circles in Fig. 8 are not determined.

The box B_4. Suppose F is a face of B_4 other than the top or front face. Then we have already determined the 16 coefficients corresponding to domain points on the edges of F (shown with $\oplus$ in Fig. 3 (left)), as well as 8 additional coefficients corresponding to the points marked with grey dots in the figure. Lemma 5.4 below shows that these coefficients determine the coefficients associated with all remaining points on F.

Now let F be the front face of B_4. The argument here is slightly different because in this case, of the eight coefficients corresponding to the grey dots in Fig. 3 (left), we currently know only seven of them, cf. Fig. 3 (right). But in this case, we can apply Lemma 5.5 to uniquely determine the coefficients corresponding to the remaining domain points on F. Note that at this point, we have not completed the computations on the top face of B_4.

The frame of B_2. We now know all of the coefficients on the frame of B_3, except for the one corresponding to the domain point marked with $\circ$ in Fig. 8 (right). Then the C^1 conditions uniquely determine the coefficients of s associated with the corners of B_2, except for corner number 2. We now compute coefficients corresponding to the domain points at the center of the edges of B_2 that do not contain corner number 2. Let $\mathcal{J}$ be the index set given in 10). Then for each tetrahedron $T := T_j$ with $j \in \mathcal{J}$, we combine C^1 smoothness with the C^2 and C^3 conditions in 10a)–10b) to get a system of three linear equations for the coefficients c^T_{3111}, c^T_{4011} and $c^{\widetilde{T}}_{3111}$, where $\widetilde{T}$ is tetrahedron which shares a principal face with T. This system corresponds to the matrix in (4.1). Carrying out this process for all tetrahedra in J yields 18 coefficients corresponding to points on B_3 (marked with grey dots in Fig. 9 (left)), and 9 coefficients corresponding to points on B_2 (marked with grey dots in Fig. 9 (right)). Note that so far the coefficients corresponding to domain points marked with circles in these two figures are not determined.

The box B_1. Using C^1 conditions and the coefficients which are already known on the frame of B_2, we can compute the coefficients of s corresponding to the domain points at the corners of B_1 numbered 4,6,7,8. Finding the coefficients of s on B_1 is equivalent to finding the coefficients of a spline in the space $\mathcal{S}^1_1(\triangle_B)$. But this reduces to the space of trivariate polynomials of degree one, which has dimension four. Moreover, it is easy to see that the values at the above four corners (which are the same as the values of the coefficients associated with those corners) uniquely determine a linear polynomial. This gives unique values to all of the coefficients associated with domain points on B_1, and also with the domain point v_0. We now work our way back outward to compute coefficients which were left aside in earlier steps.

The box B_2. Using C^1 smoothness conditions across principal faces and not involving the coefficient corresponding to corner number 2, we now solve for the coefficients corresponding to the domain points in Fig. 9 (right) marked with circles. These can then be used to compute the coefficient corresponding to corner number 2, see Fig. 7. Note that there are three other C^1 conditions (across three different principal faces) which involve this coefficient, but they are all automatically satisfied. The coefficients corresponding to the remaining domain points on B_2 can now be uniquely computed from bivariate C^1 smoothness conditions, see Fig. 5 (right).

The frame of B_3. Using C^1 smoothness across the principal face between T_1 and T_{20}, we now compute

$$c^{T_1}_{3021} = c^{T_1}_{3030} - c^{T_8}_{3021} - c^{T_5}_{3012} + 2c^{T_1}_{4020}.$$

The box B_3. First we apply Lemma 5.7 to compute all remaining coefficients corresponding to domain points on the front face F of B_3. Then using C^1 smoothness across the principal face between T_2 and T_8, we compute $c_{3111}^{T_8}$. Now we can apply Lemma 5.8 to compute all remaining coefficients corresponding to domain points on the right face of B_3. Then using C^1 smoothness across the principal face between T_1 and T_{20}, and between T_5 and T_{19}, we can compute $c_{3111}^{T_{20}}$ and $c_{3111}^{T_{19}}$, respectively. Finally, we apply Lemma 5.6 to compute the remaining coefficients on B_3.

The top face of B_4. Using a C^1 smoothness across the principal face between T_1 and T_{20}, we now compute the coefficient $c_{2112}^{T_{20}}$. We then complete the top face of B_4 using Lemma 5.4.

To establish the dimension statement, we now count the number of points in $\mathcal{M}$. It is easy to see that the number of points in the subsets described in 1)–5) of Definition 4.3 are 80, 12, 6, 96, and 6, which add to 200. □

§5. Some Bivariate Spline Spaces

In this section we collect several lemmas dealing with bivariate spline spaces which arise in the proofs of Theorems 4.3 and 6.1. Throughout this section we suppose that $F := \langle u_1, u_2, u_3, u_4 \rangle$ is a rectangle which has been partioned into four triangles by drawing in both diagonals. Let $\triangle_F$ be the associated triangulation, and let u_F be the point where the diagonals of F intersect.

Lemma 5.1. *Let $\mathcal{S}_{F,6}$ be the space of bivariate splines in $\mathcal{S}_6^1(\triangle_F)$ satisfying the smoothness conditions 2) of Definition 4.1. Then the set $\mathcal{M}_{F,6}$ of forty one domain points marked with black dots in Fig. 2 (left) is a minimal determining set for $\mathcal{S}_{F,6}$.*

Proof: Suppose that g is a spline in $\mathcal{S}_{F,6}$ whose coefficients corresponding to the points in $\mathcal{M}_{F,6}$ have been set. We now show that all remaining coefficients are *uniquely* determined by smoothness conditions. For coefficients corresponding to domain points in $D_5(u_F)$, the claim follows immediately from Lemma 5.2 below. Now for each $i = 1, \ldots, 4$, the two coefficients in $D_2(u_i)$ on the frame of F and marked with $\otimes$ in Fig. 2 (left) are uniquely determined by the C^2 smoothness at u_i. The uniqueness follows from the fact that these coefficients do not enter any other smoothness conditions defining $\mathcal{S}_{F,6}$. □

Lemma 5.2. *Let $\mathcal{S}_{F,5}$ be the space of bivariate splines in $\mathcal{S}_5^1(\triangle_F)$ satisfying the smoothness conditions 3) of Definition 4.1. Then the set $\mathcal{M}_{F,5}$ of twenty five domain points marked with black dots or with $\oplus$ in Fig. 2 (right) is a minimal determining set for $\mathcal{S}_{F,5}$.*

Proof: First we show that $\mathcal{M}_{F,5}$ is a determining set for $\mathcal{S}_{F,5}$. Suppose $g \in \mathcal{S}_{F,5}$ is such that its coefficients corresponding to points in $\mathcal{M}_{F,5}$ are zero. First, using C^1 continuity across the edges e_i shows that the four coefficients marked with $\otimes$ in Fig. 2 (right) are zero. Using the C^3 continuity across the edges e_i and Lemma 2.1 of [4], it follows that the coefficients corresponding to the points marked with ⓘ must be zero for $1 \le i \le 4$. Then by C^4 continuity at u_F, the coefficients corresponding to the points marked with ⓘ for $i = 5, 6$ also vanish. Using C^2

smoothness conditions, we see that the remaining coefficients corresponding to the twelve domain points marked with ○ must also be zero. This completes the proof that $\mathcal{M}_{F,5}$ is a determining set for $\mathcal{S}_{F,5}$. Since $\dim[\mathcal{S}_5^3(\triangle_F) \cap C^4(u_F)] = 25$ by Theorem 2.2 of [14], it follows that $\mathcal{M}_{F,5}$ is a MDS for $\mathcal{S}_{F,5}$. □

For each edge e of F, let D_e be a directional derivative corresponding to a vector orthogonal to e.

Lemma 5.3. *Let $\mathcal{S}_{F,5}$ be as in Lemma 5.2. Then any $s \in \mathcal{S}_{F,5}$ is uniquely determined by the following twenty five data:*

1) *For every vertex v of F, $D^\alpha s(v)$, $|\alpha| \le 1$,*
2) *For every edge e of F, $D_e s(v_e)$ for the midpoint v_e of e,*
3) *For every edge $e := \langle u_1, u_2 \rangle$ of F, $s(v_{i,e})$ for $i = 1, 2$, where $v_{1,e} = (3u_1 + u_2)/4$ and $v_{2,e} = (u_1 + 3u_2)/4$,*
4) *$s(u_F)$, where u_F is the center point of F.*

Proof: Clearly, the data in 1) determine the coefficients of s corresponding to domain points in the disks $D_1(u)$ for each corner of F. Moreover, using the data in 3) we can compute the coefficients c_{023}^T and c_{032}^T for each triangle $T_i := \langle u_F, u_i, u_{i+1} \rangle$. The value $s(u_F)$ determines the coefficients corresponding to u_F. Let $e_i := \langle u_i, u_{i+1} \rangle$ and let v_i be its midpoint. Now the data in 2) implies

$$\begin{aligned} D_{e_i} s(v_i) = &- \tfrac{5}{32}(c_{050}^{T_i} + c_{005}^{T_i}) + \tfrac{5}{16}(c_{140}^{T_i} + c_{104}^{T_i}) - \tfrac{25}{32}(c_{041}^{T_i} + c_{014}^{T_i}) \\ &+ \tfrac{5}{4}(c_{113}^{T_i} + c_{131}^{T_i}) - \tfrac{25}{16}(c_{032}^{T_i} + c_{023}^{T_i}) + \tfrac{15}{8} c_{122}^{T_i}, \end{aligned} \tag{5.1}$$

where all the coefficients are known except for $\{c_{122}^{T_i}, c_{131}^{T_i}, c_{113}^{T_i}\}_{i=1}^4$. Combining (5.1) with the smoothness conditions used in the proof of Lemma 5.2, we are led to the linear system

$$M(c_{122}^{T_1}, c_{122}^{T_2}, c_{122}^{T_3}, c_{122}^{T_4})^t = (r_1, r_2, r_3, r_4)^t,$$

where

$$M := \begin{pmatrix} 6 & 3 & 2 & 3 \\ 3 & 6 & 3 & 2 \\ 2 & 3 & 6 & 3 \\ 3 & 2 & 3 & 6 \end{pmatrix}, \tag{5.2}$$

and the $r_1, \ldots, r_4$ are linear combinations of known coefficients and the derivatives $\{D_{e_i} s(v_i)\}_{i=1}^4$. We have now determined all coefficients corresponding to the set $\mathcal{M}_{F,5}$ of Lemma 5.2, and so by the lemma s is uniquely determined. □

Lemma 5.4. *Let $\mathcal{S}_{F,4}$ be the space of bivariate splines $s \in \mathcal{S}_4^1(\triangle_F)$ satisfying the smoothness conditions 4) of Definition 4.1, and let $\mathcal{M}_{F,4}$ be the set of 24 domain points marked with black or grey dots, or with $\oplus$, in Fig. 3 (left). Then $\mathcal{M}_{F,4}$ is a minimal determining set for $\mathcal{S}_{F,4}$.*

Proof: First we show that $\mathcal{M}_{F,4}$ is a determining set for $\mathcal{S}_{F,4}$. Suppose $g \in \mathcal{S}_{F,4}$ is such that its coefficients corresponding to points in $\mathcal{M}_{F,4}$ are zero. By C^1

continuity, the coefficients marked with $\otimes$ must be zero. Then using the special conditions 4a)–4b), it follows that the coefficients of g must be zero for the points marked with ① and ②. Using 4c), we see that the coefficients marked with ③ are also zero. The remaining coefficients (marked with circles) must be zero by the C^1 smoothness across edges e_1 and e_3. This completes the proof that $\mathcal{M}_{F,4}$ is a determining set for $\mathcal{S}_{F,4}$, and thus that $\dim \mathcal{S}_{F,4)} \leq 24$. Now by Theorem 2.2 of [14], $\dim[\mathcal{S}_4^1(\triangle_F) \cap C^2(u_F)] = 26$. Since 4b)–4c) involve two additional smoothness conditions, we conclude that $\dim \mathcal{S}_{F,5} \geq 24$. This shows that $\dim \mathcal{S}_{F,5} = 24$ and $\mathcal{M}_{F,4}$ is a MDS for $\mathcal{S}_{F,4}$. $\square$

Lemma 5.5. *Let $\widehat{\mathcal{S}}_{F,4}$ be the space of bivariate splines $s \in \mathcal{S}_4^1(\triangle_F)$ satisfying the smoothness conditions 4) and 5) of Definition 4.1, and let $\widehat{\mathcal{M}}_{F,4}$ be the set of 23 domain points marked with black or grey dots or with $\oplus$ in Fig. 3 (right). Then $\widehat{\mathcal{M}}_{F,4}$ is a minimal determining set for $\widehat{\mathcal{S}}_{F,4}$.*

Proof: The proof is very similar to the proof of Lemma 5.4. The order of the computations is indicated by the numbers in Fig. 3 (right). $\square$

Due to their simplicity, we state the following three lemmas without proof. The order of computation of unset coefficients is indicated in Figs. 4 and 5.

Lemma 5.6. *The set $\mathcal{M}_{F,3}$ of 16 domain points marked with grey dots or with $\oplus$ in Fig. 4 (left) is a minimal determining set for $\mathcal{S}_3^1(\triangle_F)$.*

Lemma 5.7. *Let $\widetilde{\mathcal{S}}_{F,3}$ be the set of splines in $\mathcal{S}_3^1(\triangle_F)$ which satisfy condition 6) of Definition 4.1, and let $\widetilde{\mathcal{M}}_{F,3}$ be the set of 14 domain points marked with grey dots or with $\oplus$ in Fig. 4 (right). Then $\widetilde{\mathcal{M}}_{F,3}$ is a minimal determining set for $\widetilde{\mathcal{S}}_{F,3}$.*

Lemma 5.8. *Let $\widehat{\mathcal{S}}_{F,3}$ be the set of splines in $\mathcal{S}_3^1(\triangle_F)$ which satisfy condition 7) of Definition 4.1, and let $\widehat{\mathcal{M}}_{F,3}$ be the set of 15 domain points marked with grey dots or with $\oplus$ in Fig. 5 (left). Then $\widehat{\mathcal{M}}_{F,3}$ is a minimal determining set for $\widehat{\mathcal{S}}_{F,3}$.*

§6. Hermite Interpolation

In this section we describe a natural way to use the macro-element constructed in Sect. 4 to build a C^1 spline space defined on a set Ω which has been partitioned into boxes. We also show how this spline space can be used to solve associated Hermite interpolation problems. Let $\mathcal{B}$ be a collection of n_B rectangular boxes in $\mathbb{R}^3$ such that any two boxes can only intersect at a single vertex, along a common edge, or along a common face, and the union of such boxes is the set Ω. Let n_V, n_E and n_F be the number of vertices, edges, and faces of the boxes of $\mathcal{B}$, respectively. Let $\triangle_{\mathcal{B}}$ be the tetrahedral partition of Ω which is obtained by partitioning each box $B \in \mathcal{B}$ into 24 tetrahedra as described in Definition 3.1. Then we define

$$\mathcal{S}(\triangle_{\mathcal{B}}) := \{s \in C^1(\Omega) : s|_B \in \mathcal{S}_B, \quad \text{all } B \in \mathcal{B}\},$$

where $\mathcal{S}_B$ is the space introduced in Definition 4.1.

For each edge e of a box $B \in \mathcal{B}$, let $D^l_{1,e}$ and $D^l_{2,e}$ be directional derivatives of l-th order corresponding to distinct vectors orthogonal to e. For each face F of a box $B \in \mathcal{B}$, let D_F be the directional derivative corresponding to a vector orthogonal to F.

Theorem 6.1. *The dimension of the space $\mathcal{S}(\triangle_{\mathcal{B}})$ is given by*

$$\dim \mathcal{S}(\triangle_{\mathcal{B}}) = 10n_V + 9n_E + 2n_F.$$

Moreover, any spline $s \in \mathcal{S}(\triangle_{\mathcal{B}})$ is uniquely determined by the following set of nodal data*:*

1) *For every vertex v of $\mathcal{B}$, $\{D^\alpha s(v)\}_{|\alpha|\le 2}$,*
2) *For every edge e of $\mathcal{B}$, the values $s(v_e), D^2_{1,e}s(v_e), D^2_{2,e}s(v_e)$, $D_{1,e}D_{2,e}s(v_e)$, $D_{2,e}D_{1,e}s(v_e)$, for the midpoint v_e of e,*
3) *For every edge $e := \langle u_1, u_2\rangle$ of $\mathcal{B}$, the values $\{D_{1,e}s(v_{i,e}), D_{2,e}s(v_{i,e})\}_{i=1,2}$, where $v_{1,e} = (3u_1 + u_2)/4$ and $v_{2,e} = (u_1 + 3u_2)/4$,*
4) *For every face F of $\mathcal{B}$, $s(u_F)$ and $D_F s(u_F)$, where u_F is the center point of F.*

Proof: First we show that for each box $B \in \mathcal{B}$, the nodal data listed above uniquely determines a spline $s_B \in \mathcal{S}(\triangle_B)$. Let $\mathcal{M}_B$ be the minimal determining set for $\mathcal{S}(\triangle_B)$ as in Definition 4.2. Clearly, the data in 1) determine the coefficients corresponding to domain points in the disks $D_2(u)$ for each corner of B. Moreover, the values of s at the midpoints of the edges determine the coefficients c^T_{0033} for each tetrahedron in B. Next, we use the data in 3) to compute the coefficients c^T_{0123} and c^T_{0132} for each tetrahedron T in B. Moreover, the data in 4) determine the coefficients corresponding to the domain points at the centers u_F of each face of B_6 and each face of B_5.

Now consider a face $F := \langle u_1, u_2, u_3, u_4\rangle$ of B_6 as shown in Fig. 2 (left), and let $T_i := \langle v_0, u_F, u_i, u_{i+1}\rangle$, $i = 1, \dots, 4$ be the four tetrahedra sharing the face F. Let v_i be the center of $\langle u_i, u_{i+1}\rangle$, and let $a_i := \langle v_i, u_F\rangle$. Then

$$\begin{aligned} D^2_{a_i}s(v_i) = {} & \tfrac{15}{32}(c^{T_i}_{0060} + c^{T_i}_{0006}) + \tfrac{45}{16}(c^{T_i}_{0051} + c^{T_i}_{0015}) + \tfrac{225}{32}(c^{T_i}_{0042} + c^{T_i}_{0024}) \\ & + \tfrac{75}{8}(c^{T_i}_{0033} - c^{T_i}_{0141} - c^{T_i}_{0114}) + \tfrac{45}{4}c^{T_i}_{0222} - \tfrac{75}{4}(c^{T_i}_{0132} + c^{T_i}_{0123}) \\ & + \tfrac{15}{2}(c^{T_i}_{0213} + c^{T_i}_{0231}) + \tfrac{15}{8}(c^{T_i}_{0240} + c^{T_i}_{0204} - c^{T_i}_{0150} - c^{T_i}_{0105}), \end{aligned} \tag{6.1}$$

where all the coefficients are known except for $\{c^{T_i}_{0222}, c^{T_i}_{0231}, c^{T_i}_{0213}\}^4_{i=1}$. Combining (6.1) with the smoothness conditions used in the proof of Lemma 5.1, we are led to the linear system

$$M(c^{T_1}_{0222}, c^{T_2}_{0222}, c^{T_3}_{0222}, c^{T_4}_{0222})^t = (r_1, r_2, r_3, r_4)^t,$$

where M is the matrix in (5.2), and $r_1, \dots, r_4$ are linear combinations of known coefficients and the derivatives $\{D^2_{a_i}s(v_i)\}^4_{i=1}$. We have now shown that the coefficients corresponding to domain points in $\mathcal{M}_B \cap B_6$ are determined by the nodal data.

Now suppose F is a face of B_5, and let T_i, a_i, v_i be as above. Let $a_F := \langle u_F, v_0 \rangle$. Then

$$\begin{aligned} D_{a_i} D_{a_F} s(v_i) = \ & \tfrac{15}{16}(c^{T_i}_{0150} + c^{T_i}_{0105} - c^{T_i}_{1050} - c^{T_i}_{1005}) + \tfrac{90}{8}(c^{T_i}_{1122} - c^{T_i}_{0222}) \\ & + \tfrac{75}{16}(c^{T_i}_{0114} + c^{T_i}_{0141} - c^{T_i}_{1041} - c^{T_i}_{1014}) + \tfrac{75}{8}(c^{T_i}_{0132} + c^{T_i}_{0123} - c^{T_i}_{1032} - c^{T_i}_{1023}) \\ & + \tfrac{15}{8}(c^{T_i}_{1140} + c^{T_i}_{1104} - c^{T_i}_{0240} - c^{T_i}_{0204}) + \tfrac{15}{2}(c^{T_i}_{1113} + c^{T_i}_{1131} - c^{T_i}_{0213} - c^{T_i}_{0231}), \end{aligned} \tag{6.2}$$

where all the coefficients have been computed directly from nodal data or can be computed using C^1 smoothness conditions, except for $\{c^{T_i}_{1122}, c^{T_i}_{1131}, c^{T_i}_{1113}\}_{i=1}^4$. Combining (6.2) with the smoothness conditions used in Lemma 5.2, we get the linear system

$$M(c^{T_1}_{1122}, c^{T_2}_{1122}, c^{T_3}_{1122}, c^{T_4}_{1122})^t = (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4)^t,$$

where $\tilde{r}_1, \ldots, \tilde{r}_4$ are linear combinations of known coefficients and the derivatives $\{D_{a_i} D_{a_F} s(v_i)\}_{i=1}^4$. We have now shown that the coefficients corresponding to domain points in $\mathcal{M}_B \cap B_5$ are determined by the nodal data, and thus all of the coefficients corresponding to domain points in $\mathcal{M}_B$ are determined, and in fact are uniquely determined since the number of data used to compute these coefficients is exactly 200, which is the cardinality of $\mathcal{M}_B$.

Let B and $\widetilde{B}$ be two boxes sharing a face $F := \langle u_1, u_2, u_3, u_4 \rangle$, and let $s = s_B$ and $\tilde{s} = s_{\widetilde{B}}$. We now show that s and $\tilde{s}$ join with C^1 smoothness across F. The C^0 continuity is clear since the coefficients corresponding to domain points on F of both splines are computed from the same data. Now

$$D_{a_F} s|_{\langle u_F, u_i, u_{i+1} \rangle} = 6 \sum_{j+k+l=5} (c_{0,j+1,kl} - c_{1jkl}) B^5_{0jkl}, \qquad i = 1, \ldots, 4,$$

and using the fact that both sets of coefficients $\{c_{0,j+1,kl}\}$ and $\{c_{1jkl}\}$ satisfy conditions 2) and 3) of Definition 4.1, it is easy to see that $D_{a_F} s|_F \in \mathcal{S}_{F,5} := \mathcal{S}^3_5(\triangle_F) \cap C^4_F(u_F)$. Similarly, $D_{a_F} \tilde{s}|_F \in \mathcal{S}_{F,5}$. Since the nodal data in 1)–4) includes the 25 nodal data of Lemma 5.3, it follows that $D_{a_F} s|_F \equiv D_{a_F} \tilde{s}|_F$. $\square$

Theorem 6.1 defines a linear interpolation operator S mapping $C^2(\Omega)$ into $\mathcal{S}(\triangle_{\mathcal{B}})$. We note that $Sp = p$ for any polynomial p of degree six. Our next theorem shows that this operator provides optimal order approximation. Given a box $B \in \mathcal{B}$, let $|B|$ be its diameter. Given $1 \leq m$, let $W^m_\infty(B)$ be the usual Sobolev space with seminorm

$$|f|_{m,B} := \sum_{|\alpha|=m} \|D^\alpha f\|_B, \tag{6.3}$$

where D^α is the derivative operator in standard multi-index notation, and $\|\cdot\|_B$ is the ∞-norm on B. Let δ_B be the ratio of the length of the longest edge of B to the length of the shortest edge of B.

Theorem 6.2. *There exists a constant C depending only on δ_B such that for every $f \in W_\infty^{m+1}(B)$ with $2 \le m \le 6$,*

$$\|D^\alpha(f - Sf)\|_B \le C|B|^{m+1-|\alpha|}|f|_{m+1,B}, \tag{6.4}$$

for all $0 \le |\alpha| \le m$.

Proof: Fix $2 \le m \le 6$, and let $f \in W_\infty^{m+1}(B)$. By Lemma 4.3.8 of [10], there exists a polynomial $q := q_{f,B} \in \mathcal{P}_6$ such that

$$\|D^\alpha(f - q)\|_B \le |(f - q)|_{|\alpha|,B} \le K|B|^{m+1-|\alpha|}|f|_{m+1,B}, \tag{6.5}$$

were K is a constant depending only on m and δ_B. Using $Sq = q$, it is clear that

$$\|D^\alpha(f - Sf)\|_B \le \|D^\alpha(f - q)\|_B + \|D^\alpha S(f - q)\|_B.$$

In view of (6.5), it suffices to estimate the second term. By the Markov inequality [16] applied to each subtetrahedron of B,

$$\|D^\alpha S(f - q)\|_B \le K_1|B|^{-|\alpha|}\|S(f - q)\|_B,$$

where K_1 depends only on δ_B. Now let T be one of the tetrahedra in the partition $\triangle_B$. Since the associated Bernstein basis polynomials B_{ijk}^T form a partition of unity on T,

$$\|S(f - q)\|_T \le \max_{\xi \in \mathcal{D}_{6,T}} |c_\xi|,$$

where c_ξ are the associated B-coefficients of $S(f - q)|_T$. We show below that

$$|c_\xi| \le K_2(|f - q|_{0,B} + |B||f - q|_{1,B} + |B|^2|f - q|_{2,B}), \tag{6.6}$$

for all $\xi \in \mathcal{D}_{6,B}$. Inserting this in the above inequalities and using (6.5) leads immediately to (6.4).

To complete the proof, we now justify (6.6). First we consider c_ξ for ξ in the minimal determining set $\mathcal{M}$ described in Definition 4.2. As shown in the proof of Theorem 6.1, each of these coefficients can be determined from the nodal data (function values and derivatives at points on the faces of B). Except for coefficients of the form c_{0222}^T and c_{1122}^T, this is a standard computation which leads to the bound (6.6). For example, see Theorem 8.2 of [15] for coefficients corresponding to domain points in disks of the form $D_2(v)$. Coefficients of the form c_{0222}^T and c_{1122}^T are computed by solving 4×4 linear systems arising from the equations (6.1) and (6.2). Each of these systems corresponds to the matrix (5.2), and has a right-hand side which involves coefficients which have already been shown to satisfy (6.6). Thus, these coefficients also satisfy (6.6). Now the remaining coefficients of s are computed from the $\{c_\xi\}_{\xi \in \mathcal{M}}$ by smoothness conditions. For some of these coefficients, this involves solving small linear systems according to Lemma 2.1 of [4]. But this is known to be a stable process, and in particular there exists a constant K such that all coefficients computed in this way satisfy $|c_\eta| \le K \max_{\xi \in \mathcal{M}} |c_\xi|$. □

Theorem 6.2 leads immediately to the following global error bound involving the mesh size $|\triangle_{\mathcal{B}}|$, which is the diameter of the largest box in the partition $\triangle_{\mathcal{B}}$. Let $\delta := \max_{B \in \mathcal{B}} \delta_B$.

Corollary 6.3. *There exists a constant C depending only on δ such that for every $f \in W_\infty^{m+1}(\Omega)$ with $2 \le m \le 6$,*

$$\|D^\alpha(f - Sf)\|_\Omega \le C|\triangle_{\mathcal{B}}|^{m+1-|\alpha|}|f|_{m+1,\Omega}, \tag{6.7}$$

for all $0 \le |\alpha| \le m$.

§7. Remarks

Remark 7.1. Clearly, in constructing a C^1 macro-element over the partition $\triangle_B$ it is best to use the lowest degree polynomials possible. We now show that such a construction is impossible for $n < 6$. Suppose it were possible for $n = 5$. Then in order for our macro-element to join with C^1 continuity to neighboring boxes, we have to set the coefficients at all eight corners of B along with enough additional information on the faces of B to determine all of the B-coefficients corresponding to domain points on both of the boxes B_5 and B_4. But then (cf. the proof of Theorem 4.3), the C^1 continuity conditions across principal faces will determine the values of all coefficients corresponding to domain points on the frame of B_3 and also at the corners of B_2. In fact, the coefficient corresponding to any given corner of B_2 will depend explicitly on the value assigned at the associated corner of B. But it is easy to see that for a C^1 spline, it is not possible to independently set its coefficients corresponding to all eight corners of B_2.

Remark 7.2. For convenience, we list the vertices of the 24 tetrahedra in the partition $\triangle$ of the box B in the following table.

T	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
v_1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
v_2	9	9	9	9	10	10	10	10	11	11	11	11	12	12	12	12	13	13	13	13	14	14	14	14
v_3	2	3	4	1	5	6	3	2	7	8	6	5	1	4	8	7	7	5	2	1	6	8	4	3
v_4	1	2	3	4	2	5	6	3	5	7	8	6	7	1	4	8	1	7	5	2	3	6	8	4

Remark 7.3. To test our macro-element, we have written Fortran programs to compute the B-coefficients from those in the MDS. Using the programs we verified the polynomial reproduction property, and also checked the C^1 smoothness visually for random choices of the degrees of freedom.

Remark 7.4. In Theorem 4.3 we have explicitly computed the dimension of the spline space $\mathcal{S}_B$ associated with our macro-element. In general, little is known about dimension of trivariate spline spaces, even on cells. See [2,7,8,9].

Remark 7.5. Our construction of the macro-element space $\mathcal{S}_B$ in Section 4 is based on the idea of enforcing certain special smoothness conditions. This idea was used in [4,5,12,13] to create classes of C^r smooth bivariate macro-elements on triangulations.

Remark 7.6. It is easy to see that it is possible to define other superspline spaces on the partition $\triangle_B$ which can be used as C^1 macro-elements. Indeed, there are many possible choices for special smoothness conditions which will work, and in fact, if we are willing to give up symmetry of the nodal data set, it is even possible to create macro-element spaces whose dimensions are smaller than $\dim \mathcal{S}_B = 200$. We also point out that although we required our macro-element to be C^2 at the vertices of B, it is possible to define a similar macro-element which is only C^1 at the vertices.

Remark 7.7. Clearly, one of the main advantages of using a box macro-element as constructed in this paper, as compared to using tensor-product splines, is that we can work with fairly general collections of boxes as described in Section 6. In addition, we note that using tensor-product splines of total degree 9 which are built with C^1 cubic splines, we would get only order 4 approximation of smooth functions. In contrast, using our box macro-element, we are getting order 7 approximation with splines of total degree 6.

Remark 7.8. In [15] we studied a space of C^1 quintic splines defined on tetrahedral partitions of boxes which are different from the one used here. These spaces are useful for approximation, but are not macro-element spaces. C^1 quintic macro-elements based on tetrahedral partitions of octahedra were constructed in [11].

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