Quadrilateral Macro-Elements

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Abstract. Macro-elements of smoothness C^r are constructed on a triangulated quadrilateral for all $r \geq 1$ which depend only on natural derivative information.

§1. Introduction

Suppose Q is a convex quadrilateral, and that $\triangle_Q$ is the triangulation obtained by splitting Q into four triangles by drawing in the two diagonals. Let v_Q be the intersection of the diagonals.

The first macro-element on $\triangle_Q$ was the C^1 piecewise cubic macro-element constructed in [5,14]. Later, a class of C^r macro-elements on $\triangle_Q$ was constructed in [11,12]. The aim of this paper is to improve these higher smoothness macro-elements by removing unnatural degrees of freedom.

The macro-elements in [12] are based on the superspline spaces

$$\begin{aligned} &\mathcal{S}_{6m+1}^{2m+1,3m}(\triangle_Q), \qquad r = 2m, \\ &\mathcal{S}_{6m+3}^{2m+1,3m+1}(\triangle_Q), \qquad r = 2m+1, \end{aligned} \tag{1.1}$$

where in general if $\triangle$ is a triangulation of a domain Ω,

$$\begin{aligned} \mathcal{S}_d^{r,\rho}(\triangle) := \{s \in C^r(\Omega) :\ & s \text{ is a piecewise polynomial of degree } d \text{ on } \triangle, \\ & s \in C^\rho(v) \text{ for all vertices } v\}. \end{aligned} \tag{1.2}$$

As usual, $C^\rho(v)$ means that all polynomials on triangles sharing the vertex v have common derivatives up to order ρ at that vertex.

In this paper we will make use of certain subspaces of the superspline spaces (1.1) which satisfy additional supersmoothness at the vertex v_Q, as well as some other special smoothness conditions.

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The paper is organized as follows. In Sect. 2 we review some well-known Bernstein-Bézier notation and state a key lemma. In Sect. 3 we discuss the case where r is even, and in Sect. 4 we illustrate it with several examples. The case where r is odd is treated in Sect. 5, and illustrated in Sect. 6. Sect. 7 contains results on the corresponding global spline spaces, including their approximation power. Concluding remarks can be found in Sect. 8.

§2. Notation and Preliminaries

We make use of standard Bernstein-Bézier techniques. Given a triangle $T := \langle u_1, u_2, u_3 \rangle$ and an integer d, let

$$\xi_{ijk}^T := \frac{(iu_1 + ju_2 + ku_3)}{d}, \qquad i+j+k = d,$$

be the corresponding *domain points*. We will work with the usual rings and disks of domain points defined by

$$R_n^T(u_1) := \{\xi_{ijk}^T : \ i = d-n\},$$
$$D_n^T(u_1) := \{\xi_{ijk}^T : \ i \geq d-n\},$$

with similar definitions at the other vertices of T. It is well-known (see [7] for explicit formulae) that specifying the B-coefficients in the disk $D_n^T(u_1)$ of a polynomial p is equivalent to specifying the derivatives $D_x^\nu D_y^\mu p(u_1)$ for $0 \leq \nu + \mu \leq n$.

Given a triangulation $\triangle$, we are interested in spline spaces which are subsets of the space $\mathcal{S}_d^0(\triangle)$ of splines which are globally C^0 and are piecewise polynomials of degree d. The corresponding set of domain points is defined to be the union of the $\{\xi_{ijk}^T\}$ as T runs over the triangles of $\triangle$, where points on edges are not repeated. We recall that a *minimal determining set (MDS)* for a spline space $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$ is a subset $\mathcal{M}$ of the domain points associated with $\mathcal{S}_d^0(\triangle)$ such that every spline $s \in \mathcal{S}$ is uniquely determined by the set of B-coefficients which are identified with the points of $\mathcal{M}$.

We shall make extensive use of certain linear functionals defined by smoothness conditions between polynomials of degree d on adjoining triangles. These were introduced in [2], but we repeat their definition here for convenience. Suppose that $T := \langle u_1, u_2, u_3 \rangle$ and $\widetilde{T} := \langle u_4, u_3, u_2 \rangle$ are two adjoining triangles which share the edge $e := \langle u_2, u_3 \rangle$. Let s be a function whose restrictions to T and $\widetilde{T}$ are polynomials of degree d. Let c_{ijk} and $\tilde{c}_{ijk}$ be the coefficients of the B-representations of s_T and $s_{\widetilde{T}}$, respectively. Then for any $n \leq m \leq d$, we define

$$\tau_{m,e}^n s := \tilde{c}_{n,m-n,d-m} - \sum_{i+j+k=n} c_{i,j+d-m,k+m-n} B_{ijk}^n(u_4), \tag{2.1}$$

where B_{ijk}^n are the Bernstein polynomials of degree n on the triangle T.

The following lemma [2] can be used to compute certain coefficients of s on the ring $R_m(u_2)$ assuming that an appropriate set of smoothness conditions across the edge e are satisfied.

Lemma 2.1. *Suppose s is a piecewise polynomial of degree d defined on $T \cup \widetilde{T}$ and that $d, m, p, q, \tilde{q}$ are integers with $0 \le q, \tilde{q}$, $-1 \le p \le q, \tilde{q}$, and $q + \tilde{q} - p \le m \le d$. Suppose*

$$\tau_{m,e}^{n} s = 0, \qquad p + 1 \le n \le q + \tilde{q} - p, \tag{2.2}$$

and that that all of the coefficients c_{ijk} involved in these smoothness conditions are known except for

$$\begin{aligned} c_\nu &:= c_{\nu, d-r, r-\nu}, \qquad \nu = p+1, \ldots, q, \\ \tilde{c}_\nu &:= \tilde{c}_{\nu, r-\nu, d-r}, \qquad \nu = p+1, \ldots, \tilde{q}. \end{aligned} \tag{2.3}$$

Then the coefficients (2.3) are uniquely determined by (2.2).

§3. The case $r = 2m$

Let Q be a quadrilateral with vertices $v_1, \ldots, v_4$ in counterclockwise order. We define the triangles $T^{[i]} := \langle v_Q, v_i, v_{i+1} \rangle$ and edges $e_i := \langle v_i, v_Q \rangle$ for $i = 1, 2, 3, 4$, where $v_5 = v_1$ and v_Q is the point where the two diagonals of Q intersect.

Theorem 3.1. *Given $r = 2m$, let $\mathcal{S}_r(\triangle_Q)$ be the linear subspace of all splines s in $\mathcal{S}_{6m+1}^{2m+1,3m}(\triangle_Q)$ that satisfy the following set of additional smoothness conditions:*

$$s \in C^{4m}(v_Q) \tag{3.1}$$

$$\tau_{3m+i+1, e_l}^{2m+1+j} s = 0, \qquad 1 \le j \le 2i, \quad 1 \le i \le m-1, \quad l = 1, 2, 3, 4, \tag{3.2}$$

and

$$\tau_{4m+1, e_1}^{2m+1+j} s = 0, \qquad 1 \le j \le 2m. \tag{3.3}$$

Then

$$\dim \mathcal{S}_r(\triangle_Q) = 26m^2 + 22m + 4.$$

Moreover, the following set $\mathcal{M}_r$ of domain points is a MDS:

1) $D_{3m}^{T^{[i]}}(v_i)$ *for* $i = 1, 2, 3, 4$,
2) $\{\xi_{j,3m,3m-j+1}^{T^{[i]}}, \ldots, \xi_{j,3m-j+1,3m}^{T^{[i]}}\}$ *for* $j = 1, \ldots, 2m$ *and* $i = 1, 2, 3, 4$.

Proof: First we show that $\mathcal{M}_r$ is a determining set. Suppose that $s \in \mathcal{S}_r(\triangle_Q)$ and that we have set the coefficients of s corresponding to all domain points in $\mathcal{M}_r$. Then using the usual smoothness conditions, we solve for the unset coefficients corresponding to domain points in the disks $D_{3m}(v_i)$ for $i = 1, 2, 3, 4$.

We now make use of Lemma 2.1. First we compute the coefficients on the rings $R_{3m+i+1}(v_l)$ for $i = 0, \ldots, m-1$ and $l = 1, 2, 3, 4$. On the the ring $R_{3m+i+1}(v_l)$ this involves solving a system of $2(m+i)+1$ linear equations. Note that the spline satisfies all of the smoothness conditions required for the lemma, since either they are already implicit in the super-smoothness of the space, or have been explicitly enforced in the definition of $\mathcal{S}_r(\triangle_Q)$.

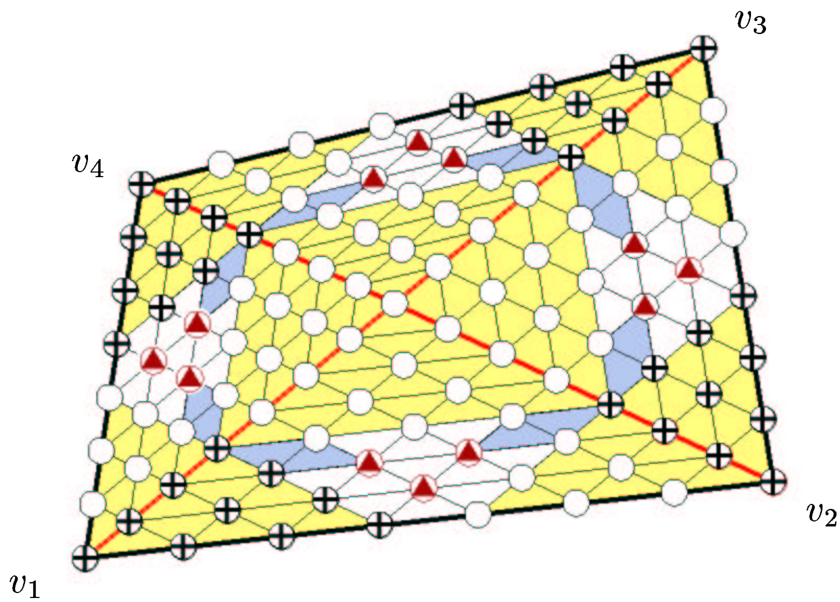

Fig. 1. The C^2 macro-element.

Using the lemma, we now compute the coefficients on $R_{4m+1}(v_1)$. We now start a sequence of calculations. First we compute the $4m$ unset coefficients corresponding to points on the edge E_0, where in general, E_i is the set of domain points in $T^{[1]} \cup T^{[2]}$ at a distance i from the edge $\langle v_1, v_3 \rangle$. Then we compute the $4m$ coefficients corresponding to points in the set $\tilde{E}_0$, where $\tilde{E}_i$ is the set of domain points in $T^{[2]} \cup T^{[3]}$ at a distance i from the edge $\langle v_2, v_4 \rangle$. The remaining coefficients in $T^{[1]} \cup T^{[2]} \cup T^{[3]}$ are computed by alternately working on the sets E_i and $\tilde{E}_i$ for $i = 1, \ldots, r$. Finally, we compute the remaining coefficients in $T^{[4]}$ from the C^r smoothness conditions.

We have shown that all coefficients of s are determined by those corresponding to the domain points in the set $\mathcal{M}_r$. This shows that $\mathcal{M}_r$ is a determining set.

To see that $\mathcal{M}_r$ is a minimal determining set, we consider the superspline space $\mathcal{S}_{6m+1}^{2m+1,4m}(\triangle_Q)$. By Theorem 2.2 in [15] the dimension of this space is $32m^2 + 18m + 4$. Our space $\mathcal{S}_r(\triangle_Q)$ is the subspace which satisfies the $4m^2 - 2m$ special conditions (3.2)–(3.3) and the supersmoothness $C^{3m}(v_i)$ for $i = 1, 2, 3, 4$. Enforcing the supersmoothness requires an additional $2m^2 - 2m$ conditions. Thus,

$$
\begin{aligned}
(32m^2 + 18m + 4) - (4m^2 - 2m) - (2m^2 - 2m)& \\
\leq \dim \mathcal{S}_r(\triangle_Q) \leq \# \mathcal{M}_r = 26m^2 + 22m + 4.&
\end{aligned}
$$

Since the expression on the left equals the one on the right, we conclude that it is equal to the dimension of $\mathcal{S}_r(\triangle_Q)$, and $\mathcal{M}_r$ is a MDS. $\square$

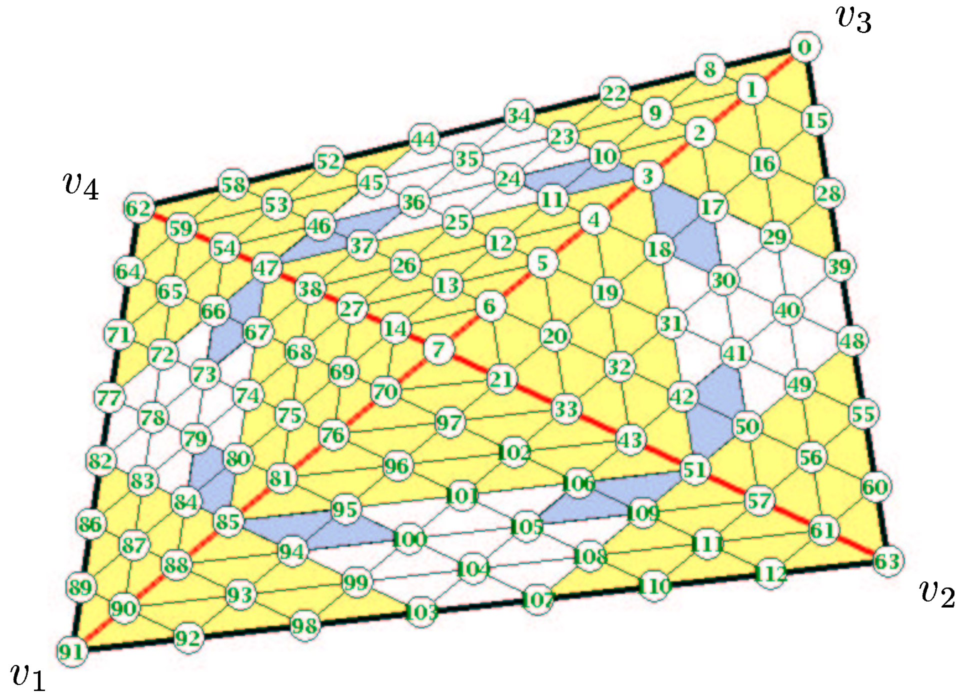

Fig. 2. Domain points for the C^2 macro-element.

§4. Examples

In this section we illustrate the construction of Sect. 2.

Example 4.1. *The space $\mathcal{S}_2(\triangle_Q)$ is the subspace of $\mathcal{S}_7^3(\triangle_Q) \cap C^4(v_Q)$ that satisfies the two special smoothness conditions corresponding to τ_{5,e_1}^4 and τ_{5,e_1}^5.*

Discussion: The dimension of $\mathcal{S}_2(\triangle_Q)$ is 52, and the MDS for this macro-element is shown in Fig. 1. It consists of 10 points in each of the disks $D_3(v_i)$ (marked with crosses) and 3 points corresponding to item 2) of Theorem 3.1 for each edge of Q (marked with triangles). After setting the coefficients in the MDS, the remaining coefficients are computed in the following order. First we use C^3 smoothness to compute the coefficients numbered 80,81,95 in Fig. 2, followed by those numbered 106,43,42, then 18,4,11, and 37,38,67. Using the two special smoothness conditions, we can now compute the coefficients numbered 74,75,76,96,101. Then using C^4 smoothness, we compute the coefficients numbered 70,7,6,5 (lying in the set E_0) and 33,21,14,27 (lying in the set $\tilde{E}_0$). Next we compute coefficients numbered 97,20,19, then 32,13,26, then 102,31 and 12,25. Finally, the coefficients numbered 68,69 are computed by standard smoothness conditions. □

Example 4.2. *The space $\mathcal{S}_4(\triangle_Q)$ is the subspace of $\mathcal{S}_{13}^{5,6}(\triangle_Q) \cap C^8(v_Q)$ that satisfies the twelve special smoothness conditions corresponding to $\{\tau_{8,e_i}^6, \tau_{8,e_i}^7\}_{i=1}^4$ and $\tau_{9,e_1}^6, \tau_{9,e_1}^7, \tau_{9,e_1}^8, \tau_{9,e_1}^9$.*

Discussion: The dimension of $\mathcal{S}_4(\triangle_Q)$ is 152, and the MDS for this macro-element

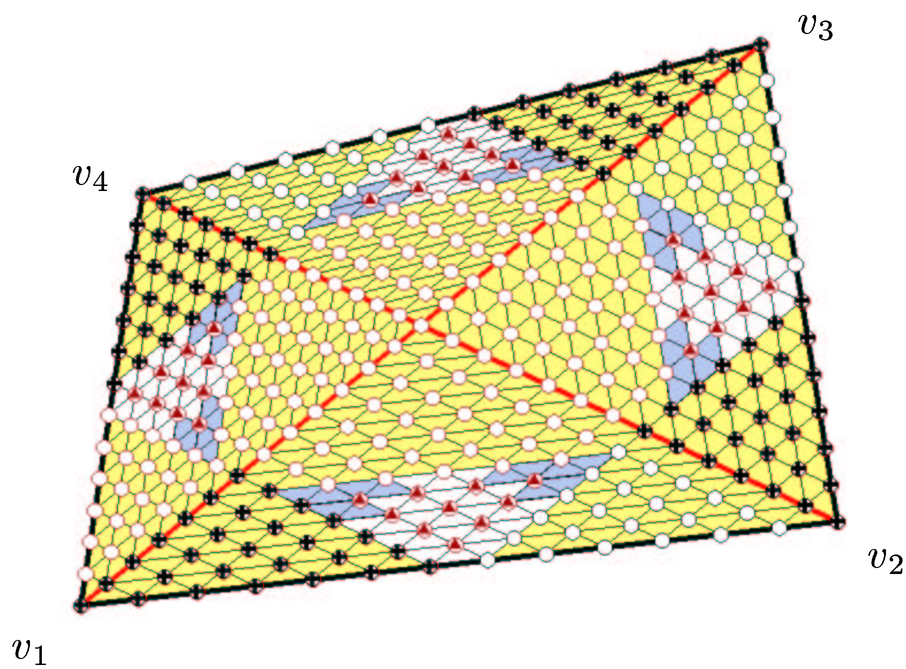

Fig. 3. The C^4 macro-element.

is shown in Fig. 3. It consists of 28 points in each of the disks $D_3(v_i)$ (marked with crosses) and 10 points corresponding to item 2) of Theorem 3.1 for each edge of Q (marked with triangles). $\square$

§5. The case $r = 2m + 1$

Theorem 5.1. *Given $r = 2m + 1$, let $\mathcal{S}_r(\triangle_Q)$ be the linear subspace of all splines s in $\mathcal{S}_{6m+3}^{2m+1,3m+1}(\triangle_Q)$ that satisfy the following set of additional smoothness conditions:*

$$s \in C^{4m+1}(v_Q) \tag{5.1}$$

$$\tau_{3m+i+2,e_l}^{2m+1+j} s = 0, \qquad 1 \le j \le 2i, \quad 1 \le i \le m-1, \quad l = 1,2,3,4, \tag{5.2}$$

$$\tau_{4m+2,e_l}^{2m+1+j} s = 0, \qquad 1 \le j \le 2m, \quad l = 1,2,3. \tag{5.3}$$

Then

$$\dim \mathcal{S}_r(\triangle_Q) = 26m^2 + 42m + 16.$$

Moreover, the following set $\mathcal{M}_r$ of domain points is a MDS:

1) $D_{3m+1}^{T^{[i]}}(v_i)$ *for* $i = 1,2,3,4$,
2) $\{\xi_{j,3m+1,3m-j+2}^{T^{[i]}}, \ldots, \xi_{j,3m-j+2,3m+1}^{T^{[i]}}\}$ *for* $j = 1, \ldots, 2m+1$ *and* $i = 1,2,3,4$.

Proof: First we show that $\mathcal{M}_r$ is a determining set. Suppose that $s \in \mathcal{S}_r(\triangle_Q)$ and that we have set the coefficients of s corresponding to all domain points in $\mathcal{M}_r$.

Then using the usual smoothness conditions, we solve for the unset coefficients corresponding to domain points in the disks $D_{3m+1}(v_i)$ for $i = 1, 2, 3, 4$.

Next we use Lemma 2.1 to compute the coefficients corresponding to points on the rings $R_{3m+i+2}(v_l)$ for $i = 0, \ldots, m-1$ and $l = 1, 2, 3, 4$. On the ring $R_{3m+i+2}(v_l)$ this involves solving a system of $2(m+i)+1$ linear equations. Then we compute coefficients on the rings $R_{4m+2}(v_l)$ for $l = 1, 2, 3$.

Using the lemma, we now compute the $4m+1$ unset coefficients corresponding to the sets E_0 and $\tilde{E}_0$ defined in the proof of Theorem 3.1. The remaining coefficients in $T^{[1]} \cup T^{[2]} \cup T^{[3]}$ are computed by alternately working on the sets E_i and $\tilde{E}_i$ for $i = 1, \ldots, r-1$. Finally, we compute the remaining coefficients in $T^{[4]}$ from the C^r smoothness conditions.

We have shown that all coefficients of s are determined by those corresponding to the domain points in the set $\mathcal{M}_r$. This shows that $\mathcal{M}_r$ is a determining set.

To see that $\mathcal{M}_r$ is a minimal determining set, we consider the superspline space $\mathcal{S}_{6m+3}^{2m+1,4m+1}(\triangle_Q)$. By Theorem 2.2 in [15] the dimension of this space is $32m^2+46m+16$. Our space $\mathcal{S}_r(\triangle_Q)$ is the subspace which satisfies the $4m^2+2m$ special conditions (5.2)–(5.3) and the supersmoothness $C^{3m+1}(v_i)$ for $i = 1, 2, 3, 4$. Enforcing the supersmoothness requires an additional $2m^2+2m$ conditions. Thus,

$$\begin{aligned}(32m^2 + 46m + 16) - (4m^2 + 2m) - (2m^2 + 2m)& \\ \leq \dim \mathcal{S}_r(\triangle_Q) \leq \#\mathcal{M}_r = 26m^2 + 42m + 16.&\end{aligned}$$

Since the expression on the left equals the one on the right, we conclude that it is equal to the dimension of $\mathcal{S}_r(\triangle_Q)$, and $\mathcal{M}_r$ is a MDS. □

§6. Examples

In this section we illustrate the construction of Sect. 2.

Example 6.1. *The space $\mathcal{S}_3(\triangle_Q)$ is the subspace of $\mathcal{S}_9^{3,4}(\triangle_Q) \cap C^5(v_Q)$ that satisfies the six special smoothness conditions corresponding to $\tau_{6,e_i}^4, \tau_{6,e_i}^5$ for $i = 1, 2, 3$.*

Discussion: The dimension of $\mathcal{S}_3(\triangle_Q)$ is 84, and the MDS for this macro-element is shown in Fig. 4. It consists of 15 points in each of the disks $D_4(v_i)$ (marked with crosses) and 6 points corresponding to item 2) of Theorem 5.1 along each edge of Q (marked with triangles). □

Example 6.2. *The space $\mathcal{S}_5(\triangle_Q)$ is the subspace of $\mathcal{S}_{15}^{5,7}(\triangle_Q) \cap C^9(v_Q)$ that satisfies the twenty special smoothness conditions corresponding to $\{\tau_{9,e_i}^6, \tau_{9,e_i}^7\}_{i=1}^4$ and $\{\tau_{10,e_i}^6, \tau_{10,e_i}^7, \tau_{10,e_i}^8, \tau_{10,e_i}^9\}_{i=1}^3$.*

Discussion: The space $\mathcal{S}_5(\triangle_Q)$ has dimension 204, and the MDS for this macro-element is shown in Fig. 5. It consists of 36 points in each of the disks $D_5(v_i)$ (marked with crosses) and 15 points corresponding to item 2) in Theorem 5.1 along each edge of Q (marked with triangles). □

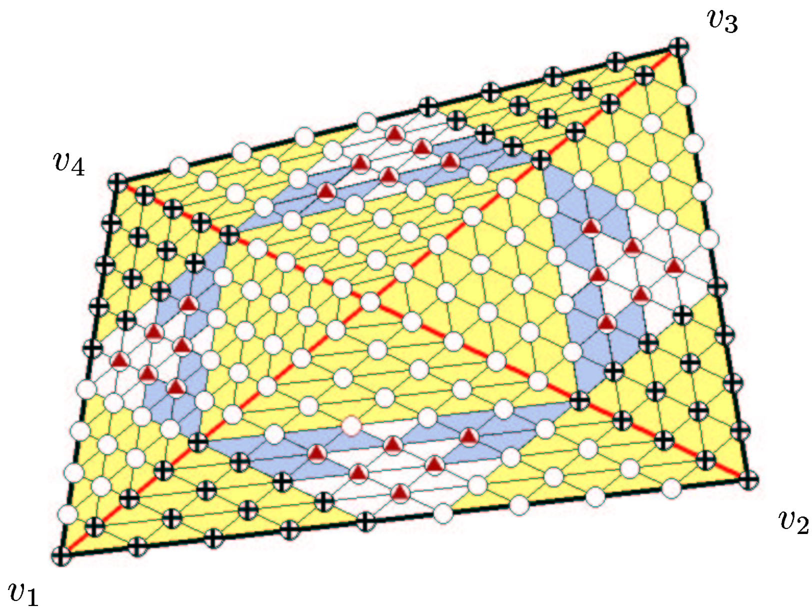

Fig. 4. The C^3 macro-element.

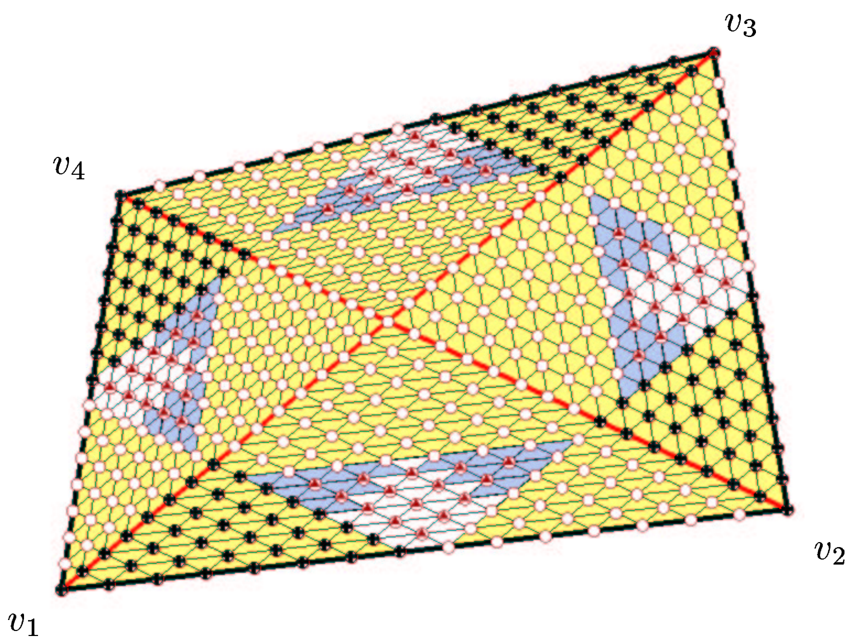

Fig. 5. The C^5 macro-element.

§7. Superspline spaces with stable bases

Let $\diamondsuit$ be a quadrangulation of a domain Ω with vertices $\{v_i\}_{i=1}^V$. Suppose E is the number of edges. Let $\diamondsuit\!\!\!\!+$ be the triangulation obtained by inserting both diagonals in each quadrilateral Q in $\Diamond$. Let

$$\mathcal{S}_r(\diamondplus) := \{s \in C^r(\Omega) : \; s|_Q \in \mathcal{S}_r(\triangle_Q) \text{ all } Q \in \Diamond\}, \tag{7.1}$$

where $\mathcal{S}_r(\triangle_Q)$ are the spaces defined in Theorems 3.1 and 5.1. Let

$$d_r = \begin{cases} 6m+1, & r = 2m, \\ 6m+3, & r = 2m+1, \end{cases} \tag{7.2}$$

and

$$\rho_r = \begin{cases} 3m, & r = 2m, \\ 3m+1, & r = 2m+1, \end{cases} \tag{7.3}$$

Theorem 7.1. *For all $r \geq 1$,*

$$\dim \mathcal{S}_r(\diamondplus) = \binom{\rho_r + 2}{2} V + \binom{r+1}{2} E. \tag{7.4}$$

Moreover, the following set $\mathcal{M}_r$ of domain points forms a MDS:

1) *for each vertex $v \in \Diamond$, choose $D^T_{\rho_r}(v)$, where T is some triangle in $\diamondplus$ with vertex at v,*
2) *for each edge $e \in \Diamond$, choose $\{\xi^T_{j,\rho_r,d_r-\rho_r-j}, \ldots, \xi^T_{j,d_r-\rho_r-j,\rho_r}\}$ for $j = 1, \ldots, r$, where T is some triangle in $\diamondplus$ sharing the edge e.*

Proof: First we show that $\mathcal{M}_r$ is a determining set. For each vertex $v \in \Diamond$, using the smoothness conditions, item 1) determines all coefficients corresponding to points in the disk $D_{\rho_r}(v)$. Similarly, if $\widetilde{T}$ is a second triangle sharing the edge e, then item 2) determines the corresponding coefficients in both T and $\widetilde{T}$. The claim then follows from Theorems 3.1 and 5.1.

To show that $\mathcal{M}_r$ is a minimal determining set, we now construct the dual basis corresponding to $\mathcal{M}_r$. For each $\xi \in \mathcal{M}_r$, let B_ξ be the unique spline in $\mathcal{S}_r(\diamondplus)$ such that

$$\lambda_\eta B_\xi = \delta_{\xi,\eta}, \qquad \eta \in \mathcal{M}_r, \tag{7.5}$$

where λ_η is the linear functional which picks off the B-coefficient corresponding to the domain point η.

In view of (7.5), the splines in $\mathcal{B} := \{B_\xi\}_{\xi \in \mathcal{M}_r}$ are linearly independent, and thus $\mathcal{B}$ forms a basis for $\mathcal{S}_r(\diamondplus)$. It follows that $\dim \mathcal{S}_r(\diamondplus) = \#\mathcal{M}_r$ which is the number in (7.4). $\square$

It is easy to see that the dual basis functions constructed in the above proof have local support. In particular,

1) If ξ is a point as in item 1) of Theorem 7.1, then $\operatorname{supp}(B_\xi)$ is contained in the union of all quadrilaterals of $\diamondplus$ sharing the vertex v.
2) If ξ is a point as in item 2) of Theorem 7.1, then $\operatorname{supp}(B_\xi)$ is contained in $Q \cup \tilde{Q}$, where e is the edge between Q and $\tilde{Q}$. (If e is a boundary edge of a quadrilateral Q, then the support is simply Q).

Lemma 7.2. *Let $\{B_\xi\}_{\xi \in \mathcal{M}}$ be the set of dual basis splines constructed in the proof of Theorem 7.1. Then there exists a constant K depending only on the smallest angle in $\diamondsuit$ such that $\|B_\xi\| \le K$ for all $\xi \in \mathcal{M}_r$.*

Proof: Fix $\xi \in \mathcal{M}_r$, and let B_ξ be the corresponding dual basis spline. We examine the size of its B-coefficients. By definition, $c_\xi = 1$ and $c_\eta = 0$ for all other $\eta \in \mathcal{M}_r$. The remaining B-coefficients of B_ξ are computed by using smoothness conditions or solving the linear systems of equations appearing in Lemma 2.1 of [2]. These involve matrices whose inverses are bounded in norm by a constant depending only on the smallest angle in $\diamondsuit$. This shows that all of the B-coefficients of B_ξ are bounded by a constant K, and the result follows. $\square$

Theorem 7.3. *The dual basis $\{B_\xi\}_{\xi \in \mathcal{M}_r}$ is a stable basis in the sense that there exist constants K_1, K_2 depending only on the smallest angle in $\diamondsuit$ such that for all choices of the coefficient vector $c = (c_\xi)_{\xi \in \mathcal{M}_r}$,*

$$K_1 \|c\|_\infty \;\le\; \| \sum_{\xi \in \mathcal{M}_r} c_\xi B_\xi \|_\infty \;\le\; K_2 \|c\|_\infty. \tag{7.6}$$

Proof: The proof follows in the same way as the proof of Theorem 2.3 of [4]. $\square$

We conclude this section with an approximation result. Given a function f in $L_1(\Omega)$ and an integer $0 \le k \le d_r$, let

$$Q_k f := \sum_{\xi \in \mathcal{M}_r} \lambda_{\xi,k} f \, B_\xi,$$

where $\lambda_{\xi,k}$ is the linear functional defined in Sect. 10 of [8].

Theorem 7.4. *Fix $1 \le p \le \infty$. Suppose f lies in the Sobolev space $W_p^{k+1}(\Omega)$ for some $0 \le k \le d_r$. Then*

$$\|D_x^\alpha D_y^\beta (f - Q_k f)\|_p \le K |\diamondsuit|^{k+1-\alpha-\beta} |f|_{k+1,p} \tag{7.7}$$

for $0 \le \alpha + \beta \le k$, where $|\diamondsuit|$ is the mesh size of $\diamondsuit$ (ie., the diameter of the largest triangle), and $|f|_{k+1,p}$ is the usual Sobolev semi-norm. If Ω is convex, then the constant K depends only on d_r, p, k, and on the smallest angle in $\diamondsuit$. If Ω is nonconvex, it also depends on the Lipschitz constant $L_{\partial\Omega}$ associated with the boundary of Ω.

Proof: The proof follows in the same way as the proof of Theorem 1.1 of [8]. $\square$

§8. Remarks

Remark 8.1. We used the java code described in [1] to check the macro-elements described in this paper, and to generate the figures. The code can be used or downloaded from `http://www.math.utah.edu/~alfeld`.

Remark 8.2. Theorem 10.1 of [10] implies that in order to obtain macro elements on $\triangle_Q$ which will join with C^r smoothness when constructed on the individual quadrilaterals of a quadrangulation, we must require supersmoothness of order ρ_r at the vertices of Q, where ρ_r is defined in (7.3). This implies that we cannot use polynomials of degree lower than the d_r given in (7.2).

Remark 8.3. While there is a unique choice of minimal degree and minimal supersmoothness at the vertices of Q, our choice of extra smoothness conditions is not the only choice which leads to macro-elements based on the natural set of degrees of freedom, *i.e.*, other sets of τ's will also work.

Remark 8.4. In view of the connection between derivatives and B-coefficients, Theorem 7.1 immediately implies that given $f \in C^{\rho_r}(\Omega)$, there exists a unique spline $s \in \mathcal{S}_r(\diamondsuit)$ which solves the Hermite interpolation problem:

$$D_x^\nu D_y^\mu s(v) = D_x^\nu D_y^\mu f(v), \qquad 0 \le \nu + \mu \le \rho_r, \qquad v \in \diamondsuit, \tag{8.1}$$

and

$$D_e^j s(\eta_{e,i}^j) = D_e^j f(\eta_{e,i}^j), \qquad 1 \le i \le j, \quad 1 \le j \le r \tag{8.2}$$

for all edges e of $\diamondsuit$. Here D_e denotes the perpendicular derivative to the edge $e := \langle u_1, u_2 \rangle$, and

$$\eta_{e,i}^j s := \frac{(j+1-i)u_1 + iu_2}{j+1}, \qquad i = 1, \ldots, j.$$

A simple application of the Bramble-Hilbert lemma shows that this interpolant satisfies the error bounds of Theorem 7.4 with $p = \infty$.

Remark 8.5. In view of Remark 8.2, it is clear that the natural set of degrees of freedom for a C^r macro-element on $\triangle_Q$ are precisely the derivative information described in (8.1)–(8.2). For a comparison with the natural degrees of freedom for smooth macro-elements defined on Clough-Tocher and Powell-Sabin splits, see [2,3,9,10].

References

1. Alfeld, P., Bivariate splines and minimal determining sets, J. Comput. Appl. Math. **119** (2000), 13–27.
2. Alfeld, P. and L. L. Schumaker, Smooth finite elements based on Clough-Tocher triangle splits, ms, 2000.
3. Alfeld, P. and L. L. Schumaker, Smooth finite elements based on Powell-Sabin triangle splits, ms, 2000.
4. Davydov, O. and L. L. Schumaker, On stable local bases for bivariate polynomial spline spaces, submitted, 1999.

5. Fraeijs de Veubeke, B., A conforming finite element for plate bending, J. Solids Structures **4** (1968), 95–108.

6. Ibrahim, A. and L. L. Schumaker, Super spline spaces of smoothness r and degree $d \geq 3r+2$, Constr. Approx. **7** (1991), 401–423.

7. Lai, M. J., On dual functionals of polynomials in B-form, J. Approx. Theory **67** (1991), 19–37.

8. Lai, M.-J. and L. L. Schumaker, On the approximation power of bivariate splines, Advances in Comp. Math. **9** (1998), 251–279.

9. Lai, M.-J. and L. L. Schumaker, Macro-elements and stable local bases for splines on Clough-Tocher triangulations, Numer. Math., to appear.

10. Lai, M.-J. and L. L. Schumaker, Macro-elements and stable local bases for splines on Powell-Sabin triangulations, manuscript, 1999.

11. Laghchim-Lahlou, M. and P. Sablonniére, Composite quadrilateral finite elements of class C^r, in *Mathematical Methods in Computer Aided Geometric Design,* T. Lyche and L. Schumaker (eds), Academic Press, New York, 1989, 313–418.

12. Laghchim-Lahlou, M. and P. Sablonniére, Quadrilateral finite elements of FVS type and class C^ρ, Num. Math. **70** (1995), 229–243.

13. Sablonnière, P. and M. Laghchim-Lahlou, Elements finis polynomiaux composé de classe C^r, C. R. Acad. Sci. Paris Sr. I Math. **316** (1993), 503–508.

14. Sander, G., Bornes supérieures et inférieures dans l'analyse matricielle des plaques en flexion-torsion, Bull. Soc. Royale Sciences Liège **33** (1964), 456–494.

15. Schumaker, L. L., Dual bases for spline spaces on cells, Comput. Aided Geom. Design **5** (1988), 277–284.