Interpolation and Scattered Data Fitting on Manifolds using Projected Powell-Sabin Splines

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Dedicated to Professor M.J.D. Powell
on the occasion of his 70th birthday

Abstract

We present methods for either interpolating data or for fitting scattered data on a two-dimensional smooth manifold Ω. The methods are based on a local bivariate Powell-Sabin interpolation scheme, and make use of a family of charts $\{(U_\xi, \phi_\xi)\}_{\xi \in \Omega}$ satisfying certain conditions of smooth dependence on ξ. If Ω is a C^2-manifold embedded into $\mathbb{R}^3$, then projections into tangent planes can be employed. The data fitting method is a two-stage method. We prove that the resulting function on the manifold is continuously differentiable, and establish error bounds for both methods for the case when the data are generated by a smooth function.

1 Introduction

Let Ω be a 2-dimensional smooth manifold. For simplicity we assume that Ω is compact and has no boundary. Suppose we are given the values of a

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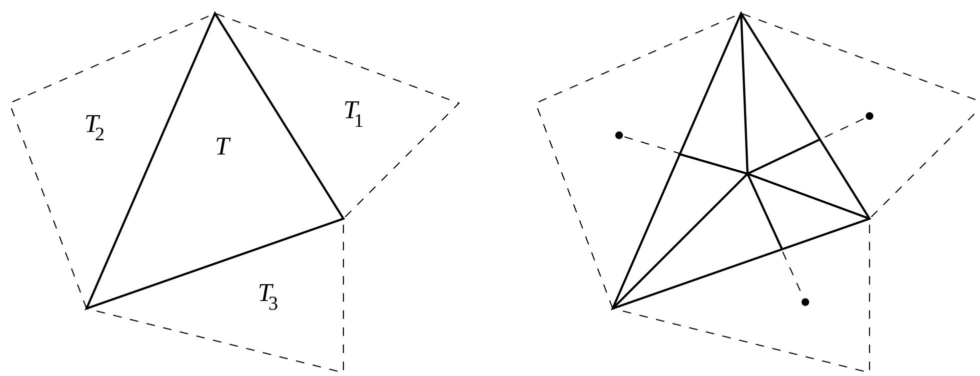
(possibly unknown) smooth function f defined on Ω at a set of points X on Ω. Our aim is to construct a function s defined on Ω that approximates f. This problem arises frequently in practice, see Remark 8.1, but there do not seem to be many methods available for general manifolds. Several methods have been developed for the case when Ω is the sphere, see Remark 8.2.

Our approach to solving this problem is as follows. Suppose we have an atlas $\Phi = \{(U_\xi, \phi_\xi)\}_{\xi\in\Omega}$ for Ω, where for each $\xi \in \Omega$, U_ξ are open sets on Ω containing ξ, and ϕ_ξ are mappings of U_ξ into $\mathbb{R}^2$. We assume that the ϕ_ξ depend smoothly on ξ in a way to be described in Definition 3.1. Then for each $\xi \in \Omega$, we map the data locations into $\phi_\xi(U_\xi) \subset \mathbb{R}^2$, and use a local bivariate Powell-Sabin spline to compute the value $s(\xi)$ of the approximating function s. This approach is related to methods introduced by Demjanovich [11, 12] and Pottmann [29], see Remarks 8.3 and 8.4.

The paper is organized as follows. In Section 2 we describe the classical bivariate piecewise quadratic spline interpolant by Powell and Sabin [30], and prove that it depends smoothly on the vertex locations and the data. In Section 3 we introduce some basic concepts and notation, including atlases, gradients, Sobolev spaces, and triangulations on manifolds. In view of the Poincaré-Hopf index theorem, local parametrizations defined by the charts (U_ξ, ϕ_ξ) in general cannot be smooth functions of ξ, see Remark 8.6. Therefore, we introduce a weaker concept of smoothness whereby ϕ_ζ for ζ close to ξ may be adjusted by local rotations or rotoinversions (see Definition 3.1). In Section 4 we present a method for constructing an interpolant to data on an arbitrary smooth 2-dimensional manifold Ω, assuming we are also given values for the gradients at each of the data locations in X. We show that the method produces a C^1 function on Ω, and give an error bound for how well it approximates smooth functions. In Section 5 we describe a two-stage scattered data fitting method which is more appropriate than interpolation for large data sets or noisy data. In the next section we give an error bound for this method. In Section 7 we specialize to the case where the manifold is embedded in $\mathbb{R}^3$. In particular, we show how to explicitly construct an atlas with the required properties using local projections into tangent planes, describe the computation of gradients in this case, and discuss certain simplifications in the algorithms. More details on our method used with this specific atlas can be found in our paper [9], which also gives numerical examples for both the sphere and for certain ring-type manifolds. We conclude the paper with remarks and references.



(a) (b)

Figure 1: Powell-Sabin split of a triangle. PStri

2 Powell-Sabin Spline Interpolant

PSsplines

Let $\triangle$ be a regular triangulation of a bounded polygonal domain $G \subset \mathbb{R}^2$, i.e., $\triangle$ is a set of pairwise disjoint open non-degenerate triangles T such that $\overline{G} = \cup_{T \in \triangle} \overline{T}$, and no vertex of a triangle lies in the interior of an edge of another triangle. In order to emphasise the difference between this definition and the more abstract notion of a triangulation of a manifold to be introduced in Section 3.4, we call $\triangle$ a *planar triangulation*. We say that $T \in \triangle$ is an *interior triangle* if none of its edges lies on the boundary of G. Otherwise, T is called a *boundary triangle*.

Given an interior triangle T in $\triangle$, let T_1, T_2, T_3 be the three triangles in $\triangle$ sharing edges with T, see Figure 1(a). The set T_{PS} of six triangles obtained by connecting the incenter of T to the incenters of T_1, T_2, T_3, and to the vertices of T is called the *Powell-Sabin-6 split* of T, see Figure 1(b). If T is a boundary triangle, then the split is the same except that whenever there is no other triangle in $\triangle$ sharing an edge e with T, the incenter of T is connected to the midpoint of e. A new triangulation of G, called the *Powell-Sabin refinement* of $\triangle$, is obtained as

$$\triangle_{PS} := \bigcup_{T \in \triangle} T_{PS}.$$

It is well known [21, 30] that the space $S_2^1(\triangle_{PS})$ of all C^1 piecewise quadratic functions with respect to $\triangle_{PS}$ has a particularly simple structure. Its dimension is three times the number of vertices of $\triangle$, and every $s \in$

$S^1_2(\triangle_{PS})$ is uniquely determined by its values and gradients at the vertices of $\triangle$. In other words, the interpolation problem

$$s(v) = a_v, \quad \nabla s(v) = \sigma_v, \quad \text{for all } v \in V, \tag{2.1}$$ PSint

has a unique solution $s \in S^1_2(\triangle_{PS})$ for any real numbers a_v and real 2-vectors $\sigma_v = [\sigma_v^{[1]}, \sigma_v^{[2]}]$, $v \in V$. Moreover, s depends *locally* on the data $\{a_v, \sigma_v\}_{v\in V}$. In particular, for each $T \in \triangle$, $s|_T$ is uniquely determined by the data at the three vertices of T only, i.e. by a_{v_i}, σ_{v_i}, $i = 1, 2, 3$, where $T = \langle v_1, v_2, v_3\rangle$.

Let $x \in G$. Then $x \in T$ for some $T \in \triangle$. For later use, we need to investigate the differentiability of the Powell-Sabin spline $s(x)$ as a function of *all* of the parameters that determine it. These include not only the data a_v, σ_v at the vertices, but also the locations of the vertices. Due to the nature of the Powell-Sabin-6 split, the locations of the vertices of the immediate neighbours of T also influence $s(x)$. Moreover, when studying differentiabilty, we also need to take into account all parameters that have influence on $s(x)$ if the triangulation is perturbed, further extending the set of 'active' parameters if x lies on the boundary of T, or at one of its vertices. We set

$$V_x = \{v \in V : x \in \overline{T} \text{ for some } T \in \triangle \text{ with a vertex at } v\},$$

and denote by $\tilde{V}_x$ the set of those vertices in $V \setminus V_x$ that form a triangle in $\triangle$ with a pair of vertices in V_x.

smoothpar **Theorem 2.1.** *Let $s \in S^1_2(\triangle_{PS})$ be the Powell-Sabin spline interpolant that solves the problem (2.1). The value $s(x)$ of s at a fixed point $x \in G$ is a continuously differentiable function of the following parameters: (a) the coordinates of the vertices v and the data values $a_v, \sigma_v^{[1]}, \sigma_v^{[2]}$ for all $v \in V_x$, and (b) the coordinates of the vertices in $\tilde{V}_x$.*

Proof. Clearly, a sufficiently small perturbation of the vertices of $\triangle$ does not produce any degenerate or overlapping triangles in either $\triangle$ or $\triangle_{PS}$. Moreover, the vertices of $\triangle_{PS}$ depend smoothly on the locations of the vertices of $\triangle$ since the incenter of a triangle is an analytic function of its vertices. If x lies in a triangle $T \in \triangle_{PS}$, then it will still be in the same triangle after a small perturbation of the vertices. If, however, x is on an edge or at a vertex of $\triangle_{PS}$, then it can move into a triangle attached to that edge or vertex. In any case, after a small perturbation the set $V_x \cup \tilde{V}_x$ either remains the same or becomes smaller. As discussed above, $s(x)$ depends only on parameters in (a) and (b) even if *all* vertices $v \in V$ and corresponding data values are slightly perturbed.

As a quadratic polynomial, each $s|_T$, $T \in \triangle_{PS}$, has a unique expansion

$$s|_T = \sum_{i+j+k=2} c^T_{ijk} B^{T,2}_{ijk}$$

with respect to the Bernstein basis polynomials $B^{T,2}_{ijk} := \frac{2}{i!j!k!} b_1^i b_2^j b_3^k$, where b_1, b_2, b_3 are the barycentric coordinates relative to T. Note that c^T_{200}, c^T_{020} and c^T_{002} are the values of s at the vertices of T. Referring to the Bernstein-Bézier analysis of the Powell-Sabin element [6, 21], it is easy to see that the Bézier coefficients c_{ijk} are analytic functions of the data values and vertices of $\triangle$. Clearly, the value $B^{T,2}_{ijk}(x)$ is an analytic function of the coordinates of the vertices of T (where we do not need to assume that $x \in \overline{T}$). This immediately implies that $s(x) = \sum_{i+j+k=2} c^T_{ijk} B^{T,2}_{ijk}(x)$ is an analytic function of the parameters in (a) and (b) if x lies in a triangle $T \in \triangle_{PS}$.

Suppose that x lies on an edge e of $\triangle_{PS}$ shared by two triangles $T_1, T_2 \in \triangle_{PS}$, but not at a vertex. A small perturbation of the parameters will leave x in the interior of $\overline{T_1} \cup \overline{T_2}$. Note that C^1 smoothness of the spline s implies $s|_{T_2} - s|_{T_1} = \alpha \ell_e^2$, where $\ell_e = 0$ is the linear equation describing the straight line going through e, and $\alpha \in \mathbb{R}$. Thus,

$$s(x) = p(x) + \alpha \psi(x),$$

where

$$p(x) = \sum_{i+j+k=2} c^{T_1}_{ijk} B^{T_1,2}_{ijk}(x), \quad \psi(x) = \begin{cases} 0, & \text{if } x \in \overline{T_1}, \\ \ell_e^2(x), & \text{if } x \in \overline{T_2}. \end{cases}$$

As discussed above, $p(x)$ is an analytic function of the parameters in (a) and (b). We now show that $\psi(x)$ is a C^1 function of the parameters. Without loss of generality we may write the equation for ℓ_e in the normalized form $\ell_e(x) = x^{[1]} \cos b + x^{[2]} \sin b + c$, where b and c are analytic functions of the coordinates of the vertices of e. It follows that $\psi(x)$ is a composition of the analytic function $\ell_e(x)$ with the univariate truncated power function

$$t_+^2 = \begin{cases} 0, & \text{if } t \le 0, \\ t^2, & \text{if } t > 0. \end{cases}$$

This is a C^1 function, and hence $\psi(x)$ is a C^1 function of the parameters. Finally, we claim that α is also an analytic function. To see this, we denote by w the vertex of T_2 not shared by T_1. We have $s(w) = p(w) + \alpha \ell_e^2(w)$, which implies

$$\alpha = \frac{s(w) - p(w)}{(w^{[1]} \cos b + w^{[2]} \sin b + c)^2}.$$

This expression is an analytic function of the parameters since $s(w)$ is a Bézier coefficient of $s|_{T_2}$.

Finally, we consider the case where x coincides with a vertex v of $\triangle_{PS}$. If the parameters are perturbed, then x will lie in the *cell* $C = \cup_{i=1}^n \overline{T_i}$ formed by the set $\{T_i\}_{i=1}^n$ of all triangles in $\triangle_{PS}$ attached to v, numbered in counterclockwise order. We consider only the case where v is an interior vertex of $\triangle_{PS}$ since the boundary case is similar and simpler. Let $v_1, \ldots, v_n$ be the set of all vertices of $\triangle_{PS}$ attached to v, where T_i has vertices v, v_i, v_{i+1}, with $v_{n+1} := v_1$. We write e_i for the edge $\langle v, v_i \rangle$, $i = 1, \ldots, n$. By a rotation of the coordinate system, we may assume that no edge e_i is parallel to the first coordinate axis. Then $v_i^{[1]} - v^{[1]} \neq 0$, $i = 1, \ldots, n$. Following [31], we obtain

$$s(x) = p(x) + \sum_{i=1}^n \alpha_i \psi_i(x), \qquad (2.2) \quad \boxed{\texttt{cellform}}$$

where

$$p(x) = \sum_{i+j+k=2} c_{ijk}^{T_n} B_{ijk}^{T_n,2}(x),$$

$$\psi_i(x) = \begin{cases} 0, & \text{if } x \in \overline{T_j},\ 1 \leq j < i, \\ [x^{[2]} - v^{[2]} + \sigma_i (x^{[1]} - v^{[1]})]^2, & \text{if } x \in \overline{T_j},\ i \leq j \leq n, \end{cases} \qquad (2.3) \quad \boxed{\texttt{psi}}$$

$$\sigma_i = -(v_i^{[2]} - v^{[2]})/(v_i^{[1]} - v^{[1]}),$$

and the coefficients α_i satisfy

$$\sum_{i=1}^n \alpha_i \sigma_i^k = 0, \quad k = 0, 1, 2. \qquad (2.4) \quad \boxed{\texttt{acond}}$$

Note that (2.4) follows from the fact that $s|_{T_n} = p$, and so

$$\sum_{i=1}^n \alpha_i \psi_i(x) = 0 \quad \text{if } x \in \overline{T_n}.$$

By (2.2),

$$s(v_{j+1}) = p(v_{j+1}) + \sum_{i=1}^{j} \alpha_i \psi_i(v_{j+1}), \quad j = 1, \dots, n,$$

which shows that α_i are analytic functions of the parameters. However, except for ψ_1, the functions ψ_i are not even continuous, so to complete the proof we directly compute bounds on the change of $s(x)$ and its derivatives with respect to the parameters as the parameters are perturbed. Let

$$\tilde{s}(x) := s(x) - p(x) = \sum_{i=1}^{n} \alpha_i \psi_i(x).$$

Then $\tilde{s}(x) = 0$ before the perturbation of the parameters. By (2.3) it follows that

$$|\tilde{s}(x)| \le A(x^{[2]} - v^{[2]})^2 + B(x^{[1]} - v^{[1]})^2, \tag{2.5}$$

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with

$$A = 2\sum_{i=1}^{n} |\alpha_i|, \quad B = 2\sum_{i=1}^{n} |\alpha_i| \sigma_i^2,$$

whenever the parameters are perturbed. Since A and B are bounded, we conclude that the change in $\tilde{s}(x)$ as parameters are perturbed is bounded in terms of the changes in the parameters, which proves the continuity of $\tilde{s}(x)$ and $s(x)$. Moreover, since (2.5) does not include first order terms, it follows that the differential of $\tilde{s}(x)$ with respect to the parameters exists and is zero whenever x coincides with v. The differential also exists if x is in a small neighborhood of v, but does not coincide with it, since in this case x is either in a triangle or on an edge, which are the situations we have considered above. Let ν be any of the parameters in (a) or (b) in the statement of the theorem. From (2.3) we obtain that

$$\left| \frac{\partial \tilde{s}(x)}{\partial \nu} \right| \le C(x^{[2]} - v^{[2]}) + D(x^{[1]} - v^{[1]}),$$

with some bounded C, D. This shows that the partial derivatives of $\tilde{s}(x)$ go to zero as the change in parameters goes to zero. We conclude that $\tilde{s}(x)$ and $s(x)$ are countinuously differentiable. ■

3 Manifolds: Preliminaries

prelim

3.1 Atlases

atlas

Let Ω be a compact 2-dimensional smooth manifold without boundary. For each $\xi \in \Omega$, suppose that U_ξ is an open subset of Ω containing ξ, and that $\phi_\xi : U_\xi \to \mathbb{R}^2$ is a homeomorphism between U_ξ and an open subset of $\mathbb{R}^2$. Suppose also that for every $\xi, \zeta \in \Omega$, $\phi_\zeta \circ \phi_\xi^{-1} : \phi_\xi(U_\zeta \cap U_\xi) \to \phi_\zeta(U_\zeta \cap U_\xi)$ is a C^1 mapping whenever $U_\xi \cap U_\zeta \neq \emptyset$. Then, according to standard terminology, see e.g. [19], $\Phi = \{(U_\xi, \phi_\xi)\}_{\xi\in\Omega}$ is an *atlas* for Ω, and (U_ξ, ϕ_ξ), $\xi \in \Omega$, are its *charts*. We emphasize that we need a chart for each $\xi \in \Omega$, rather than simply a covering of Ω by charts, as is usually required of an atlas. Moreover, we suppose that the charts depend smoothly on ξ in the sense of Definition 3.1 below. Let

$$B_\xi := \phi_\xi(U_\xi), \quad \phi_{\zeta\xi} := \phi_\zeta \circ \phi_\xi^{-1}.$$

Then B_ξ is an open set in $\mathbb{R}^2$, and $\phi_{\zeta\xi}$ is an invertible C^1 mapping between $\phi_\xi(U_\zeta \cap U_\xi) \subset B_\xi$ and $\phi_\zeta(U_\zeta \cap U_\xi) \subset B_\zeta$.

A real function f defined in a neighborhood of a point $\xi \in \Omega$ is said to be *continuous* (C^0) at ξ if one of its *local representations* $f \circ \phi_\zeta^{-1}$ is continuous at $\phi_\zeta(\xi)$ for some ζ with $\xi \in U_\zeta$. Similarly, we say that f is *continuously differentiable* (C^1) at ξ provided one of its *local representations* $f \circ \phi_\zeta^{-1}$ is continuously differentiable at $\phi_\zeta(\xi)$ for some ζ with $\xi \in U_\zeta$. Since all $\phi_{\zeta\xi}$ are C^1 mappings, *every* local representation $f \circ \phi_\zeta^{-1}$ will then be C^0 (resp. C^1) at $\phi_\zeta(\xi)$.

For a C^1 function f defined in a neighborhood U of $\zeta \in \Omega$, we also define $J_\zeta(f) : U \cap U_\zeta \to \mathbb{R}^{2\times 2}$ by

$$J_\zeta(f)(\mu) := J(f \circ \phi_\zeta^{-1})(\phi_\zeta(\mu)), \qquad \mu \in U \cap U_\zeta,$$

where for any smooth function $g : \mathbb{R}^2 \to \mathbb{R}^2$, $g = [g^{[1]}, g^{[2]}]^T$, $J(g)$ denotes its Jacobian

$$J(g) := \begin{bmatrix} \frac{\partial g^{[1]}}{\partial x^{[1]}} & \frac{\partial g^{[1]}}{\partial x^{[2]}} \\ \frac{\partial g^{[2]}}{\partial x^{[1]}} & \frac{\partial g^{[2]}}{\partial x^{[2]}} \end{bmatrix}.$$

We write

$$J_{\zeta\xi} := J_\xi(\phi_\zeta), \quad \text{on } U_\zeta \cap U_\xi,$$

so that

$$J_{\zeta\xi}(\mu) = J_\xi(\phi_\zeta)(\mu) = J(\phi_{\zeta\xi})(\phi_\xi(\mu)), \qquad \mu \in U_\zeta \cap U_\xi,$$

is the Jacobian of $\phi_{\zeta\xi}$ evaluated at $\phi_\xi(\mu)$. Since $\phi_{\zeta\xi}^{-1} = \phi_{\xi\zeta}$, the well-known properties of the Jacobian imply

$$[J_{\zeta\xi}(\mu)]^{-1} = J_{\xi\zeta}(\mu). \tag{3.1}$$ jacobrelat

smoothness **Definition 3.1.** We say that the *charts* (U_ξ, ϕ_ξ) *depend smoothly on* ξ if $\phi_\xi(\xi)$ is a C^1 function of ξ, and for each $\xi \in \Omega$ there is an open neighborhood $\tilde{U}_\xi$ of ξ such that the following conditions hold:

- $\tilde{U}_\xi \subset U_\zeta$ for all ζ sufficiently close to ξ.
- For any ζ sufficiently close to ξ, there is a rotation or rotoinversion (a rotation followed by a flip) $r_\zeta : \mathbb{R}^2 \to \mathbb{R}^2$ about $\phi_\zeta(\zeta)$ such that for any $\mu \in \tilde{U}_\xi$, both $(r_\zeta \circ \phi_\zeta)(\mu)$ and $J_\mu(r_\zeta \circ \phi_\zeta)(\mu)$ are C^1 functions of ζ at $\zeta = \xi$.

If the charts of a C^1 atlas $\Phi = \{(U_\xi, \phi_\xi)\}_{\xi\in\Omega}$ depend smoothly on ξ, then Φ is said to be *admissible.*

It is shown in Section 7 that if Ω is a 2-dimensional manifold embedded in $\mathbb{R}^3$, then an admissible atlas can be defined using local orthogonal projections onto tangent planes. More specific examples of admissible atlases are given in Remarks 8.10 and 8.11 for the sphere and the torus.

Note that removing local transformations r_ζ from Definition 3.1 would result in severe restrictions on the topology of the manifold Ω. This is related to the famous 'hairy ball' theorem, see Remark 8.6.

3.2 Gradients

grad

Suppose that f is a continuously differentiable real-valued function on Ω. For any $\xi \in \Omega$, we write $f_\xi := f \circ \phi_\xi^{-1}$. Then $f_\xi : B_\xi \to \mathbb{R}$ is a bivariate C^1 function. For any $\mu \in U_\xi$, we write

$$\nabla_\xi f(\mu) := \nabla f_\xi(\phi_\xi(\mu))$$

for the value of the gradient $\nabla f_\xi := \left[\frac{\partial f_\xi}{\partial x^{[1]}}, \frac{\partial f_\xi}{\partial x^{[2]}}\right]$ of f_ξ at $\phi_\xi(\mu)$. If $U_\zeta \cap U_\xi \neq \emptyset$, then $f_\xi = f_\zeta \circ \phi_{\zeta\xi}$ on $\phi_\xi(U_\zeta \cap U_\xi)$, and by the chain rule,

$$\nabla_\xi f(\mu) = \nabla_\zeta f(\mu)\, J_{\zeta\xi}(\mu), \qquad \mu \in U_\xi \cap U_\zeta. \tag{3.2}$$ gradtrans

3.3 Sobolev spaces

Given a continuous function $f : \Omega \to \mathbb{R}$, we define its maximum norm to be $\|f\|_{C(\Omega)} := \max_{\xi\in\Omega} |f(\xi)|$. Given $r \geq 1$, we say that f belongs to the *Sobolev space* $W^r_\infty(\Omega)$ provided $f_\xi \in W^r_\infty(B_\xi)$ for all $\xi \in \Omega$. We define the *Sobolev norm* on $W^r_\infty(\Omega)$ by

$$\|f\|_{W^r_\infty(\Omega)} := \max_{\xi\in\Omega} \|f_\xi\|_{W^r_\infty(B_\xi)}.$$

This definition of the Sobolev norm for the functions defined on the manifold Ω is equivalent to the standard definitions, see [13].

3.4 Manifold Triangulations

tri

Given a set $\mathcal{V}$ of points in Ω, let $\mathcal{T}$ be a set of triples $\tau = \{v, u, w\}$ of points $v, u, w \in \mathcal{V}$ such that

- any two triples have at most two common points,
- any pair of points in $\mathcal{V}$ belong to at most two different triples in $\mathcal{T}$, and
- for any $v \in \mathcal{V}$, the set of all triples containing v forms a *cell*, i.e. $\{\tau \in \mathcal{T} : v \in \tau\} = \{\tau_i\}_{i=1}^n$ for some $n \geq 3$, where $\tau_i = \{v, v_i, v_{i+1}\}$, with distinct $v_1, \ldots, v_n$, and $v_{n+1} = v_1$.

If these conditions are satisfied, we say that $\mathcal{T}$ is a *triangulation* of Ω with *vertices* $\mathcal{V}$. We say that two vertices v_1, v_2 are *connected* in $\mathcal{T}$ if there is a triple $\tau \in \mathcal{T}$ containing both v_1 and v_2. This definition of a triangulation of a manifold Ω is described by connectivity of vertices only, and does not involve "edges" or "triangles" on Ω. Indeed, $\mathcal{T}$ is essentially an *abstract simplicial complex* [25] with vertices in Ω.

It is well known that any compact 2-dimensional manifold Ω can be triangulated (see [20, 24]), i.e. there exists a finite triangulation $\mathcal{T}$ of Ω in the above sense along with a corresponding partition of Ω into homeomorphic images of triangles, similar to a planar triangulation. Such a partition is also called a triangulation of Ω, but since we never make use of it in this paper, there will be no confusion with our definition.

Given $\xi \in \Omega$, assuming that all vertices of $\tau = \{v, u, w\} \in \mathcal{T}$ are in U_ξ, we denote by $\phi_\xi(\tau)$ the (open) planar triangle in B_ξ with vertices $\phi_\xi(u), \phi_\xi(v), \phi_\xi(w)$. Note that the triangle $\phi_\xi(\tau)$ may be degenerate.

consistency **Definition 3.2.** Let $\Phi = \{(U_\xi, \phi_\xi)\}_{\xi\in\Omega}$ be an admissible atlas for Ω. Let $\tilde{U}_\xi$, $\xi \in \Omega$, be the open neighborhoods in Definition 3.1. We say that a triangulation $\mathcal{T}$ of Ω is *consistent* with Φ provided that for any $\xi \in \Omega$, there is a finite subset $\triangle_\xi \subset \{\phi_\xi(\tau) : \tau \subset \tilde{U}_\xi\}$ such that

- every triangle $T \in \triangle_\xi$ is non-degenerate,
- $\triangle_\xi$ is a planar triangulation of $P_\xi := \cup_{T\in\triangle_\xi}\overline{T}$,
- $\phi_\xi(\xi)$ lies in the interior of P_ξ.

Let $\mathcal{V}_\xi$ be the set of vertices of all τ such that $\phi_\xi(\xi)$ lies in the closure of $\phi_\xi(\tau)$, i.e.

$$\mathcal{V}_\xi := \{v \in \mathcal{V} \cap U_\xi : \phi_\xi(\xi) \in \overline{\phi_\xi(\tau)} \text{ for some } \tau \in \mathcal{T} \text{ with a vertex at } v\}.$$

For a consistent triangulation $\mathcal{T}$, if ξ is a vertex in $\mathcal{V}$, then $\mathcal{V}_\xi$ consists of ξ and all vertices connected to it. For any point $\xi \in \Omega\setminus\mathcal{V}$ the set $\mathcal{V}_\xi$ contains either three or four points, depending on whether $\phi_\xi(\xi)$ belongs to the interior of a triangle in $\triangle_\xi$, or lies on a common edge of two such triangles. We say that ξ is an *interior point* or respectively an *edge point* with respect to the triangulation $\mathcal{T}$. We have $\phi_\xi(\mathcal{V}_\xi) \subset P_\xi$ in any case, and hence $\mathcal{V}_\xi \subset \tilde{U}_\xi$.

cVxi **Lemma 3.3.** *Let $\mathcal{T}$ be a triangulation consistent with an admissible atlas Φ for Ω. Then for any $\xi \in \Omega$, $\mathcal{V}_\zeta \subseteq \mathcal{V}_\xi$ for all $\zeta \in \Omega$ sufficiently close to ξ.*

Proof. According to Definition 3.1, for all ζ sufficiently close to ξ, $(r_\zeta\circ\phi_\zeta)(\mu)$ is a C^1 function of ζ at $\zeta = \xi$, for appropriately chosen transformations r_ζ and for any $\mu \in \tilde{U}_\xi$. Assume $v \in \mathcal{V}_\zeta$. Then $\phi_\zeta(\zeta) \in \overline{\phi_\zeta(\tau)}$ for some $\tau = \{v_1, v_2, v_3\} \in \mathcal{T}$ with $v_1 = v$. We set $v_i(\zeta) = (r_\zeta \circ \phi_\zeta)(v_i)$, $i = 1, 2, 3$, and let $T(\zeta) = \langle v_1(\zeta), v_2(\zeta), v_3(\zeta)\rangle$ be the triangle in $\mathbb{R}^2$ with vertices $v_1(\zeta), v_2(\zeta), v_3(\zeta)$. We have $\phi_\zeta(\zeta) = (r_\zeta \circ \phi_\zeta)(\zeta) \in \overline{T(\zeta)}$. Since $v_i \in \mathcal{V}_\zeta \subset \tilde{U}_\xi$, $i = 1, 2, 3$, the functions $v_i(\zeta)$ are continuous at $\zeta = \xi$. By Definition 3.1, $\phi_\zeta(\zeta)$ is also a continuous function of ζ. In the limit we obtain $\phi_\xi(\xi) \in \overline{T(\xi)}$, which implies $\phi_\xi(\xi) \in \overline{\phi_\xi(\tau)}$ and hence $v \in \mathcal{V}_\xi$. ∎

It is easy to see that the set of all interior points with respect to a consistent triangulation is an open subset of Ω. Indeed, if ξ is an interior point, then $\mathcal{V}_\xi$ consists of just three vertices. For any ζ sufficiently close to ξ, $\mathcal{V}_\zeta \subset \mathcal{V}_\xi$ and $\mathcal{V}_\zeta$ cannot have fewer than three vertices. Hence $\mathcal{V}_\zeta = \mathcal{V}_\xi$ and

ζ is an interior point. In addition to consistency, we will need the following assumption specifically related to the Powell-Sabin spline:

for every $\xi \in \Omega$, if $\phi_\xi(\xi) \in \overline{T}$ for a $T \in \triangle_\xi$, then $\triangle_\xi$ also includes three triangles sharing edges with T. (3.3) fine1

We extend $\mathcal{V}_\xi$ to $\tilde{\mathcal{V}}_\xi$ by adding to $\mathcal{V}_\xi$ the vertices of the triangles described in (3.3). Definition 3.2 implies that $\tilde{\mathcal{V}}_\xi \subset \tilde{U}_\xi$. For a consistent triangulation $\mathcal{T}$, we define the *mesh size* of $\mathcal{T}$ as the length of the longest edge in the triangles in the set $\{\triangle_\xi\}_{\xi\in\Omega}$.

4 An Interpolation Method

PS

Let $\mathcal{T}$ be a triangulation of Ω consistent with an admissible atlas $\Phi = \{(U_\xi, \phi_\xi)\}_{\xi\in\Omega}$. We assume that $\mathcal{T}$ is fine enough for (3.3) to hold. Let $D := \{a_v, \sigma_v\}_{v\in\mathcal{V}}$, where a_v are real numbers and $\sigma_v = [\sigma_v^{[1]}, \sigma_v^{[2]}]$ are 2-vectors. We now show how to construct a C^1 function $s_\mathcal{T}$ defined on Ω that satisfies the interpolation conditions

$$s_\mathcal{T}(v) = a_v, \quad \nabla_v s_\mathcal{T}(v) = \sigma_v, \qquad \text{all } v \in \mathcal{V}. \tag{4.1}$$ interp

eval **Algorithm 4.1.** *Given $\xi \in \Omega$, compute $s_\mathcal{T}(\xi)$:*

1. *Let $T := \langle w_1, w_2, w_3\rangle$ be a triangle in $\triangle_\xi$ such that $\phi_\xi(\xi) \in \overline{T}$, and let $T_1 := \langle w_4, w_3, w_2\rangle$, $T_2 := \langle w_5, w_1, w_3\rangle$, and $T_3 := \langle w_6, w_2, w_1\rangle$ be the three triangles in $\triangle_\xi$ sharing edges with T, see Figure 1(a).*

2. *Let T_{PS} be the Powell-Sabin split of T into six triangles obtained by connecting the incenter w of T to the incenters of T_1, T_2, T_3, and to the vertices w_1, w_2, w_3, see Figure 1(b).*

3. *Let $g_i := \sigma_{v_i} J_{v_i\xi}(v_i)$, where $v_i = \phi_\xi^{-1}(w_i)$, for $i = 1, 2, 3$.*

4. *Let $s_\mathcal{T}(\xi)$ be the value at $\phi_\xi(\xi)$ of the Powell-Sabin C^1 quadratic spline s_ξ defined on T_{PS} that satisfies $s_\xi(w_i) = a_{v_i}$ and $\nabla s_\xi(w_i) = g_i$ for $i = 1, 2, 3$.*

Since the Powell-Sabin interpolant in step 4) is uniquely defined by the values $\{(a_{v_i}, g_i)\}_{i=1}^3$ at the vertices $\{w_i\}_{i=1}^3$, it follows that $s_\mathcal{T}$ is uniquely defined by the data D. By construction, $s_\mathcal{T}$ satisfies (4.1).

`smooth` **Theorem 4.2.** *The interpolant $s_{\mathcal{T}}$ is a C^1 function on the manifold Ω.*

Proof. We may assume without loss of generality that for any $\zeta \in \Omega$ the point $\phi_\zeta(\zeta)$ is the origin in $\mathbb{R}^2$, since otherwise we may replace ϕ_ζ by $\phi_\zeta - \phi_\zeta(\zeta)$, and s_ζ by $s_\zeta(\cdot + \phi_\zeta(\zeta))$, which coincides with the Powell-Sabin spline computed with respect to the shifted version of the local triangulation $\triangle_\zeta$.

Fix $\xi \in \Omega$. Since we are assuming that the charts of Φ depend smoothly on ξ, it follows that for any ζ sufficiently close to ξ there is a rotation or rotoinversion $r_\zeta : \mathbb{R}^2 \to \mathbb{R}^2$ about the origin (an orthogonal linear transformation of the plane) such that both $(r_\zeta \circ \phi_\zeta)(\mu)$ and $J_\mu(r_\zeta \circ \phi_\zeta)(\mu)$, as functions of ζ, are continuously differentiable at $\zeta = \xi$ as soon as $\mu \in \tilde{U}_\xi$. Without loss of generality we assume that r_ξ is the identity. Recall from Section 3.4 that $\mathcal{V}_\xi$ denotes the set of all vertices $v \in \mathcal{V} \cap U_\xi$ such that $\phi_\xi(\xi) \in \overline{T}$ for a triangle $T \in \triangle_\xi$ attached to $\phi_\xi(v)$. Since $\mathcal{T}$ is consistent with the atlas Φ, by Lemma 3.3 we may choose a neighborhood $U \subset U_\xi$ of ξ such that $\mathcal{V}_\zeta \subseteq \mathcal{V}_\xi \subset \tilde{U}_\xi$ for all $\zeta \in U$. Moreover, according to Definition 3.1, we may choose a smaller U to ensure that $\tilde{U}_\xi \subset U_\zeta$ for all $\zeta \in U$. For any $\zeta \in U$ it follows by Definition 3.2 that all points in $\phi_\zeta(\mathcal{V}_\zeta)$ are vertices of $\triangle_\zeta$. Clearly, $\triangle_\zeta$ includes all triangles $\phi_\zeta(\tau)$ for $\tau \in \mathcal{T}$ with vertices in $\mathcal{V}_\zeta$. In view of (3.3), it also includes images of the triangles having a pair of vertices in $\mathcal{V}_\zeta$. Moreover, $\tilde{\mathcal{V}}_\xi \subset \tilde{U}_\xi \subset U_\zeta$.

Now for any $\zeta \in U$ and each vertex $v \in \tilde{\mathcal{V}}_\xi$, set $v(\zeta) = (r_\zeta \circ \phi_\zeta)(v)$. The functions $v(\zeta)$ are continuously differentiable at $\zeta = \xi$. Let $\tilde{\triangle}_\zeta$ denote the triangulation obtained by applying r_ζ to $\triangle_\zeta$, and let $\tilde{\triangle}_{\zeta,TP}$ be the Powell-Sabin split of $\tilde{\triangle}_\zeta$. Since r_ζ is an orthogonal transformation, it is easy to see that $s_{\mathcal{T}}(\zeta)$ can be computed as the value at the origin of the Powell-Sabin spline $\tilde{s}_\zeta$ defined on $\tilde{\triangle}_{\zeta,TP}$ that interpolates the values $\{(a_v, g(\zeta))\}$ at the vertices $v(\zeta)$ for all $v \in \tilde{\mathcal{V}}_\xi$, where $g(\zeta) = \sigma_v J_{v\zeta}(v) J(r_\zeta^{-1})$. Since $J_{v\zeta}(v) J(r_\zeta^{-1}) = [J_v(r_\zeta \circ \phi_\zeta)(v)]^{-1}$ is continuously differentiable with respect to ζ at $\zeta = \xi$, and since by Theorem 2.1 the value of the Powell-Sabin interpolant s_ζ depends smoothly on the vertex locations and the data, we conclude that $s_{\mathcal{T}}$ is continuously differentiable at ξ. ∎

Suppose that $s_{\mathcal{T}}(f)$ is the interpolant corresponding to the data

$$a_v := f(v), \quad \sigma_v := \nabla_v f(v), \qquad \text{all } v \in \mathcal{V},$$

where f is a smooth function defined on Ω. We now show that $s_{\mathcal{T}}(f)$ approximates f to order $\mathcal{O}(h^3)$, where h is the mesh size of $\mathcal{T}$, i.e., the length of the longest edge in the triangles in the set $\{\triangle_\xi\}_{\xi\in\Omega}$. Let α be the *smallest angle* appearing in the triangles in this set.

PSerror **Theorem 4.3.** *Let $f \in W^3_\infty(\Omega)$. Then*

$$\|f - s_{\mathcal{T}}(f)\|_{C(\Omega)} \le K\, h^3 \|f\|_{W^3_\infty(\Omega)}, \tag{4.2}$$ ferror

where K is a constant depending only on α.

Proof. Fix $\xi \in \Omega$, and let s_ξ be the bivariate Powell-Sabin spline defined on the triangulation $\triangle_\xi$ that interpolates the values $\{(a_{v_i}, g_i)\}_{i=1}^3$ at the vertices $\{w_i\}_{i=1}^3$ of the triangle in $\triangle_\xi$ containing ξ as described in Algorithm 4.1. Then

$$f_\xi(w_i) = f(v_i) = a_{v_i} = s_\xi(w_i), \qquad i = 1, 2, 3,$$

and by (3.2)

$$\nabla f_\xi(w_i) = \nabla_\xi f(v_i) = \nabla_{v_i} f(v_i)\, J_{v_i\xi}(v_i) = \sigma_{v_i} J_{v_i\xi}(v_i) = \nabla s_\xi(w_i),$$

$i = 1, 2, 3$.

Thus, s_ξ interpolates the function values and gradients of f_ξ at w_1, w_2, w_3. It follows from well-known error bounds for bivariate Powell-Sabin interpolation [30] (see also [21]) that

$$|f_\xi(\xi) - s_\xi(\xi)| \le K_1 h^3 \|f_\xi\|_{W^3_\infty(B_\xi)},$$

where K_1 is a constant depending only on the smallest angle in $\triangle_\xi$. By the definition of the Sobolev norm on Ω, taking the maximium over $\xi \in \Omega$ gives (4.2). ■

5 A Two-Stage Data Fitting Method

scheme

In practice we are frequently given only values of an unknown function f at a set X of scattered data points on the manifold Ω. In this case we can use a two-stage method to construct an approximation. First we select a consistent triangulation $\mathcal{T}$ of Ω satisfying (3.3). Let $\mathcal{V}$ be the set of vertices of $\mathcal{T}$. Note that we do not require that the vertices be located at the data points of X, and the number of vertices may be much smaller than the number of data points.

In the *first stage* of the algorithm we compute approximations to the values $\{f(v), \nabla_v f(v)\}_{v \in \mathcal{V}}$ based on the data $\{f(\xi)\}_{\xi \in X}$. We perform these calculations in the sets $B_v \subset \mathbb{R}^2$ using techniques available for local fitting of bivariate data. To carry this out, we suppose that

$$X \text{ is sufficiently dense so that } X \cap U_v \neq \emptyset \text{ for each } v \in \mathcal{V}. \tag{5.1}$$

fine2

Experience with the bivariate case [10] suggests that for each $v \in \mathcal{V}$, we compute both $a_v \approx f(v)$ and $\sigma_v \approx \nabla_v f(v)$ by averaging several estimates of the same quantities based on different sets of nearby data. It follows from the consistency of $\mathcal{T}$ that for each vertex $v \in \mathcal{V}$, all vertices of $\mathcal{T}$ connected to v belong to the set U_v.

1st **Algorithm 5.1.** *Given* $\{f(\xi)\}_{\xi \in X}$, *compute* $\{a_v, \sigma_v\}_{v \in \mathcal{V}}$

1. *For each* $v \in \mathcal{V}$,
   (a) *Let* $v_0 := v$, *and let* $v_1, v_2, \ldots, v_n \in \mathcal{V}$ *be the set of vertices of* $\mathcal{T}$ *connected to* v. *Let* $\tilde{v}_i = \phi_v(v_i)$, $i = 1, \ldots, n$.
   (b) *Choose a set* $\tilde{X}_v \subset \phi_v(X \cap U_v)$ *of points in* B_v *near* $\phi_v(v)$.
   (c) *Compute a bivariate approximation* p_v *defined on* B_v *based on the data* $\{f_v(\tilde{\xi})\}_{\tilde{\xi} \in \tilde{X}_v}$, *where* $f_v := f \circ \phi_v^{-1}$.
   (d) *Store the numbers* $a_{v,v_i} := p_v(\tilde{v}_i)$ *and vectors* $\sigma_{v,v_i} := \nabla p_v(\tilde{v}_i)\, J_{vv_i}(v_i)$ *for* $i = 0, \ldots, n$.

2. *For each* $v \in \mathcal{V}$, *set*

$$a_v := \frac{1}{n+1} \sum_{i=0}^{n} a_{v_i,v}, \quad \sigma_v := \frac{1}{n+1} \sum_{i=0}^{n} \sigma_{v_i,v}.$$

In the *second stage* of the algorithm we construct our approximant $s_{\mathcal{T}}$ as the interpolant (4.1) to the data $\{a_v, \sigma_v\}_{v \in \mathcal{V}}$ obtained from Algorithm 5.1.

We have not specified how $\mathcal{T}$ is selected and how the steps 1(b) and 1(c) of Algorithm 5.1 are to be performed. However, the overall performance of the two-stage method will depend significantly on the particular techniques used in these steps. Numerical examples in our paper [9] make use of recently developed adaptive techniques based on local least squares fitting by bivariate polynomials and radial basis functions [7, 8, 10].

6 An Error Bound for the Two-Stage Method

error

Suppose that f is a smooth function defined on Ω, and that $s_{\mathcal{T}} = s_{\mathcal{T}}(f)$ is the approximant of f constructed in the previous section based on measurements $\{f(\xi)\}_{\xi \in X}$ of f at some scattered set X of data points on Ω. Let h be the mesh size of the triangulation $\mathcal{T}$. In this section we show that $s_{\mathcal{T}}$ approximates f to order h^3 as $h \to 0$ provided that for each vertex v of $\mathcal{T}$, the local approximation p_v of $f_v := f \circ \phi_v^{-1}$ approximates the function value $f_v(v)$ to order at least h^3, and the first derivatives of f_v at v to order h^2.

Given $\xi \in \Omega$, let $\triangle_\xi$ be the associated planar triangulation in B_ξ, as in Section 3.4. Let $\kappa(\xi)$ be the maximum of $\max\{\|J_{\xi v}(v)\|_2, \|J_{v\xi}(v)\|_2\}$ over all $v \in \mathcal{V} \cap U_\xi$ such that $\phi_\xi(\xi)$ belongs to the closure of a triangle of $\triangle_\xi$ attached to $\phi_\xi(v)$. We assume that

$$\kappa := \sup_{\xi \in \Omega} \kappa(\xi) < \infty. \tag{6.1}$$ kappa

For each $v \in \mathcal{V}$, let N_v be the union of all triangles of $\triangle_v$ attached to v, and let p_v be the bivariate approximation to f_v, as in Algorithm 5.1.

b1 **Theorem 6.1.** *Let $f \in W_\infty^3(\Omega)$. Then*

$$\begin{aligned} \|f - s_{\mathcal{T}}\|_{C(\Omega)} \quad &\leq \quad K \Big[h^3 \|f\|_{W_\infty^3(\Omega)} \\ &+ \quad \max_{v \in \mathcal{V}} \big\{ \|f_v - p_v\|_{C(N_v)} + h \|\nabla f_v - \nabla p_v\|_{C(N_v)} \big\} \Big], \end{aligned}$$

where K is a constant depending only on κ and the smallest angle α.

Proof. Let $\xi \in \Omega$. We have $|f(\xi) - s_{\mathcal{T}}(\xi)| = |f_\xi(\xi) - s_\xi(\xi)|$, where $f_\xi := f \circ \phi_\xi^{-1}$ and s_ξ is the bivariate Powell-Sabin interpolating spline on the planar triangulation $\triangle_\xi$, computed using the values a_{v_i} and vectors σ_{v_i} corresponding to the vertices $\{w_i := \phi_\xi(v_i)\}_{i=1}^3$, of the triangle T in B_ξ that contains $\phi_\xi(\xi)$. Recall that a_{v_i}, σ_{v_i} are computed in the first stage using Algorithm 5.1. By Algorithm 4.1,

$$s_\xi(w_i) = a_{v_i}, \quad \nabla s_\xi(w_i) = \sigma_{v_i} J_{v_i \xi}(v_i), \quad i = 1, 2, 3.$$

Let $\hat{s}$ be the bivariate Powell-Sabin spline on the triangulation $\triangle_\xi$ interpolating the exact values and gradients of f_ξ, i.e.,

$$\hat{s}(w_i) = \hat{a}_{v_i} = f_\xi(w_i) = f(v_i), \quad \nabla \hat{s}(w_i) = \hat{\sigma}_{v_i} = \nabla f_\xi(w_i) = \nabla_\xi f(v_i),$$

for $i = 1, 2, 3$. Then by Theorem 4.3,

$$|f_\xi(\xi) - \hat{s}(\xi)| \le K_1 h^3 \|f\|_{W^3_\infty(\Omega)},$$

where K_1 is a constant depending only on α. Now by a standard argument involving the cardinal functions of Powell-Sabin interpolation,

$$|\hat{s}(\xi) - s_\xi(\xi)| \le K_2 \max_{i=1,2,3} \Big\{ |\hat{a}_{v_i} - a_{v_i}| + h\|\hat{\sigma}_{v_i} - \sigma_{v_i} J_{v_i\xi}(v_i)\|_2 \Big\},$$

with a constant K_2 depending only on α. For each $i = 1, 2, 3$,

$$\hat{a}_{v_i} - a_{v_i} = f(v_i) - \frac{1}{n+1}\sum_{j=0}^{n} a_{u_j,v_i} = \frac{1}{n+1}\sum_{j=0}^{n} \big[f_{u_j}(w_{ij}) - p_{u_j}(w_{ij}) \big],$$

where $u_0 = v_i$, the $u_1, \ldots, u_n$ are the vertices of $\mathcal{T}$ connected to v_i, and $w_{ij} = \phi_{u_j}(v_i)$. Since $w_{ij} \in N_{u_j}$, it follows that

$$|f_{u_j}(w_{ij}) - p_{u_j}(w_{ij})| \le \|f_{u_j} - p_{u_j}\|_{C(N_{u_j})},$$

and hence

$$|\hat{a}_{v_i} - a_{v_i}| \le \max_{v\in\mathcal{V}} \|f_v - p_v\|_{C(N_v)}, \qquad i = 1, 2, 3.$$

Similarly, by (3.2) and the definition of σ_{v_i},

$$\begin{aligned}
\hat{\sigma}_{v_i} - \sigma_{v_i} J_{v_i\xi}(v_i) &= \nabla_\xi f(v_i) - \frac{1}{n+1}\sum_{j=0}^{n} \sigma_{u_j,v_i} J_{v_i\xi}(v_i) \\
&= \frac{1}{n+1}\sum_{j=0}^{n} \Big(\nabla_{v_i} f(v_i) - \sigma_{u_j,v_i} \Big) J_{v_i\xi}(v_i).
\end{aligned}$$

Since $\sigma_{u_j,v_i} = \nabla p_{u_j}(w_{ij})\, J_{u_j v_i}(v_i)$ and $\nabla_{u_j} f(v_i) = \nabla f_{u_j}(w_{ij})$, (3.2) implies

$$\begin{aligned}
\nabla_{v_i} f(v_i) - \sigma_{u_j,v_i} &= \Big(\nabla_{u_j} f(v_i) - \nabla p_{u_j}(w_{ij}) \Big) J_{u_j v_i}(v_i) \\
&= \Big(\nabla f_{u_j}(w_{ij}) - \nabla p_{u_j}(w_{ij}) \Big) J_{u_j v_i}(v_i).
\end{aligned}$$

Hence

$$\begin{aligned}
\|\nabla_{v_i} f(v_i) - \sigma_{u_j,v_i}\|_2 &\le \|\nabla f_{u_j}(w_{ij}) - \nabla p_{u_j}(w_{ij})\|_2 \, \|J_{u_j v_i}(v_i)\|_2 \\
&\le \kappa \|\nabla f_{u_j} - \nabla p_{u_j}\|_{C(N_{u_j})},
\end{aligned}$$

and

$$\|\hat{\sigma}_{v_i} - \sigma_{v_i} J_{v_i\xi}(v_i)\|_2 \le \kappa^2 \max_{v\in\mathcal{V}} \|\nabla f_v - \nabla p_v\|_{C(N_v)}.$$

Combining the above inequalities, we get the desired estimate. ■

7 C^2-Manifolds Embedded in $\mathbb{R}^3$

R3

In this section we examine the case when Ω is an arbitrary compact 2-dimensional C^2-manifold embedded in $\mathbb{R}^3$. Our main task is to show how to construct an atlas for Ω that satisfies the smoothness assumptions of Section 3.1. More details on our method for scattered data fitting on surfaces, including extensive numerical tests, can be found in [9].

Throughout this section, we write $\langle \cdot, \cdot \rangle$ for the usual inner product in $\mathbb{R}^3$, and $\|a\|_2$ for the Euclidean norm of any 3-vector a.

7.1 Projection atlas

projections

Since Ω is embedded in $\mathbb{R}^3$, it can be represented locally as a regular level surface of a C^2 function of three variables. More precisely, each point $\xi \in \Omega$ has a neighborhood $\mathcal{G}_\xi$ in $\mathbb{R}^3$ such that $\mathcal{G}_\xi \cap \Omega = F_\xi^{-1}(0)$, where $F_\xi : \mathcal{G}_\xi \to \mathbb{R}$ is a C^2 function with nonzero gradient ∇F_ξ everywhere in $\mathcal{G}_\xi \cap \Omega$, see [19]. Then $n_\xi := \nabla F_\xi(\xi)/\|\nabla F_\xi(\xi)\|_2$ is a *normal vector* to Ω at ξ. Moreover, the *tangent plane* Γ_ξ is the unique plane in $\mathbb{R}^3$ that contains ξ and is orthogonal to n_ξ. Clearly, for all $\zeta \in \mathcal{G}_\xi \cap \Omega$, a normal vector to Ω at ζ can also be computed as $\nabla F_\xi(\zeta)/\|\nabla F_\xi(\zeta)\|_2$. It coincides with either n_ζ or $-n_\zeta$. Clearly, $\langle n_\xi, \nabla F_\xi(\zeta)\rangle > 0$ for all $\zeta \in \mathcal{G}_\xi \cap \Omega$.

We are now ready to define an atlas associated with Ω. For each $\xi \in \Omega$, let U_ξ be the connected component of the open set $\{\zeta \in \Omega : \langle n_\xi, n_\zeta\rangle \neq 0\}$ that contains ξ. Then U_ξ is an open neighborhood of ξ. Clearly, the *orthogonal projection* $\pi_\xi : U_\xi \to \Gamma_\xi$ defined by

$$\pi_\xi(\zeta) = \zeta + \langle \xi - \zeta, n_\xi\rangle\, n_\xi, \qquad \zeta \in U_\xi,$$

is invertible. Assuming that $\gamma_\xi^{[1]}, \gamma_\xi^{[2]}$ are orthogonal unit vectors in Γ_ξ such that $\gamma_\xi^{[1]} \times \gamma_\xi^{[2]} = n_\xi$, we can also write

$$\pi_\xi(\zeta) = \xi + \langle \zeta - \xi, \gamma_\xi^{[1]}\rangle\, \gamma_\xi^{[1]} + \langle \zeta - \xi, \gamma_\xi^{[2]}\rangle\, \gamma_\xi^{[2]}.$$

Define ϕ_ξ by the formula

$$\phi_\xi(\zeta) := [\langle \zeta - \xi, \gamma_\xi^{[1]}\rangle, \langle \zeta - \xi, \gamma_\xi^{[2]}\rangle]^T, \qquad \zeta \in U_\xi.$$

We call $\Phi = \{U_\xi, \phi_\xi\}_{\xi\in\Omega}$ the *projection atlas* associated with Ω. The remainder of this subsection is devoted to a proof that Φ satisfies the hypotheses of Section 3.1.

proadm **Theorem 7.1.** *The projection atlas Φ is an admissible atlas for Ω.*

Proof. By the choice of U_ξ, ϕ_ξ is invertible. Consider the coordinate system for $\mathbb{R}^3$ with coordinate vectors $\gamma_\xi^{[1]}, \gamma_\xi^{[2]}, n_\xi$ and origin ξ. For any $\mu \in U_\xi$, the equation $F_\mu = 0$ determines an implicit function $x^{[3]} = \delta_\mu(x^{[1]}, x^{[2]})$ in a neighborhood of $\phi_\xi(\mu)$, such that

$$\phi_\xi^{-1}(x^{[1]}, x^{[2]}) = \xi + x^{[1]}\gamma_\xi^{[1]} + x^{[2]}\gamma_\xi^{[2]} + \delta_\mu(x^{[1]}, x^{[2]})\, n_\xi.$$

Since $\langle n_\xi, \nabla F_\mu(\mu)\rangle = \langle n_\xi, n_\mu\rangle \|\nabla F_\mu(\mu)\|_2 \neq 0$, the implicit function theorem implies that $\delta_\mu(x^{[1]}, x^{[2]})$ is a C^2 function in a neighborhood of $\phi_\xi(\mu)$. Assuming $\mu \in U_\xi \cap U_\zeta$, we also have

$$\phi_{\zeta\xi}(x^{[1]}, x^{[2]}) = (\phi_\zeta \circ \phi_\xi^{-1})(x^{[1]}, x^{[2]}) = [\phi_{\zeta\xi}^{[1]}(x^{[1]}, x^{[2]}), \phi_{\zeta\xi}^{[2]}(x^{[1]}, x^{[2]})]^T,$$

where for $i = 1, 2$,

$$\phi_{\zeta\xi}^{[i]}(x^{[1]}, x^{[2]}) = \langle \xi - \zeta + x^{[1]}\gamma_\xi^{[1]} + x^{[2]}\gamma_\xi^{[2]} + \delta_\mu(x^{[1]}, x^{[2]})\, n_\xi, \gamma_\zeta^{[i]}\rangle \tag{7.1}$$ phizx

in a neighborhood of $\phi_\xi(\mu)$. Therefore $\phi_{\zeta\xi} : \phi_\xi(U_\xi \cap U_\zeta) \to \phi_\zeta(U_\xi \cap U_\zeta)$ is a C^2 mapping.

For later use, we now obtain explicit formulas for the Jacobian $J_{\zeta\xi}(\xi) := J(\phi_{\zeta\xi})(\phi_\xi(\xi))$ as defined in Section 3.1, and its determinant. By the above construction, the implicit function $x^{[3]} = \delta_\xi(x^{[1]}, x^{[2]})$ is C^2 in a neighborhood of the origin $\phi_\xi(\xi)$. Moreover, it vanishes together with its gradient at the origin. Hence, by (7.1),

$$J_{\zeta\xi}(\xi) = [\langle \gamma_\zeta^{[i]}, \gamma_\xi^{[j]}\rangle]_{i,j=1,2}. \tag{7.2}$$ Jzx

Clearly, the determinant of this matrix is the projection of $n_\zeta = \gamma_\zeta^{[1]} \times \gamma_\zeta^{[2]}$ on n_ξ, i.e.,

$$\det J_{\zeta\xi}(\xi) = \langle n_\zeta, n_\xi\rangle. \tag{7.3}$$ detJzx

It remains to show that the charts (U_ξ, ϕ_ξ) depend smoothly on ξ in the sense of Definition 3.1. To this end, we fix $\xi \in \Omega$ and choose $\tilde U_\xi \subset U_\xi$ to be an open neighborhood of ξ such that the closure of $\tilde U_\xi$ is a compact set contained in U_ξ. Then $\inf_{\mu \in \tilde U_\xi} |\langle n_\xi, n_\mu\rangle| > 0$ and hence $\langle n_\zeta, n_\mu\rangle \neq 0$ for all $\mu \in \tilde U_\xi$ and all ζ in some neighborhood of ξ. This implies that $\tilde U_\xi \subset U_\zeta$ for all ζ sufficiently close to ξ.

For any $\zeta \in U_\xi$ we define a coordinate system in Γ_ζ with origin ζ and orthonormal coordinate vectors $\tilde{\gamma}_\zeta^{[1]}, \tilde{\gamma}_\zeta^{[2]}$, where

$$\tilde{\gamma}_\zeta^{[1]} = \hat{\gamma}_\zeta^{[1]}/\|\hat{\gamma}_\zeta^{[1]}\|_2, \quad \hat{\gamma}_\zeta^{[1]} = \gamma_\xi^{[1]} - \langle \gamma_\xi^{[1]}, \nabla F_\xi(\zeta)\rangle \nabla F_\xi(\zeta),$$

$$\tilde{\gamma}_\zeta^{[2]} = \tilde{n}_\zeta \times \tilde{\gamma}_\zeta^{[1]}, \quad \tilde{n}_\zeta = \nabla F_\xi(\zeta)/\|\nabla F_\xi(\zeta)\|_2.$$

Set

$$\tilde{\phi}_\zeta(\mu) := [\langle \mu - \zeta, \tilde{\gamma}_\zeta^{[1]}\rangle, \langle \mu - \zeta, \tilde{\gamma}_\zeta^{[2]}\rangle]^T, \qquad \mu \in \tilde{U}_\xi \subset U_\zeta.$$

Since the coordinate system $\gamma_\zeta^{[1]}, \gamma_\zeta^{[2]}$ can be obtained from $\tilde{\gamma}_\zeta^{[1]}, \tilde{\gamma}_\zeta^{[2]}$ by an orthogonal linear transformation $r_\zeta : \mathbb{R}^2 \to \mathbb{R}^2$,

$$\tilde{\phi}_\zeta = r_\zeta \circ \phi_\zeta.$$

Since F_ξ is a C^2 function and $\nabla F_\xi(\zeta) \neq 0$ for all $\zeta \in U_\xi$, it follows that $\tilde{\phi}_\zeta(\mu)$, as a function of ζ, is continuously differentiable at $\zeta = \xi$.

Finally, for a fixed $\mu \in \tilde{U}_\xi$ consider $M(\zeta) := J_\mu(r_\zeta \circ \phi_\zeta)(\mu) = J(r_\zeta) J_{\zeta\mu}(\mu)$. In view of (7.2),

$$M(\zeta) = J(r_\zeta)[\langle \gamma_\zeta^{[i]}, \gamma_\mu^{[j]}\rangle]_{i,j=1,2} = [\langle \tilde{\gamma}_\zeta^{[i]}, \gamma_\mu^{[j]}\rangle]_{i,j=1,2},$$

which is continuously differentiable at $\zeta = \xi$, as required. ■

7.2 Projected gradients

pgrad

Let $f \in C^1(\Omega)$, and let $f_\xi = f \circ \phi_\xi^{-1}$. Since Ω is embedded in $\mathbb{R}^3$, the gradient $\nabla f_\xi = \left[\frac{\partial f_\xi}{\partial x^{[1]}}, \frac{\partial f_\xi}{\partial x^{[2]}}\right]$ of f_ξ can be identified with the 3-vector

$$\operatorname{grad} f_\xi = \frac{\partial f_\xi}{\partial x^{[1]}} \gamma_\xi^{[1]} + \frac{\partial f_\xi}{\partial x^{[2]}} \gamma_\xi^{[2]}$$

lying in the tangent plane $\Gamma_\xi \subset \mathbb{R}^3$. Adopting a notation similar to that in Section 3.2, we write

$$\operatorname{grad}_\xi f(\mu) := (\operatorname{grad} f_\xi)(\phi_\xi(\mu)), \qquad \mu \in U_\xi,$$

for the gradient of f_ξ evaluated at $\phi_\xi(\mu)$. We call $\operatorname{grad}_\xi f(\mu)$ the *projected gradient* of f at μ. It is easy to see that $\operatorname{grad}_\xi f(\xi)$ coincides with the standard gradient of a function on a 2-surface in $\mathbb{R}^3$, as defined for example in [33, p. 96]. We also need projected gradients when $\mu \neq \xi$.

pgradtrans **Lemma 7.2.** *For any $\xi \in \Omega$ and $\zeta \in U_\xi$, the projected gradient* $\mathrm{grad}_\zeta f(\zeta)$ *is the orthogonal projection of* $\mathrm{grad}_\xi f(\zeta)$ *onto Γ_ζ. In particular,*

$$\mathrm{grad}_\zeta f(\zeta) = \mathrm{grad}_\xi f(\zeta) - \langle \mathrm{grad}_\xi f(\zeta), n_\zeta \rangle \, n_\zeta, \tag{7.4}$$ pro1

and

$$\mathrm{grad}_\xi f(\zeta) = \mathrm{grad}_\zeta f(\zeta) - \frac{\langle \mathrm{grad}_\zeta f(\zeta), n_\xi \rangle}{\langle n_\zeta, n_\xi \rangle} \, n_\zeta, \qquad \text{if } \langle n_\zeta, n_\xi \rangle \neq 0, \tag{7.5}$$ pro2

where n_ζ and n_ξ are the unit normal vectors to Γ_ζ and Γ_ξ, respectively.

Proof. We have

$$\mathrm{grad}_\xi f(\zeta) = \frac{\partial f_\xi}{\partial x^{[1]}} \Big(\phi_\xi(\zeta)\Big) \gamma_\xi^{[1]} + \frac{\partial f_\xi}{\partial x^{[2]}} \Big(\phi_\xi(\zeta)\Big) \gamma_\xi^{[2]}.$$

Its projection onto Γ_ζ is therefore

$$\begin{aligned} &\left(\frac{\partial f_\xi}{\partial x^{[1]}} \Big(\phi_\xi(\zeta)\Big) \langle \gamma_\xi^{[1]}, \gamma_\zeta^{[1]} \rangle + \frac{\partial f_\xi}{\partial x^{[2]}} \Big(\phi_\xi(\zeta)\Big) \langle \gamma_\xi^{[2]}, \gamma_\zeta^{[1]} \rangle \right) \gamma_\zeta^{[1]} \\ &+ \left(\frac{\partial f_\xi}{\partial x^{[1]}} \Big(\phi_\xi(\zeta)\Big) \langle \gamma_\xi^{[1]}, \gamma_\zeta^{[2]} \rangle + \frac{\partial f_\xi}{\partial x^{[2]}} \Big(\phi_\xi(\zeta)\Big) \langle \gamma_\xi^{[2]}, \gamma_\zeta^{[2]} \rangle \right) \gamma_\zeta^{[2]}. \end{aligned}$$

This last expression coincides with $\mathrm{grad}_\zeta f(\zeta)$, since

$$\nabla_\zeta f(\zeta) = \nabla_\xi f(\zeta) J_{\xi\zeta}(\zeta) = \nabla_\xi f(\zeta) \, [\langle \gamma_\xi^{[i]}, \gamma_\zeta^{[j]} \rangle]_{i,j=1,2}$$

by (3.2) and (7.2).

The formulas (7.4) and (7.5) for the projection and inverse projection, respectively, follow immediately. ■

7.3 Consistent triangulations

triR3

As mentioned in Section 3.4, every compact 2-dimensional smooth manifold Ω admits a triangulation $\mathcal{T}$. Let Ω be embedded into $\mathbb{R}^3$, and let Φ be the projection atlas for it. For any $\epsilon > 0$ there is a triangulation $\mathcal{T}$ of Ω consistent with Φ and with the mesh size $h < \epsilon$. Clearly, any triangulation will be consistent with Φ if it is sufficiently fine in the sense that for each $\xi \in \Omega$ there is a triangle $\tau \in \mathcal{T}$ with vertices in $\tilde{U}_\xi$ such that $\phi_\xi(\xi) \in \overline{\phi_\xi(\tau)}$, and the maximum distance in $\mathbb{R}^3$ between connected vertices of $\mathcal{T}$ is sufficiently small. See [2] for a construction of suitable triangulations using sufficiently dense samples of points on 2-dimensional manifolds embedded in $\mathbb{R}^3$.

Recall that $\tilde{U}_\xi$ of Section 7.1 may be any open set whose closure is a compact subset of U_ξ. Therefore Definition 3.2 can be simplified when Φ is the projection atlas by removing any reference to $\tilde{U}_\xi$, and requiring that $\triangle_\xi \subset \{\phi_\xi(\tau) : \tau \subset U_\xi\}$ instead of $\triangle_\xi \subset \{\phi_\xi(\tau) : \tau \subset \tilde{U}_\xi\}$, see [9].

interp-fit

7.4 Interpolation and data fitting

Using projected gradients, we can reformulate the interpolation problem

$$s_\mathcal{T}(v) = a_v, \quad \nabla_v s_\mathcal{T}(v) = \sigma_v, \qquad \text{all } v \in \mathcal{V}, \tag{7.6}$$ interp1

of (4.1) as

$$s_\mathcal{T}(v) = a_v, \quad \operatorname{grad}_v s_\mathcal{T}(v) = c_v, \qquad \text{all } v \in \mathcal{V}, \tag{7.7}$$ interpR3

where $c_v = \sigma_v^{[1]}\gamma_\xi^{[1]} + \sigma_v^{[2]}\gamma_\xi^{[2]}$ is a vector in Γ_v.

An advantage of the formulation (7.7) over (7.6) is that each c_v is determined by just three real numbers (the Cartesian coordinates of c_v), whereas σ_v requires two real numbers and the tangent vectors $\gamma_\xi^{[1]}, \gamma_\xi^{[2]}$. Thus, when defining $s_\mathcal{T}$ by (7.7), we do not need any coordinate systems in the tangent planes Γ_v.

Clearly, Algorithms 4.1 and 5.1 can now be formulated without reference to any coordinate systems in the tangent planes provided we use projected gradients. Assume the data is given as $\{a_v, c_v\}_{v\in\mathcal{V}}$, where a_v are real numbers and c_v are 3-vectors in Γ_v. Then to construct a C^1 function $s_\mathcal{T}$ defined on Ω that satisfies the interpolation conditions (7.7), we simply apply Algorithm 4.1, where we replace the formula for g_i in step 3 by $g_i := c_{v_i} - \frac{\langle c_{v_i}, n_\xi\rangle}{\langle n_{v_i}, n_\xi\rangle} n_{v_i}$, and require in step 4 that s_ξ interpolates the values $\{a_{v_i}\}_{i=1}^3$ and the gradients corresponding to the tangent vectors $\{g_i\}_{i=1}^3$ at the vertices $\{w_i\}_{i=1}^3$. Algorithm 5.1 describing the first stage of the two-stage data fitting method can also be reformulated by replacing the vectors σ_{v,v_i} in step 1d by $c_{v,v_i} := \operatorname{grad} p_v(\tilde{v}_i) - \langle \operatorname{grad} p_v(\tilde{v}_i), n_{v_i}\rangle n_{v_i}$ for $i = 0, \ldots, n$, and by taking the average of $c_{v_i,v}$'s instead of $\sigma_{v_i,v}$'s in step 2. Precise formulations can be found in [9].

Theorems 4.3 and 6.1 give error bounds for our interpolation and scattered data fitting methods, respectively, in terms of the mesh size h of $\mathcal{T}$, i.e., the length of the longest edge in the triangles in the set $\{\triangle_\xi\}_{\xi\in\Omega}$, the smallest angle α appearing in the triangles in this set, and the parameter κ defined in (6.1). In the case of surfaces embedded in $\mathbb{R}^3$, there are more convenient parameters to play the role of h, α, κ. Let us define the *mesh size* $\tilde{h}$ as the maximum distance in $\mathbb{R}^3$ between any pair of vertices $v \in \mathcal{V}$ connected in $\mathcal{T}$. By actually connecting these pairs of vertices by straight line segments, we obtain a 2-dimensional triangulation in $\mathbb{R}^3$. Let $\tilde{\alpha}$ be the *smallest angle* appearing in its triangles. Let, furthermore, $\tilde{\kappa}(\xi)$ be the maximum of $\langle n_\xi, n_v\rangle^{-1}$ over all $v \in \mathcal{V} \cap U_\xi$ such that $\xi = \phi_\xi(\xi)$ belongs to the closure of a triangle of $\triangle_\xi$ attached to $\phi_\xi(v)$, and let $\tilde{\kappa} = \max_{\xi\in\Omega} \tilde{\kappa}(\xi)$. Assuming Φ is the projection atlas, it is not difficult to see that (a) $h \le \tilde{h}$, (b) $\tilde{\alpha} > 0$ implies $\alpha > 0$ if $\tilde{h}$ is sufficiently small, and (c) $\kappa < \infty$ if and only if $\tilde{\kappa} < \infty$. (Note that (c) follows from (3.1), (7.2) and (7.3).) Thus, in the case of the projection atlas, Theorems 4.3 and 6.1 can be reformulated with $\tilde{h}, \tilde{\alpha}, \tilde{\kappa}$ in place of h, α, κ, see [9].

8 Remarks

hist **Remark 8.1.** The problem of fitting functions defined on surfaces arises in many applications, see for example [1, 3, 4, 5, 11, 12, 14, 15, 16, 22, 23, 28, 29, 32, 34], and references therein. Used parametrically, such functions can be applied to the problem of modelling surfaces of arbitrary topological type from point clouds, see for example [17, 18, 35].

sph **Remark 8.2.** Many of the papers mentioned in the above remark deal with the sphere in $\mathbb{R}^3$. For a survey of interpolation and scattered data fitting methods on the sphere, see [14]. For some specific methods, see [16, 22, 23, 28, 32, 34].

Dem **Remark 8.3.** The method of this paper is closely related to work of Demjanovich [11, 12]. He also computes an interpolant s at a point ξ on the manifold by using local charts (U_ξ, ϕ_ξ) and finite element interpolation in $\phi_\xi(U_\xi)$. A key difference is that for each evaluation point ξ, his method involves interpolation of the original function values and derivatives assigned to certain points in $\phi_\xi(U_\xi)$ determined by the finite element scheme, whereas in our methods we only interpolate projected gradients corresponding to the vertices of the underlying triangulation $\mathcal{T}$, compare steps 3 and 4 of Algorithm 4.1. Therefore, our interpolation operator only requires function values and gradients at the vertices of $\mathcal{T}$, which makes it possible to design a two-stage scattered data fitting method. Only one of the methods in [11, 12] (based on interpolation with Courant hat functions) has similar properties for general manifolds, but it does not produce a C^1 interpolant.

Pot **Remark 8.4.** The special case of our method for surfaces in $\mathbb{R}^3$ (Section 7) is also closely related to work of Pottmann [29], which also makes use of projected gradients. (It is not difficult to see that equation (7.5) describes the π-*transform* of [29].) However, instead of using local approximation methods to estimate gradients, he constructs a kind of minimum norm network.

CT **Remark 8.5.** Here we have made use of the standard bivariate C^1 quadratic Powell-Sabin macro-element to solve the interpolation problem in the tangent plane. Its key feature is that it is constructed from only nine pieces of data, the values and gradients at the three vertices of the macro-triangle. Using the same data, we can also construct an interpolant based on the classical C^1 *reduced Clough-Tocher macro-element*. It is based on a split of the macro-triangle into three subtriangles (typically using the barycenter), and is a cubic polynomial on each piece. Along each edge its cross derivative is restricted to be a linear polynomial. Yet another possibility is a modified quadratic *Powell-Sabin macro-element on a 12-split* [30], where the cross derivatives are assumed linear rather than piecewise linear on the edges of the macro-triangles. Note that with either the Clough-Tocher or Powell-Sabin-12 macro-element, the assumption (3.3) will not be needed.

hairy_ball **Remark 8.6.** It follows from the Poincaré-Hopf index theorem [26] that any continuous tangent vector field on a compact differentiable manifold without boundary vanishes at a point unless the Euler number of the manifold is zero. Recall that the Euler number of an oriented surface of genus g is $2-2g$. Thus, the Euler number is zero for the torus, but is nonzero for surfaces of other topological types, including the sphere. For the sphere, this is just the 'hairy ball' theorem which states that there is no nonvanishing continuous tangent vector field on the sphere, and explains why we need to use local rotations r_ζ in Definition 3.1. Indeed, if the charts (U_ξ, ϕ_ξ) for Ω can be found such that Definition 3.1 holds with r_ζ being the identity in all cases, then one can easily construct a smooth tangent vector field on Ω by taking unit vectors corresponding to partial derivatives of all local parametrizations. Then the Poincaré-Hopf theorem implies that the Euler number of Ω is zero, which is a severe restriction on the topology of the manifold.

Whitney **Remark 8.7.** By the Whitney immersion theorem, any 2-dimensional manifold can be immersed in $\mathbb{R}^3$. Clearly, the projection atlas can be used on these immersions, where the correct tangent planes have to be chosen for the points of self-intersection. For any 2-dimensional C^2-submanifold of $\mathbb{R}^n$, $n > 3$, arguing as in Section 7.1, we can use local orthogonal projections on tangent planes to define an admissible atlas in the sense of Definition 3.1. This construction is also applicable to arbitrary 2-dimensional C^2-manifolds in view of the Whitney embedding theorem, which says that any smooth 2-dimensional manifold can be smoothly embedded into $\mathbb{R}^4$.

boundary **Remark 8.8.** For the sake of simplicity, in this paper we consider only compact 2-dimensional manifolds without boundary. Clearly, our method is local, and therefore can be used on non-compact manifolds that have a countable basis for their topology. Indeed, by a theorem of Radó, such manifolds can be triangulated such that every point has a neighborhood that meets only finitely many triangles [24]. The method is also applicable to manifolds with boundary. However, in this case we have to assume that all points on the boundary of Ω are edge points with respect to the triangulation $\mathcal{T}$, see Section 3.4. Definition 3.2 needs obvious adjustments for the points ξ on the boundary, requiring that $\phi_\xi(\xi)$ is on the boundary of P_ξ instead of its interior.

Rn **Remark 8.9.** To extend our method to higher dimensional manifolds, one would need a construction of local C^1 interpolants in n variables, with $n > 2$, completely determined by the function and gradient values at vertices, and depending smoothly on this data and the vertex locations. Similarly, to obtain C^2 or higher smoothness C^r with our scheme, a local bivariate C^r interpolant determined by the function and gradient values at vertices is needed, whereas all known macro-elements of higher smoothness [21] require higher order derivative values.

sphe **Remark 8.10.** If Ω is the 2-dimensional sphere $\mathbb{S}^2$, then an admissible atlas $\{(U_\xi, \phi_\xi)\}_{\xi \in \mathbb{S}^2}$ can be defined using either central or stereographic projections onto the tangent planes rather than the orthogonal projections as in Section 7.1. The central projection has the property that the edge points with respect to any consistent triangulation are segments of great circles on $\mathbb{S}^2$. The advantage of the stereographic projection is that U_ξ can be chosen to be $\mathbb{S}^2 \setminus \{-\xi\}$ provided $-\xi$ is the center of the stereographic projection that defines ϕ_ξ. Therefore very coarse triangulations, for example the one defined by a tetrahedron inscribed in the sphere, are consistent with the atlas.

toru **Remark 8.11.** In the case when Ω is the torus $\mathbb{T}^2$, the following simple atlas is admissible. The torus with inner radius $R - r$ and outer radius $R + r$ is defined parametrically by the equations

$$\begin{aligned} x &= (R + r\cos v)\cos u, \\ y &= (R + r\cos v)\sin u, \\ z &= r\sin v, \end{aligned}$$

with $u, v \in [0, 2\pi)$. For each ξ on the torus, let u_ξ, v_ξ be its parameter values. We can reparametrize the torus as

$$\begin{aligned} x &= (R + r\cos(v_\xi + v))\cos(u_\xi + u), \\ y &= (R + r\cos(v_\xi + v)))\sin(u_\xi + u), \\ z &= r\sin(v_\xi + v)), \end{aligned}$$

with $u, v \in [-\pi, \pi)$, and define the chart (U_ξ, ϕ_ξ) by setting $\phi_\xi(\zeta) = (u, v) \in \mathbb{R}^2$, where u, v are the parameter values of ζ, and letting U_ξ be the set of all points ζ with $\phi_\xi(\zeta) \in (-\pi, \pi)^2$. This family of charts depends smoothly on ξ in the sense of Definition 3.1, where no local transformations r_ζ are needed. Moreover, the transition mappings $\phi_\zeta \circ \phi_\xi^{-1}$ are the shifts $(u, v) \mapsto (u + u_\xi - u_\zeta, v + v_\xi - v_\zeta)$. Hence, all Jacobians $J_{\zeta\xi}$ are unit matrices, which makes the transformations in Step 3 of Algorithm 4.1 and in Step 1(d) of Algorithm 5.1 trivial. Note that using this atlas with our method is equivalent to interpreting the data on $\mathbb{T}^2$ as periodic data on $\mathbb{R}^2$, constructing a periodic triangulation, and interpolating or approximating the data by the ordinary Powell-Sabin spline.

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