Multi-sided Macro-Element Spaces Based on Clough-Tocher Triangle Splits with Applications to Hole Filling

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Abstract. C^r macro-element spaces are constructed on polygonal domains with an arbitrary number of sides. The spaces consist of polynomial supersplines defined on triangulations which have been partially refined with Clough-Tocher splits. In addition to giving dimension formulae, minimal determining sets, and nodal bases, we derive error bounds for the corresponding Hermite interpolation operators. A number of examples are presented to show how the spaces can be used to fill n-sided holes.

§1. Introduction

Before describing the problem of interest in this paper, we need to recall some standard notation. Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain which may contain holes. We do not require that Ω be connected. Suppose $\triangle$ is a triangulation of Ω. Given integers $0 \le r \le d$, let

$$\mathcal{S}_d^r(\triangle) := \{s \in C^r(\Omega) : s|_T \in \mathcal{P}_d, \text{ for all } T \in \triangle\}$$

be the classical space of polynomial splines of degree d and smoothness r, where $\mathcal{P}_d$ is the space of bivariate polynomials of degree at most d. Given $r \le \rho \le d$, we write

$$\mathcal{S}_d^{r,\rho}(\triangle) := \{s \in \mathcal{S}_d^r(\triangle) : s \in C^\rho(v) \text{ for all vertices } v \text{ of } \triangle\}$$

for the corresponding space of supersplines, where as usual, $s \in C^\rho(v)$ means that the polynomials in the set $\{s|_T : v \text{ is a vertex of } T\}$ have common derivatives up to order ρ at v.

Let H be an n-sided polygonal domain whose edges are either edges of $\triangle$ or are not contained in Ω. Fig. 1 and Fig. 2 show some examples. We define

$$\theta_i := \begin{cases} 1, & \text{if the edge } \langle v_i, v_{i+1}\rangle \text{ is an edge of } \triangle, \\ 0, & \text{otherwise.} \end{cases}$$

where $v_1, \ldots, v_n$ are the vertices of H. Let $\theta = (\theta_1, \ldots, \theta_n)$. Our aim in this paper is to solve the following problem.

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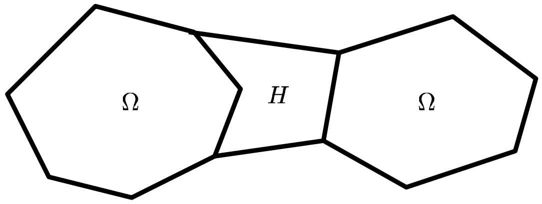

Fig. 1. The extension problem.

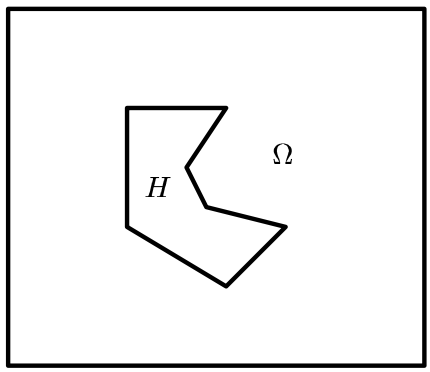

Fig. 2. The hole filling problem.

Problem 1.1. *Suppose $\triangle$ is a triangulation of the set Ω. Find a triangulation $\triangle_{\theta,H}$ of H and a spline space $\mathcal{S}_{\theta,H} \subset \mathcal{S}_d^{r,\rho}(\triangle_{\theta,H})$ such that for every $s \in \mathcal{S}_d^{r,\rho}(\triangle)$, there exists a spline $s_{\theta,H} \in \mathcal{S}_{\theta,H}$ satisfying:*

1) *the derivatives of s and $s_{\theta,H}$ agree up to order ρ at each of the vertices shared by H and Ω,*
2) *the normal derivatives of s and $s_{\theta,H}$ agree up to order r across each of the edges shared by H and Ω.*

We can think of the spline s_H as extending the spline s from $\mathcal{S}_d^{r,\rho}(\triangle)$ to $\mathcal{S}_d^{r,\rho}(\widetilde{\triangle})$, where $\widetilde{\triangle} := \triangle \cup \triangle_{\theta,H}$. The extended spline has the same supersmoothness as the original spline, but is defined on all of $\widetilde{\Omega} := \Omega \cup H$. The case where H is in the interior of Ω is particularly interesting. In this case we say that H defines a hole in Ω, and solving Problem 1.1 is equivalent to filling the hole H.

Finding a spline space $\mathcal{S}_{\theta,H} \subset \mathcal{S}_d^{r,\rho}(\triangle_{\theta,H})$ solving Problem 1.1 can be also be regarded as the problem of creating an n-sided macro-element which takes information from $m \leq n$ sides.

Definition 1.2. *We say that the spline space $\mathcal{S}_{\theta,H} \subset \mathcal{S}_d^{r,\rho}(\triangle_{\theta,H})$ is a θ-*macro-element space *provided it solves Problem 1.1.*

If H is a three-sided hole in Ω, then Problem 1.1 can be solved using classical macro-element spaces, at least for those values of r, ρ, d where such spaces are known. In this paper we make use of the Clough-Tocher (CT) macro-element spaces constructed in [2]. They are subspaces of the superspline spaces $\mathcal{S}_{6m+1}^{2m,3m}(\triangle)$ with $m \geq 1$ and $\mathcal{S}_{6m+3}^{2m+1,3m+1}(\triangle)$ with $m \geq 0$. In our notation, such macro-element spaces can be regarded as (1,1,1)-macro-element spaces, see Sect. 4 below. Our construction of general n-sided macro element spaces will use them along with some new (1,1,0)-macro-element spaces to be introduced here.

The paper is organized as follows. In Sect. 2 we collect some well-known facts from the Bernstein-Bézier theory. In Sect. 3 we review minimal determining sets for spline spaces. The (1,1,1)-macro-element spaces are described in Sect. 4 , and the new (1,1,0) macro-element spaces are intoduced in Sect. 5. In Sect. 6 we describe θ-macro-element spaces, and in Sect. 7 we introduce the extension operator used to solve the main problem. In Sect. 8 we describe nodal degrees of freedom for our macro-element spaces. Error bounds and numerical examples can be found in Sects. 9–11, and concluding remarks are given in Sect. 12.

§2. Preliminaries

We will make extensive use of well-established Bernstein-Bézier techniques, cf. [1–9]. As is well known, any polynomial p of degree d on a triangle $T := \langle u_1, u_2, u_3\rangle$ can be written in B-form

$$p = \sum_{i+j+k=d} c_{ijk}^T B_{ijk}^d,$$

where B_{ijk}^d are the Bernstein basis polynomials of degree d associated with T. In particular, if (α, β, γ) are the barycentric coordinates of any point $u \in \mathbb{R}^2$ in terms of the triangle T, then

$$B_{ijk}^d(u) := \frac{d\,!}{i\,!\,j\,!\,k\,!}\alpha^i\beta^j\gamma^k, \qquad i+j+k=d. \tag{2.1}$$

As usual, we associate the coefficient c_{ijk}^T with the domain point

$$\xi_{ijk}^T := \frac{(iu_1 + ju_2 + ku_3)}{d}, \qquad i+j+k=d.$$

We write $\mathcal{D}_T := \{\xi_{ijk}^T\}_{i+j+k=d}$. The domain point ξ_{ijk}^T is said to be distance i to the edge $\langle u_2, u_3\rangle$. It is distance j from $\langle u_3, u_1\rangle$ and distance k from $\langle u_1, u_2\rangle$. We will work with the usual rings and disks of domain points defined by

$$R_n^T(u_1) := \{\xi_{ijk}^T : i = d-n\},$$
$$D_n^T(u_1) := \{\xi_{ijk}^T : i \geq d-n\},$$

with similar definitions at the other vertices of T. If v is a vertex of $\triangle$, the sets $R_n(v)$ and $D_n(v)$ are defined to be the unions of the rings $R_n^T(v)$ and disks $D_n^T(v)$, respectively, taken over all triangles T attached to v.

Suppose that $T := \langle u_1, u_2, u_3 \rangle$ and $\widetilde{T} := \langle u_4, u_3, u_2 \rangle$ are two adjoining triangles which share the edge $e := \langle u_2, u_3 \rangle$. Let c_{ijk} and $\tilde{c}_{ijk}$ be the coefficients of the B-representations of $p := s|_T$ and $\tilde{p} := s|_{\widetilde{T}}$ of some piecewise polynomial s defined on $T \cup \widetilde{T}$. Then following [2], for any $n \le m \le d$, we let

$$\tau^n_{m,e} s := \tilde{c}_{n,m-n,d-m} - \sum_{i+j+k=n} c_{i,j+d-m,k+m-n} B^n_{ijk}(u_4), \tag{2.2}$$

where B^n_{ijk} are the Bernstein polynomials of degree n on the triangle T. We refer to the domain points $\xi^T_{n,d-m,m-n}$ and $\xi^{\widetilde{T}}_{n,m-n,d-m}$ as the tips of the smoothness condition. As observed in [2], s is C^r continuous across the edge e if and only if

$$\tau^n_{m,e} s = 0, \qquad n \le m \le d, \quad 0 \le n \le r.$$

Assuming that the coefficients of p are known and that $\tilde{p}$ joins p with C^r continuity, the smoothness conditions can be used to compute the coefficients $\tilde{c}_{n,m-n,d-m}$ of $\tilde{p}$ for $0 \le n \le r$. They can also be used in situations where some of the coefficients of both p and $\tilde{p}$ are known and others are unknown, as shown in the following lemma for computing coefficients on the ring $R_m(u_2)$.

Lemma 2.1. [2] *Suppose T and $\widetilde{T}$ are as above, and the edge $\langle u_2, u_3 \rangle$ is nondegenerate with respect to u_2, i.e., the points u_1, u_2, u_4 are not collinear. Suppose that all coefficients of the polynomials p and $\tilde{p}$ corresponding to domain points in the disk $D_m(u_2)$ are known except for the following coefficients on $R_m(u_2)$:*

$$\begin{aligned} c_\nu &:= c_{\nu,d-m,m-\nu}, \qquad && \nu = \ell+1, \ldots, q, \\ \tilde{c}_\nu &:= \tilde{c}_{\nu,m-\nu,d-m}, \qquad && \nu = \ell+1, \ldots, \tilde{q}, \end{aligned} \tag{2.3}$$

for some $\ell, m, q, \tilde{q}$ with $0 \le q, \tilde{q}$, $-1 \le \ell \le q, \tilde{q}$, and $q + \tilde{q} - \ell \le m \le d$. Then these coefficients are uniquely determined by the smoothness conditions

$$\tilde{c}_{n,m-n,d-m} = \sum_{i+j+k=n} c_{i,j+d-m,k+m-n} B^n_{ijk}(u_4), \qquad \ell + 1 \le n \le q + \tilde{q} - \ell. \tag{2.4}$$

§3. Minimal determining sets

Let $\mathcal{S}^0_d(\triangle)$ be the space of continuous splines of degree d on a triangulation $\triangle$, and let $\mathcal{D}_{d,\triangle}$ be the union of the sets of domain points associated with the triangles of $\triangle$. Then it is well known that each spline in $\mathcal{S}^0_d(\triangle)$ is uniquely determined by its set of B-coefficients $\{c_\xi\}_{\xi \in \mathcal{D}_{d,\triangle}}$. In particular, the coefficients of the polynomial $s|_T$ are precisely $\{c_\xi\}_{\xi \in \mathcal{D}_{d,\triangle} \cap T}$.

We recall that a determining set for a spline space $\mathcal{S} \subseteq \mathcal{S}^0_d(\triangle)$ is a subset $\mathcal{M}$ of the set of domain points $\mathcal{D}_{d,\triangle}$ such that if $s \in \mathcal{S}$ and $c_\xi = 0$ for all $\xi \in \mathcal{M}$, then $c_\xi = 0$ for all $\xi \in \mathcal{D}_{d,\triangle}$, *i.e.*, $s \equiv 0$. A determining set $\mathcal{M}$ is called a minimal determining set (MDS) for $\mathcal{S}$ if there is no smaller determining set. It is known that $\mathcal{M}$ is a MDS for $\mathcal{S}$ if and only if every spline $s \in \mathcal{S}$ is uniquely determined by its set of B-coefficients $\{c_\xi\}_{\xi \in \mathcal{M}}$. A MDS $\mathcal{M}$ is called stable provided for each $\theta > 0$ there is a constant K_θ such that for any spline $s \in \mathcal{S}$ defined over a triangulation whose smallest angle is at least θ, $\max_{\xi \in \mathcal{D}_{d,\triangle}} |c_\xi| \le K_\theta \max_{\xi \in \mathcal{M}} |c_\xi|$.

§4. (1,1,1)-macro element spaces

For convenience, we recall the construction of (1,1,1)-macro-elements given in [2]. Given a triangle $T := \langle v_1, v_2, v_3\rangle$, let v_T be the barycenter of T. Then we define the associated Clough-Tocher split T_{CT} to consist of the three triangles $T^{[i]} := \langle v_T, v_i, v_{i+1}\rangle$ for $i = 1,2,3$, where we identify $v_4 = v_1$. We write e_i for the edge $\langle v_i, v_T\rangle$ for $i = 1,2,3$. Given $r \ge 1$, let $m = \lfloor r/2 \rfloor$ and

$$(\rho, \mu, d) = \begin{cases} (3m+1, 5m+2, 6m+3), & r = 2m+1, \\ (3m, 5m+1, 6m+1), & r = 2m. \end{cases} \tag{4.1}$$

Let

$$\mathcal{S}_d^{r,\rho,\mu}(T_{CT}) := \{s \in \mathcal{S}_d^{r,\rho}(T_{CT}) : s \in C^\mu(v_T)\}. \tag{4.2}$$

Theorem 4.1. [2] *Let $\mathcal{S}_r(T_{CT})$ be the linear subspace of all splines s in $\mathcal{S}_d^{r,\rho,\mu}(T_{CT})$ satisfying the additional smoothness conditions*

$$\tau_{\rho+i+1,e_1}^{2m+1+i+j} s = 0, \qquad 1 \le j \le i, \quad 1 \le i \le r-m-1, \tag{4.3}$$

$$\tau_{\rho+i+1,e_2}^{2m+1+i+j} s = 0, \qquad 1 \le j \le i, \quad 1 \le i \le r-m-1, \tag{4.4}$$

$$\tau_{2r+i,e_1}^{\rho+i+j} s = 0, \qquad 1 \le j \le m-i+1, \quad 1 \le i \le m, \tag{4.5}$$

$$\tau_{2r+i,e_2}^{\rho+i+j} s = 0, \qquad 1 \le j \le m-i, \quad 1 \le i \le m-1. \tag{4.6}$$

Then

$$\dim \mathcal{S}_r(T_{CT}) = \begin{cases} \frac{39m^2+63m+24}{2}, & r = 2m+1, \\ \frac{39m^2+33m+6}{2}, & r = 2m. \end{cases}$$

Moreover, the following set $\mathcal{M}_r$ of domain points is a stable MDS:

1) $D_\rho^{T^{[i]}}(v_i)$, *for* $i = 1,2,3$,
2) $\{\xi_{j,\rho,d-\rho-j}^{T^{[i]}}, \dots, \xi_{j,d-\rho-j,\rho}^{T^{[i]}}\}_{j=1}^r$, *for* $i = 1,2,3$.

§5. (1,1,0)-macro element spaces

Given $r \ge 1$, let m, ρ, μ, d be as in (4.1).

Theorem 5.1. *Let $\widehat{\mathcal{S}}_r(T_{CT})$ be the linear subspace of all splines s in $\mathcal{S}_d^{r,\rho,\mu}(T_{CT})$ that satisfy the following set of additional smoothness conditions:*

$$\tau_{i,e_2}^{j} s = 0, \qquad i-r+m+1 \le j \le 2i-2r-1, \quad \rho+2 \le i \le 2r, \tag{5.1}$$

$$\tau_{i,e_2}^{j} s = 0, \qquad i-r+m+1 \le j \le i, \quad 2r+1 \le i \le d, \tag{5.2}$$

$$\tau_{\rho+i,e_3}^{2m+j+i} s = 0, \qquad 1 \le j \le r-m-i, \quad 1 \le i \le r-m-1. \tag{5.3}$$

Then

$$\dim \widehat{\mathcal{S}}_r(T_{CT}) = \begin{cases} \frac{37m^2+57m+22}{2}, & r = 2m+1, \\ \frac{37m^2+31m+6}{2}, & r = 2m. \end{cases} \tag{5.4}$$

Moreover, the following set $\widehat{\mathcal{M}}_r$ of domain points is a stable MDS:

1) $D_\rho^{T^{[i]}}(v_i)$, *for* $i = 1, 2, 3$,
2) $\{\xi^{T^{[i]}}_{j,\rho,d-\rho-j}, \ldots, \xi^{T^{[i]}}_{j,d-\rho-j,\rho}\}_{j=1}^{r}$, *for* $i = 1, 2$,
3) $\{\xi^{T^{[1]}}_{r+j,m+i-j+1,\rho-i}\}_{j=1}^{\lfloor\frac{i-r+m+1}{2}\rfloor} \cup \{\xi^{T^{[2]}}_{r+j,\rho-i,m+i-j+1}\}_{j=1}^{\lfloor\frac{i-r+m}{2}\rfloor}$, *for* $i = r - m + 1, \ldots, r - 1$,
4) $\{\xi^{T^{[1]}}_{i+j,\rho-j+1,\rho-i}\}_{j=1}^{\lfloor\frac{r-i+m+1}{2}\rfloor} \cup \{\xi^{T^{[2]}}_{i+j,\rho-i,\rho-j+1}\}_{j=1}^{\lfloor\frac{r-i+m}{2}\rfloor}$, *for* $i = r, \ldots, \rho - 1$.

Proof: First we show that $\widehat{\mathcal{M}}_r$ is a determining set. Suppose that we set the coefficients c_ξ of $s \in \widehat{\mathcal{S}}_r(T_{CT})$ to zero for all $\xi \in \widehat{\mathcal{M}}_r$. Then we claim that all other coefficients must be zero. First, by the C^ρ supersmoothness at the vertices, all coefficients corresponding to domain points in the disks $D_\rho(v_j)$ must be zero for $j = 1, 2, 3$. Now we use Lemma 2.1 to solve for the unset coefficients corresponding to domain points on the rings $R_{\rho+i}(v_2)$ for $i = 1, \ldots, \rho + 1$. Each step involves solving a homogeneous system of equations. Since $d = 2\rho + 1$, this shows that all coefficients corresponding to domain points in $T^{[1]} \cup T^{[2]}$ must be zero. By the C^μ supersmoothness at v_T, it follows that all of the coefficients of s corresponding to domain points in $T^{[3]} \cap D_\mu(v_T)$ must also be zero. Then enforcing the smoothness conditions listed in (5.3) implies that the remaining undetermined coefficients in $T^{[3]}$ are zero. We have shown that all coefficients of s must be zero, and thus $\widehat{\mathcal{M}}_r$ is a determining set.

To show that $\widehat{\mathcal{M}}_r$ is a minimal determining set, we now show that its cardinality is equal to the dimension of $\widehat{\mathcal{S}}_r(T_{CT})$. First suppose $r = 2m + 1$. It is easy to check that $\#\widehat{\mathcal{M}}_r = (37m^2 + 57m + 22)/2$. Now consider the superspline space $\mathcal{S}^{2m+1,5m+2}_{6m+3}(T_{CT})$. By Theorem 2.2 in [11], the dimension of this space is $(46m^2 + 68m + 24)/2$. Our space $\widehat{\mathcal{S}}_r(T_{CT})$ is the subspace which satisfies the $3m^2 + 4m + 1$ special conditions (5.1)–(5.3) and the supersmoothness $C^{3m+1}(v_i)$ for $i = 1, 2, 3$. Enforcing this supersmoothness requires an additional $3(m^2+m)/2$ conditions, and thus

$$\begin{aligned}\frac{46m^2 + 68m + 24}{2} - \frac{6m^2 + 8m + 2}{2} - \frac{3(m^2 + m)}{2} &\le \dim \widehat{\mathcal{S}}_r(T_{CT}) \\ &\le \frac{37m^2 + 57m + 22}{2}.\end{aligned}$$

Since the expression on the left equals the one on the right, we conclude that it is equal to the dimension of $\widehat{\mathcal{S}}_r(T_{CT})$, and $\widehat{\mathcal{M}}_r$ is a MDS.

We now consider the case $r = 2m$, where $\#\widehat{\mathcal{M}}_r = (37m^2 + 31m + 6)/2$. By Theorem 2.2 in [11] the dimension of the superspline space $\mathcal{S}^{2m,5m+1}_{6m+1}(T_{CT})$ is $(46m^2 + 34m + 6)/2$. Our space $\widehat{\mathcal{S}}_r(T_{CT})$ is the subspace which satisfies the $3m^2$ special conditions (5.1)–(5.3) and the supersmoothness $C^{3m}(v_i)$ for $i = 1, 2, 3$. Enforcing this supersmoothness requires an additional $3(m^2+m)/2$ conditions, and

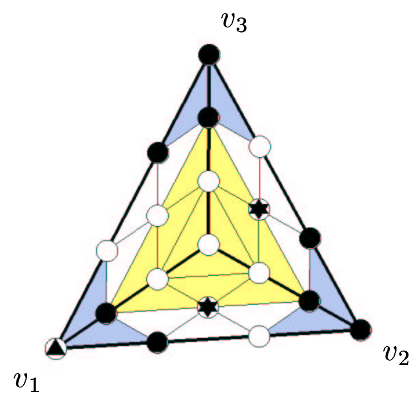

Fig. 3. The C^1 macro-element $\widehat{\mathcal{S}}_1(T_{CT})$.

thus

$$\frac{46m^2+34m+6}{2}-\frac{6m^2}{2}-\frac{3(m^2+m)}{2}\le \dim \widehat{\mathcal{S}}_r(T_{CT}) \le \frac{37m^2+31m+6}{2}.$$

Since the expression on the left equals the one on the right, we conclude that it is equal to the dimension of $\widehat{\mathcal{S}}_r(T_{CT})$, and $\widehat{\mathcal{M}}_r$ is a MDS.

Finally, we claim that $\widehat{\mathcal{M}}_r$ is stable. This follows from the fact that once we set the coefficients of a spline $s \in \widehat{\mathcal{S}}_r(T_{CT})$, the remaining unset coefficients can be computed in the order described above either directly from smoothness conditions (known to be a stable process) or from Lemma 2.1. The latter computation involves solving a non-singular linear system whose determinant (and thus the constant K of stability) depends only the smallest angle in T. $\square$

Example 5.2. *The macro-element space $\widehat{\mathcal{S}}_1(T_{CT})$ is the subspace of all splines $s \in \mathcal{S}_3^{1,1,2}(T_{CT})$ satisfying $\tau_{3,e_2}^3 s = 0$.*

Discussion: By Theorem 5.1, the dimension of $\widehat{\mathcal{S}}_1(T_{CT})$ is 11, and a MDS $\widehat{\mathcal{M}}_1$ is given by

1) $D_1^{T^{[i]}}(v_i)$, for $i = 1, 2, 3$,
2) $\{\xi_{111}^{T^{[1]}}, \xi_{111}^{T^{[2]}}\}$.

This MDS is illustrated in Fig. 3, where the tip of the special smoothness condition is marked with a black triangle. This point is in $\widehat{\mathcal{M}}_1$. The other points in item 1) are marked with black dots, while those corresponding to 2) are marked with stars. $\square$

Example 5.3. *The macro-element space $\widehat{\mathcal{S}}_2(T_{CT})$ is the subspace of all splines $s \in \mathcal{S}_7^{2,3,6}(T_{CT})$ satisfying $\tau_{i,e_2}^i s = 0$ for $i = 5, 6, 7$.*

Discussion: By Theorem 5.1, the dimension of $\widehat{\mathcal{S}}_2(T_{CT})$ is 37, and a MDS $\widehat{\mathcal{M}}_2$ is given by

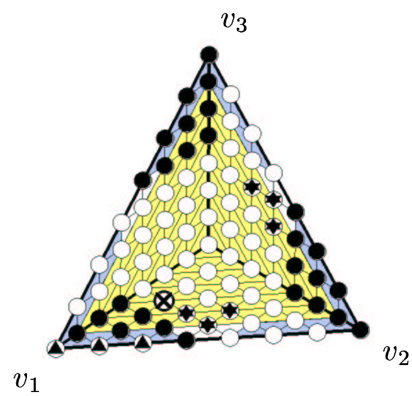

Fig. 4. The C^2 macro-element $\widehat{\mathcal{S}}_2(T_{CT})$.

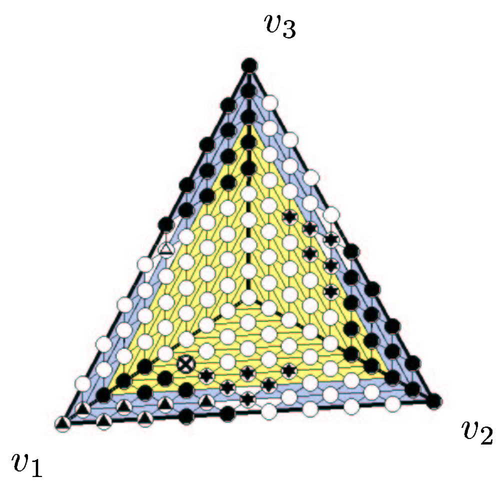

Fig. 5. The C^3 macro-element $\widehat{\mathcal{S}}_3(T_{CT})$.

1) $D_3^{T^{[i]}}(v_i)$, for $i = 1, 2, 3$,
2) $\{\xi_{133}^{T^{[i]}}, \xi_{232}^{T^{[i]}}, \xi_{223}^{T^{[i]}}\}_{i=1}^{2}$,
3) $\xi_{331}^{T^{[1]}}$.

This MDS is illustrated in Fig. 4, where the the tips of the three special smoothness conditions are marked with black triangles. These points are in $\widehat{\mathcal{M}}_2$. The other points in 1) are marked with black dots, while those in 2) are marked with stars. The point in 3) is marked with $\otimes$. $\square$

Example 5.4. *The macro-element space $\widehat{\mathcal{S}}_3(T_{CT})$ is the subspace of all splines $s \in \mathcal{S}_9^{3,4,7}(T_{CT})$ satisfying $\tau_{6,e_2}^5 s = 0$, $\tau_{i,e_2}^{i-1} s = \tau_{i,e_2}^{i} s = 0$ for $i = 7, 8, 9$, and $\tau_{5,e_3}^4 s = 0$.*

Discussion: By Theorem 5.1, the dimension of $\widehat{\mathcal{S}}_3(T_{CT})$ is 58, and the set $\widehat{\mathcal{M}}_3$ consisting of the points

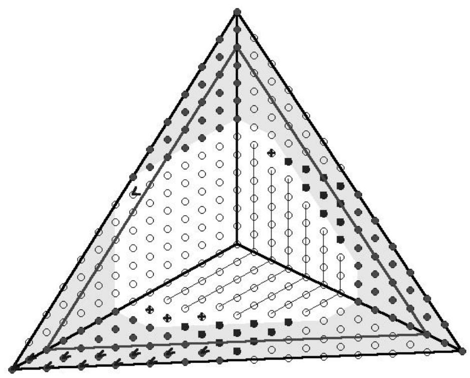

Fig. 6. The C^4 macro-element $\widehat{\mathcal{S}}_4(T_{CT})$.

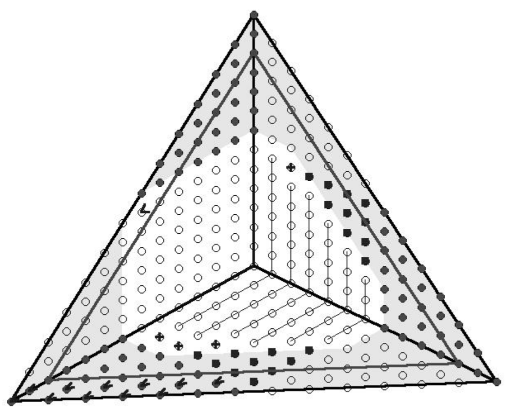

Fig. 7. The C^5 macro-element $\widehat{\mathcal{S}}_5(T_{CT})$.

1) $D_4^{T^{[i]}}(v_i)$ for $i = 1, 2, 3$,
2) $\{\xi_{144}^{T^{[i]}}, \xi_{243}^{T^{[i]}}, \xi_{234}^{T^{[i]}}, \xi_{342}^{T^{[i]}}, \xi_{333}^{T^{[i]}}, \xi_{324}^{T^{[i]}}\}_{i=1}^{2}$,
3) $\xi_{441}^{T^{[1]}}$.

is a MDS. This set is illustrated in Fig. 5, where the tips of the seven special smoothness conditions across edge e_2 are marked with black triangles. These points are in $\widehat{\mathcal{M}}_3$. The tip of the special smoothness condition across edge e_3 is marked with an open triangle. This point is not in $\widehat{\mathcal{M}}_3$. The other points in 1) are marked with black disks, while those in 2) are marked with stars. The point in 3) is marked with $\otimes$. $\square$

The analogous diagrams for our C^4 and C^5 macro-elements are shown in Fig. 6 and Fig. 7.

§6. θ-macro-element spaces

Given H and θ as in Problem 1.1, let $\triangle^{(0)}$ be an initial triangulation of H. As a means of creating extra degrees of freedom if so desired (see Remark 12.4), we allow the possibility that $\triangle^{(0)}$ may have some vertices in the interior of H. Let n_V^0, n_E^0, n_T^0 be the number of vertices, edges, and triangles of $\triangle^{(0)}$. For example, we can take $\triangle^{(0)}$ to be the Delaunay triangulation of H whose vertices coincide with the vertices of H, in which case $n_V^0 = n$, $n_E^0 = 2n-3$ and $n_T^0 = n-2$.

We now give an algorithm for creating a triangulation $\triangle_{\theta,H}$ which is suitable for constructing a θ-macro-element space associated with H. In addition to defining $\triangle_{\theta,H}$, the algorithm defines an ordering $T_1, \ldots, T_{n_T^0}$ of the triangles of the triangulation $\triangle^{(0)}$. It also classifies the edges of $\triangle^{(0)}$ by assigning an integer $0 \le \kappa_e \le 5$ to each edge. The value of κ_e will later guide the construction of an associated macro-element space.

Algorithm 6.1.

1) *For all* $1 \le i \le n$ *with* $\theta_i = 1$, *mark the edge* $\langle v_i, v_{i+1} \rangle$ *of* H *and the vertices* v_i *and* v_{i+1}.
2) *For* $i = 1$ *until* n_T^0, *find a triangle* T *in* $\triangle^{(i-1)}$ *with a maximal number of marked edges, say* $e_1, \ldots, e_{m_i}$. *We suppose these edges are numbered in counterclockwise order, and are oriented in a counterclockwise direction around the triangle. Then*
   a) *set* $T_i := T$,
   b) *if* $m_i = 1$, *set* $\kappa_{e_1} = \begin{cases} 4, & \text{if the vertex opposite } e_1 \text{ is marked,} \\ 5, & \text{otherwise,} \end{cases}$
   c) *if* $m_i = 2$, *set* $\kappa_{e_1} = 3$ *and* $\kappa_{e_2} = 2$,
   d) *if* $m_i = 3$, *set* $\kappa_{e_1} = \kappa_{e_2} = \kappa_{e_3} = 1$,
   e) *define* $\triangle^{(i)} = \triangle^{(i-1)} \setminus \{T_i\}$, *and mark the edges and vertices of* $\triangle^{(i)} \cap T_i$.
3) *Define* $\triangle_{\theta,H}$ *to be the triangulation which results from applying the Clough-Tocher split to each triangle* T_i *with* $m_i \ge 2$, *for* $i = 1, \ldots, n_T^0$.
4) *Set* $\kappa_e = 0$ *for all remaining edges of* $\triangle^{(0)}$.

We illustrate Algorithm 6.1 in the following two examples.

Example 6.2. *Let* H *be the four-sided polygonal domain shown in Fig. 8 (left), and suppose* $\theta = (1,1,1,1)$. *Let* $\triangle^{(0)}$ *be the initial triangulation of* H *shown in the figure on the left, where* $n_V^0 = 4$, $n_E^0 = 5$, *and* $n_T^0 = 2$.

Discussion: Applying Algorithm 6.1, we get $m_1 = 2$ and $m_2 = 3$ and the triangulation $\triangle_{\theta,H}$ shown in Fig. 8 (right). □

Example 6.3. *Let* H *be the seven-sided polygonal domain shown in Fig. 9 (left), and let* $\theta = (1,1,1,0,0,1,1)$. *Let* $\triangle^{(0)}$ *be the initial triangulation shown in the figure on the left, where* $n_V^0 = 7$, $n_E^0 = 11$, *and* $n_T^0 = 5$. *The edges of* H *with* $\theta_i = 1$ *are drawn with heavy lines, while those with* $\theta_i = 0$ *are drawn with thinner lines.*

Discussion: Algorithm 6.1 gives $m_1 = m_2 = m_3 = m_4 = 2$ and $m_5 = 1$, and the triangulation $\triangle_{\theta,H}$ shown in Fig. 9 (right). □

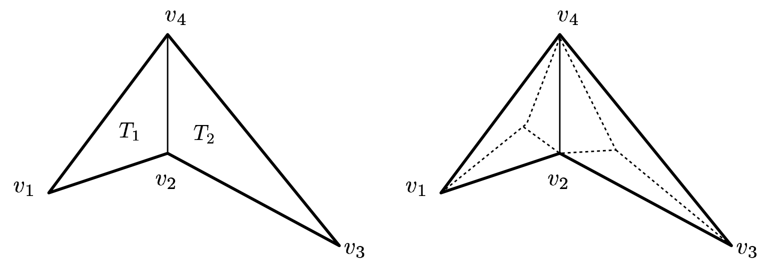

Fig. 8. The construction of $\triangle_{\theta,H}$ for Example 6.2.

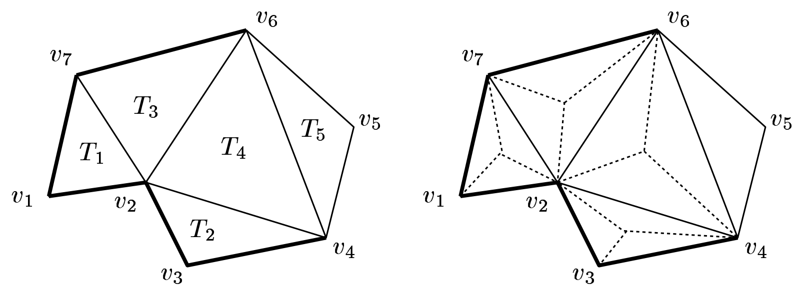

Fig. 9. The construction of $\triangle_{\theta,H}$ for Example 6.3.

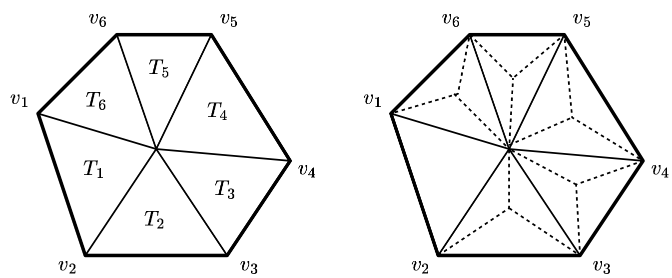

Fig. 10. The construction of $\triangle_{\theta,H}$ for Example 6.4.

Example 6.4. *Let H be the six-sided polygonal domain shown in Fig. 10 (left), and let $\theta = (1, 1, 1, 1, 1, 1)$. Let $\triangle^{(0)}$ be the initial triangulation shown in the figure on the left, where $n_V^0 = 7$, $n_E^0 = 12$, and $n_T^0 = 6$.*

Discussion: Algorithm 6.1 gives $m_1 = 1$, $m_2 = \cdots = m_5 = 2$, and $m_6 = 3$. The associated triangulation $\triangle_{\theta,H}$ is shown in Fig. 10 (right). $\square$

We are now ready to define our θ-macro-element spline space. Let $\mathcal{E}_I$ be the subset of edges of $\triangle^{(0)}$ which lie in the interior of H. In the following definition it is important that the edges of $\mathcal{E}_I$ have specific orientations. We assume that if e is the edge between two triangles T_i and T_j with $i < j$ (in the numbering assigned by Algorithm 6.1), then e is oriented counterclockwise with respect to triangle T_j.

Let μ be as defined in (4.1).

Definition 6.5. *Let $\mathcal{S}_{\theta,H}$ be the subspace of splines s in $\mathcal{S}_d^{r,\rho}(\triangle_{\theta,H})$ satisfying the following additional smoothness conditions.*

For each edge $e \in \mathcal{E}_I$,

1) *if $\kappa_e = 2$,*
   a) $\tau_{d-\rho+i,e}^{r+j} s = 0$, *for* $j = 1, \ldots, \lfloor \frac{i-r+m}{2} \rfloor$ *and* $i = r-m+1, \ldots, r-1$,
   b) $\tau_{d-\rho+i,e}^{i+j} s = 0$, *for* $j = 1, \ldots, \lfloor \frac{r-i+m}{2} \rfloor$ *and* $i = r, \ldots, \rho-1$.
2) *if $\kappa_e = 3$,*
   a) $\tau_{d-m-i+j-1,e}^{r+j} s = 0$, *for* $j = 1, \ldots, \lfloor \frac{i-r+m+1}{2} \rfloor$ *and* $i = r-m+1, \ldots, r-1$,
   b) $\tau_{d-\rho+j-1,e}^{i+j} s = 0$, *for* $j = 1, \ldots, \lfloor \frac{r-i+m+1}{2} \rfloor$ *and* $i = r, \ldots, \rho-1$.
3) *if $\kappa_e = 4$,*
   a) $\tau_{\rho+j,e}^{i} s = 0$, *for* $j = 1, \ldots, i$ *and* $i = r+1, \ldots, \rho$.
4) *if $\kappa_e = 5$,*
   a) $\tau_{\rho+j,e}^{i} s = 0$, *for* $j = 1, \ldots, i$ *and* $i = r+1, \ldots, \rho$,
   b) $\tau_{i+j,e}^{i} s = 0$, *for* $j = 0, \ldots, d-i$ *and* $i = \rho+1, \ldots, d$.

For each triangle T_ℓ with $m_\ell = 2$,

1) *$s \in C^\mu(w_\ell)$, where w_ℓ is the split point inserted in T_ℓ,*
2) *s satisfies the smoothness conditions (5.1)–(5.3) on the interior edges of the CT-split of T_ℓ,*

For each triangle T_ℓ with $m_\ell = 3$,

1) *$s \in C^\mu(w_\ell)$, where w_ℓ is the split point inserted in T_ℓ,*
2) *s satisfies the smoothness conditions (4.3)–(4.6) on the interior edges of the CT-split of T_ℓ.*

We now turn to the task of constructing a minimal determining set for the spline space $\mathcal{S}_{\theta,H}$. Our minimal determining set will consist of two parts, $\mathcal{M}_V$ and $\mathcal{M}_E$. These will be constructed by the following two algorithms.

Algorithm 6.6. *Let $\mathcal{V}_1$ be the set of vertices that lie on an edge e_i of H with $\theta_i = 1$. Set*

$$\mathcal{M}_V := \bigcup_{v \in \mathcal{V}_1} D_\rho^{T_v}(v),$$

where T_v is some triangle of $\triangle_{\theta,H}$ attached to v.

We now define a set $\mathcal{M}_E$ of points associated with the marked boundary edges of H. Let $\mathcal{E}_B$ be the set of such edges.

Algorithm 6.7. *Let $\mathcal{M}_E$ be the set of domain points obtained by the following steps. For each edge $e := \langle u_2, u_3 \rangle$ in $\mathcal{E}_B$, let $T_e := \langle u_1, u_2, u_3 \rangle$ be the triangle in $\triangle_{\theta,H}$ containing e, and include*

1) $\xi^{T_e}_{j,\rho,d-\rho-j}, \ldots, \xi^{T_e}_{j,d-\rho-j,\rho}$ *for* $j = 1, \ldots, r$.
2) *if* $\kappa_e = 2$,
   a) $\xi^{T_e}_{r+j,\rho-i,m+i-j+1}$, *for* $j = 1, \ldots, \lfloor \frac{i-r+m}{2} \rfloor$ *and* $i = r-m+1, \ldots, r-1$,
   b) $\xi^{T_e}_{i+j,\rho-i,\rho-j+1}$, *for* $j = 1, \ldots, \lfloor \frac{r-i+m}{2} \rfloor$ *and* $i = r, \ldots, \rho-1$.
3) *if* $\kappa_e = 3$,
   a) $\xi^{T_e}_{r+j,m+i-j+1,\rho-i}$, *for* $j = 1, \ldots, \lfloor \frac{i-r+m+1}{2} \rfloor$ *and* $i = r-m+1, \ldots, r-1$,
   b) $\xi^{T_e}_{i+j,\rho-j+1,\rho-i}$, *for* $j = 1, \ldots, \lfloor \frac{r-i+m+1}{2} \rfloor$ *and* $i = r, \ldots, \rho-1$.
4) *if* $\kappa_e = 4$,
   a) $\xi^{T_e}_{i,d-\rho-j,\rho+j-i}$, *for* $j = 1, \ldots, i$ *and* $i = r+1, \ldots, \rho$.
5) *if* $\kappa_e = 5$,
   a) $\xi^{T_e}_{i,d-\rho-j,\rho+j-i}$, *for* $j = 1, \ldots, i$ *and* $i = r+1, \ldots, \rho$,
   b) $\xi^{T_e}_{i,d-i-j,j}$, *for* $j = 0, \ldots, d-i$ *and* $i = \rho+1, \ldots, d$.

We are now ready to state our main theorem. Let $n_\theta := \sum_{i=1}^n \theta_i$, and let n_i be the number of edges e of H with $\kappa_e = i$ for $i = 2, \ldots, 5$. Set

$$\alpha_2 := \sum_{i=r-m+1}^{r-1} \left\lfloor \frac{i-r+m}{2} \right\rfloor + \sum_{i=r}^{\rho-1} \left\lfloor \frac{r-i+m}{2} \right\rfloor$$

$$\alpha_3 := \sum_{i=r-m+1}^{r-1} \left\lfloor \frac{i-r+m+1}{2} \right\rfloor + \sum_{i=r}^{\rho-1} \left\lfloor \frac{r-i+m+1}{2} \right\rfloor$$

$$\alpha_4 := \binom{d+2}{2} - 3\binom{\rho+2}{2} - \binom{r+1}{2},$$

$$\alpha_5 := \binom{d+2}{2} - 2\binom{\rho+2}{2} - \binom{r+1}{2}.$$

Let n_1 be the cardinality of $\mathcal{V}_1$.

Theorem 6.8. *The set*

$$\mathcal{M}_{\theta,H} := \mathcal{M}_V \cup \mathcal{M}_E$$

is a minimal determining set for $\mathcal{S}_{\theta,H}$, and

$$\dim \mathcal{S}_{\theta,H} = n_1 \binom{\rho+2}{2} + n_\theta \binom{r+1}{2} + \sum_{i=2}^{5} n_i \alpha_i. \tag{6.1}$$

Proof: Suppose s is a spline in $\mathcal{S}_{\theta,H}$, and that all of its coefficients corresponding to domain points in the set $\mathcal{M}_{\theta,H}$ have been set. We now show that all remaining coefficients are uniquely determined by smoothness conditions. We examine the triangles $T_1, \ldots, T_{n_T^0}$ in order. Suppose the coefficients of s have already been determined for triangles $T_1, \ldots, T_{\ell-1}$. To show that s is uniquely determined on T_ℓ, we consider three cases depending on the value of m_ℓ associated with T_ℓ. Here we use the notation of Sect. 4.

Suppose $m_\ell = 1$, and let $T_\ell := \langle u_1, u_2, u_3 \rangle$ where $e := \langle u_2, u_3 \rangle$ is the marked edge. Suppose $\kappa_e = 4$. Then the coefficients in the disks $D_\rho(u_i)$ for $i = 1, 2, 3$ are either set or are uniquely determined by the C^ρ smoothness at the u_i. Moreover, all coefficients of the form $c_{ijk}^{T_\ell}$ with $0 \le i \le r$ are either set or determined by the C^r smoothness across e. The remaining α_4 coefficients of s corresponding to domain points in T_ℓ are either set or are uniquely determined by the smoothness conditions listed in 3) of Definition 6.5. If $\kappa_e = 5$, then we first compute $D_\rho(u_i)$ from C^ρ smoothness for $i = 2, 3$. The remaining α_5 coefficients are either set or are computed using the smoothness conditions in 4) of Definition 6.5.

Now consider the case $m_\ell = 2$. Suppose $T_\ell := \langle u_1, u_2, u_3 \rangle$, and that $e_1 := \langle u_1, u_2 \rangle$ and $e_2 := \langle u_2, u_3 \rangle$ are the two marked edges in counterclockwise order. Let $T_\ell^{[1]} := \langle u_c, u_1, u_2 \rangle$, $T_\ell^{[2]} := \langle u_c, u_2, u_3 \rangle$, and $T_\ell^{[3]} := \langle u_c, u_3, u_1 \rangle$, where u_c is the center of T_ℓ. The coefficients in the disks $D_\rho(u_i)$ are either set or are uniquely determined by the C^ρ smoothness at u_i for $i = 1, 2, 3$. Moreover, all coefficients corresponding to domain points ξ_{ijk} in $T_\ell^{[\nu]}$ with $0 \le i \le r$ are either set or are determined by the C^r smoothness across e_ν for $\nu = 1, 2$. Now the coefficients listed in 3) and 4) of Theorem 5.1 are either set or can be computed from the smoothness conditions listed in 1) and 2) of Definition 6.5. But then Theorem 5.1 shows that all remaining coefficients of s corresponding to domain points in the triangles $T_\ell^{[\nu]}$ for $\nu = 1, 2, 3$ are uniquely determined.

Finally, suppose $m_\ell = 3$. Suppose T_ℓ and $T_\ell^{[\nu]}$ are as above. In this case, the coefficients in the disks $D_\rho(u_i)$ are either set or are uniquely determined by the C^ρ smoothness at u_i for $i = 1, 2, 3$. Moreover, all coefficients corresponding to domain points ξ_{ijk} in $T_\ell^{[\nu]}$ with $0 \le i \le r$ are either set or are determined by the C^r smoothness across e_ν for $\nu = 1, 2, 3$. Then all remaining coefficients of s are uniquely determined by Theorem 4.1.

This completes the proof that $\mathcal{M}_{\theta,H}$ is a minimal determining set for $\mathcal{S}_{\theta,H}$. But then the dimension of $\mathcal{S}_{\theta,H}$ is equal to the cardinality of $\mathcal{M}_{\theta,H}$, which is easily seen to be the number in (6.1). $\square$

§7. The extension operator

We now present a solution to Problem 1.1.

Theorem 7.1. *Given Ω, H, θ and an initial triangulation $\triangle^{(0)}$ of H, let $\triangle_{\theta,H}$ be the triangulation produced by Algorithm 6.1. Let $\mathcal{S}_{\theta,H}$ be the spline space of Definition 6.5, and let $\widetilde{\triangle} = \triangle \cup \triangle_{\theta,H}$. Then for every spline $s \in \mathcal{S}_d^{r,\rho}(\triangle)$, there exists a unique spline $\tilde{s} \in \mathcal{S}_d^{r,\rho}(\widetilde{\triangle})$, such that $\tilde{s}|_\Omega = s$ and $\tilde{s}|_H \in \mathcal{S}_{\theta,H}$, and $\tilde{s}$ satisfies the* *following additional smoothness conditions. For each edge* $e_\ell := \langle v_\ell, v_{\ell+1}\rangle$ *shared by* H *and* Ω,

1) *if* $\kappa_{e_\ell} = 2$,
   a) $\tau^{r+j}_{d-\rho+i,e_\ell}\tilde{s} = 0$, *for* $j = 1, \ldots, \lfloor\frac{i-r+m}{2}\rfloor$ *and* $i = r-m+1, \ldots, r-1$,
   b) $\tau^{i+j}_{d-\rho+i,e_\ell}\tilde{s} = 0$, *for* $j = 1, \ldots, \lfloor\frac{r-i+m}{2}\rfloor$ *and* $i = r, \ldots, \rho-1$.
2) *if* $\kappa_{e_\ell} = 3$,
   a) $\tau^{r+j}_{d-m-i+j-1,e_\ell}\tilde{s} = 0$, *for* $j = 1, \ldots, \lfloor\frac{i-r+m+1}{2}\rfloor$ *and* $i = r-m+1, \ldots, r-1$.
   b) $\tau^{i+j}_{d-\rho+j-1,e_\ell}\tilde{s} = 0$, *for* $j = 1, \ldots, \lfloor\frac{r-i+m+1}{2}\rfloor$ *and* $i = r, \ldots, \rho-1$.
3) *if* $\kappa_{e_\ell} = 4$,
   a) $\tau^{i}_{\rho+j,e_\ell}\tilde{s} = 0$, *for* $j = 1, \ldots, i$ *and* $i = r+1, \ldots, \rho$.
4) *if* $\kappa_{e_\ell} = 5$,
   a) $\tau^{i}_{\rho+j,e_\ell}\tilde{s} = 0$, *for* $j = 1, \ldots, i$ *and* $i = r+1, \ldots, \rho$,
   b) $\tau^{i}_{i+j,e_\ell}\tilde{s} = 0$, *for* $j = 0, \ldots, d-i$ *and* $i = \rho+1, \ldots, d$.

Proof: First we show that the smoothness conditions listed in 1)–4) uniquely determine the coefficients of a spline $g \in \mathcal{S}_{\theta,H}$. It suffices to show how to compute the coefficients of g associated with the minimal determining set $\mathcal{M}_V \cup \mathcal{M}_E$ of Theorem 6.8. Using the C^ρ smoothness at the boundary vertices of H, we can compute the coefficients of g corresponding domain points in $D_\rho(v)$ for all v in the set $\mathcal{V}$ of Algorithm 6.6. We have now computed all coefficients corresponding to $\mathcal{M}_V$. Now using the C^r smoothness across the boundary edges of H along with the smoothness conditions listed in 1)–4), we can compute the coefficients of g corresponding to the domain points in $\mathcal{M}_E$. It is easy to check that each coefficient in $\mathcal{M}_{\theta,H}$ corresponds to exactly one smoothness condition, and thus the above construction is unique. Moreover, the construction also guarantees that if we set $\tilde{s}|_{\triangle_{\theta,H}} = g$, then $\tilde{s} \in \mathcal{S}^{r,\rho}_d(\widetilde{\triangle})$. $\square$

Theorem 7.1 defines a linear operator Q mapping $\mathcal{S}^{r,\rho}_d(\triangle)$ into $\mathcal{S}^{r,\rho}_d(\widetilde{\triangle})$. Note that $Qp = p$ for every polynomial of degree d. We give error bounds for this operator in Sect. 9.

§8. Nodal degrees of freedom

In this section we describe a set of nodal degrees of freedom for our macro-element space. Let D_x and D_y be the usual partial derivatives, and let δ_t be the point evaluation functional associated with the point t. If $e := \langle u, v\rangle$ is an oriented edge of H, we denote the unit derivative normal to e by D_e. Let

$$\Lambda^j_e := \left\{\frac{(j+1-i)u + iv}{j+1}\right\}_{i=1}^{j}. \tag{8.1}$$

Let $\mathcal{V}_1$ be as in Algorithm 6.6, and let

$$\mathcal{N}_V := \bigcup_{v \in \mathcal{V}_1} \{\delta_v D_x^\alpha D_y^\beta\}_{0 \le \alpha+\beta \le \rho}.$$

Theorem 8.1. *Each spline $s \in \mathcal{S}_{\theta,H}$ is uniquely determined by the nodal values $\{\lambda s\}_{\lambda \in \mathcal{N}}$, where $\mathcal{N} := \mathcal{N}_V \cup \mathcal{N}_E$, and $\mathcal{N}_E$ is the following set of nodal functionals: for each edge e of H,*

1) *if $\kappa_e \ge 1$, $\{\delta_t D_e^j : t \in \Lambda_e^j\}$, for $j = 1, \ldots, r$.*
2) *if $\kappa_e = 2$, $\{\delta_t D_e^j : t \in \Lambda_e^{\nu_j}\}$ for $j = r+1, \ldots, \rho-1$, where $\nu_j := \rho - j$.*
3) *if $\kappa_e = 3$, $\{\delta_t D_e^j : t \in \Lambda_e^{\nu_j}\}$ for $j = r+1, \ldots, \rho$, where $\nu_j := \rho - j + 1$.*
4) *if $\kappa_e = 4$, $\{\delta_t D_e^j : t \in \Lambda_e^j\}$, for $j = r+1, \ldots, \rho$.*
5) *if $\kappa_e = 5$,*
   a) *$\{\delta_t D_e^j : t \in \Lambda_e^j\}$ for $j = r+1, \ldots, \rho$.*
   b) *$\{\delta_t D_e^j : t \in \Lambda_e^{d-j+1}\}$ for $j = \rho+1, \ldots, d$.*

Proof: Given $s \in \mathcal{S}_{\theta,H}$, we show that prescribing values for $\{\lambda s\}_{\lambda \in \mathcal{N}}$ uniquely determines all of the coefficients of s corresponding to the domain points in the minimal determining set $\mathcal{M}$ of Theorem 6.8. It is well known that setting the values of $\{D_x^\alpha D_y^\beta s(v)\}_{0 \le \alpha+\beta \le \rho}$ uniquely determines the coefficients of s in the disk $D_\rho(v)$, and it follows that the values of $\{\lambda s\}_{\lambda \in \mathcal{N}_V}$ uniquely determine the coefficients of s corresponding to the domain points in $\mathcal{M}_V$. To compute the coefficients of s corresponding to $\mathcal{M}_E$, we make use of the values of the normal derivatives across edges. For each edge e with $\kappa_e \ge 1$, we can use the value of the normal derivative of order one to compute the one unknown coefficient corresponding to a domain point at a distance of 1 from the edge. Then assuming we have computed all coefficients corresponding to domain points at distances $2, \ldots, j-1$ from the edge, we can use the values of the j-th derivative at j points on e to compute the unknown coefficients corresponding to domain points at distance j from the edge. This involves solving a nonsingular system of j linear equations in the j unknowns. At this point we have shown that all of the coefficients corresponding to the domain points in item 1) of Algorithm 6.7 have been determined. We now consider the computation of coefficients corresponding to domain points which lie at distances $r+1 \le j \le \rho$ from an edge e with $\kappa_e = 2$ or $\kappa_e = 3$. Fix $r+1 \le j \le \rho$, and suppose that the coefficients have already been computed for all domain points in $\mathcal{M}_E$ within a distance of $j-1$ of such edges. Then using Lemma 2.1, we now compute all coefficients at a distance j from such edges which can be determined from smoothness conditions around vertices. Then the remaining coefficients corresponding to domain points at a distance j can be computed by solving a linear system of size $\nu_j \times \nu_j$. The situation is slightly simpler when $\kappa_e = 4$. In this case we can solve $j \times j$ systems to find coefficients at a distance j from e for all $j = r+1, \ldots, \rho$. When $\kappa = 5$, this can be continued for all $j = \rho+1, \ldots, d$, using systems of size $d-j+1 \times d-j+1$. $\square$

§9. Error bound for macro-element interpolation

Given H and θ, let $\mathcal{S}_{\theta,H}$ be the corresponding macro-element subspace of $\mathcal{S}_d^{r,\rho}(\triangle_{\theta,H})$ in Definition 6.5. Then for any f which has at least d derivatives at all points on the boundary of H, as required in the statement of Theorem 8.1, there exists a unique spline $s_f \in \mathcal{S}_{\theta,H}$ for which $\lambda s_f = \lambda f$ for all $\lambda \in \mathcal{N}$, where $\mathcal{N}$ is the set of functionals of the theorem. This defines a linear projector P mapping $W_\infty^d(H)$ onto $\mathcal{S}_{\theta,H}$. We now give an error bound for how well $Pf := s_f$ approximates f on H. Let $|\triangle_{\theta,H}|$ be the mesh size of $\triangle_{\theta,H}$, i.e., the diameter of the largest triangle in $\triangle_{\theta,H}$. We write $|f|_{d+1,H}$ for the usual Sobolev semi-norm.

Theorem 9.1. *Suppose f lies in the Sobolev space $W_\infty^{d+1}(H)$. Then*

$$\|D_x^\alpha D_y^\beta (f - Pf)\|_H \le K |\triangle_{\theta,H}|^{d+1-\alpha-\beta} |f|_{d+1,H}, \tag{9.1}$$

for $0 \le \alpha + \beta \le d$. The constant K depends only on the smallest angle in $\triangle_{\theta,H}$ and on the number of triangles in $\triangle_{\theta,H}$.

Proof: Let $f \in W_\infty^{d+1}(H)$. Then there exists a polynomial $q := q_f \in \mathcal{P}_d$ such that

$$\|D_x^\alpha D_y^\beta (f - q)\|_H \le K_1 |\triangle_{\theta,H}|^{d+1-\alpha-\beta} |f|_{d+1,H}, \tag{9.2}$$

where K_1 depends only on the number of triangles in $\triangle_{\theta,H}$ and the smallest angle in $\triangle_{\theta,H}$. Since $Pq = q$,

$$\|D_x^\alpha D_y^\beta (f - Pf)\|_H \le \|D_x^\alpha D_y^\beta (f - q)\|_H + \|D_x^\alpha D_y^\beta P(f - q)\|_H.$$

To estimate the second term, suppose c_ξ are the B-coefficients of $P(f-q)$. We now show that

$$|c_\xi| \le K_2 \sum_{i=0}^{d} |\triangle_{\theta,H}|^i |f - q|_{i,H}, \qquad \text{for all } \xi \in \mathcal{D}_{\triangle_{\theta,H}}, \tag{9.3}$$

where K_2 is a constant depending only on the smallest angle in $\triangle_{\theta,H}$ and on the number of triangles in $\triangle_{\theta,H}$. Clearly, this assertion is true if ξ belongs to the minimal determining set $\mathcal{M}_{\theta,H}$ of Theorem 6.8, since for each such ξ the corresponding coefficient can be computed directly from values of f and its derivatives at points on the boundary of H. Now as shown in Theorem 6.8, all remaining coefficients c_ξ are computed using smoothness conditions across interior edges of $\triangle_{\theta,H}$. We claim that for these coefficients, we have

$$|c_\xi| \le K_3 \max_{\eta \in \mathcal{M}_{\theta,H}} |c_\eta|, \tag{9.4}$$

where K_3 is a constant depending only on the smallest angle and the number of triangles in $\triangle_{\theta,H}$. Indeed, each time we use either a smoothness condition or Lemma 2.1, the newly computed coefficients are bounded in size by a constant times the size of the previously computed coefficients.

Let T be a triangle in $\triangle_{\theta,H}$. Then since the Bernstein basis polynomials form a partition of unity,

$$\|P(f-q)\|_T \le \max_{\xi \in \mathcal{D}_T} |c_\xi|.$$

By the Markov inequality, cf. [14],

$$\|D_x^\alpha D_y^\beta P(f-q)\|_T \le K_4 |T|^{-(\alpha+\beta)} \|P(f-q)\|_T,$$

where K_4 depends on the smallest angle in T. Combining the above inequalities leads immediately to (9.1). □

If H and θ are such that the nodal determining set for $\mathcal{S}_{\theta,H}$ only involves derivatives up to order $m < d$, then the analog of Theorem 9.1 also holds for functions in $W_\infty^{k+1}(H)$ for all $m \le k \le d$. We emphasize that the error bound in Theorem 9.1 applies only to holes with a small diameter and which are filled with a small number of triangles. It is not meant for larger holes such as those in several of the examples of Sect. 11 below.

§10. Interpolation on polygonal tilings

Although our study of n-sided macro elements was motivated mostly by the hole-filling problem, it is possible to use n-sided macro-elements to construct spline interpolants on a polygonal domain Ω which has been partitioned into smaller polygons. Suppose that $\triangle = \bigcup_{i=1}^N H_i$ is a partition of Ω, where for each i, H_i is an n_i-sided polygon. In the simplest case we can take all of the H_i to be three-sided, in which case $\triangle$ is a triangulation, but here we allow more general n-sided polygons, and in fact allow a mixture of polygons with differing numbers of sides and differing shapes. Given a function f, for each H_i we suppose we are given the nodal data needed to construct the corresponding interpolant on H_i, where data is given on all n_i sides of H_i, i.e., $\theta_i = (1, \ldots, 1)$. This leads to a spline s_f defined on all of Ω. Let α_i be the smallest angle in the triangles defining the macro-element on H_i.

Theorem 10.1. *Suppose f lies in the Sobolev space $W_\infty^{d+1}(\Omega)$. Then*

$$\|D_x^\alpha D_y^\beta (f - s_f)\|_\Omega \le K |\triangle|^{d+1-\alpha-\beta} |f|_{d+1,\Omega}, \tag{10.1}$$

for all $0 \le \alpha + \beta \le d$. The constant K depends only on $m := \max\{n_i\}_{i=1}^N$ and $\alpha := \min\{\alpha_i\}_{i=1}^N$.

Proof: The result follows immediately from Theorem 9.1 applied to each polygon H_i. □

Theorem 10.1 shows that if $\triangle^{(k)}$ is a sequence of partitions of Ω with $m^{(k)} \le M$ and $\alpha^{(k)} \ge \delta$ for all k, then spline interpolation on $\triangle^{(k)}$ provides asymptotically optimal order approximation in the context of polynomial splines of degree d. It is also possible to prove a similar result where some of the macro-elements do not use information on all edges. But to make this work, we have to control the lengths

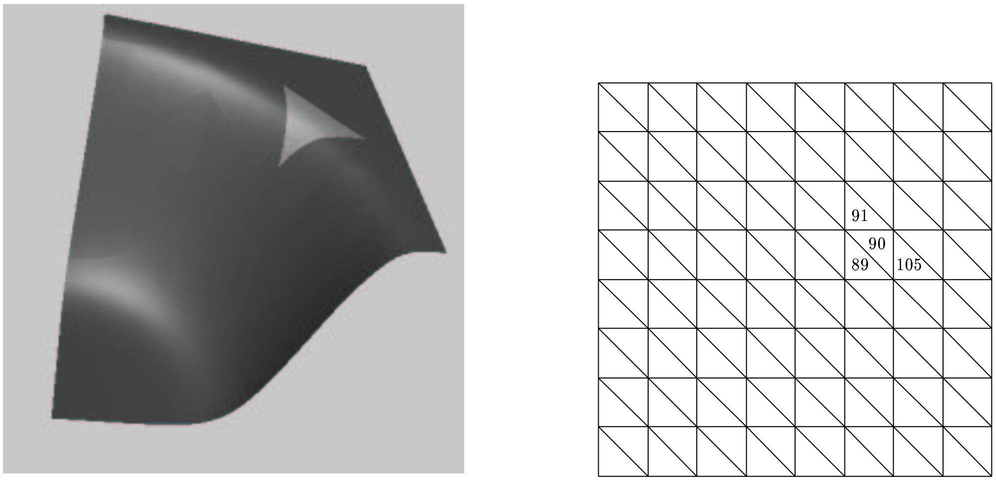

Fig. 11. Filling the hole in Example 11.1.

of chains of propagation by first interpolating on a subset of polygons with full information, and then use the elements without full information only sparingly (as was done for example in recent papers on Lagrange interpolation with splines, see e.g. [8,9] and references therein).

§11. Filling n-sided holes

In this section we illustrate the use of our macro-elements for filling n-sided holes. In all of our examples we take Ω to be the unit square $\widetilde{\Omega}$ with a hole H cut in it. In particular, we define $\widetilde{\triangle}$ to be a type 1 triangulation of $\widetilde{\Omega}$ with 128 triangles, see Fig. 11 (right). For later reference, we number these triangle in lexicographical order, starting with the triangle which is closest to the origin and moving in the y-direction, then in the x-direction. We create the holes by removing some of these triangles as marked in the figures. All of the examples are based on C^1 cubic splines, where the intial spline is obtained by first interpolating f with a spline s_f constructed using the C^1 cubic Clough-Tocher macro-element on each triangle, then restricting it to $\Omega \setminus H$. We define the initial triangulation $\triangle^{(0)}$ of H to be the set of triangles removed to form H, and set $\theta = (1, 1, \ldots, 1, 1)$. For the first three examples we take $f(x, y) = (1 + 2e^{-3(9\sqrt{x^2+y^2}-6.7)})^{-\frac{1}{2}}$ as the test function. For comparison purposes, we numerically evaluated $f - s_f$ on a grid of 10000 points and computed both the maximum norm $\|f - s_f\|_{\widetilde{\Omega}} = 1.2900\,(-2)$ and the average l_1 norm $\|f - s_f\|_1 = 7.3404\,(-4)$ defined as the sum of the absolute errors divided by 10000.

Example 11.1. *Let H be the 6–sided polygonal hole in the interior of $\widetilde{\Omega}$ obtained by removing the four triangles numbered 89, 90, 91, 105 as in Fig. 11 (right).*

Discussion: Using Algorithm 6.1, we process the triangles in the order 89, 91, 105, then 90. In the first three triangles we use 2-sided macro-elements, while in the last triangle we use the standard 3-sided macro-element. Fig. 11 (left) shows the

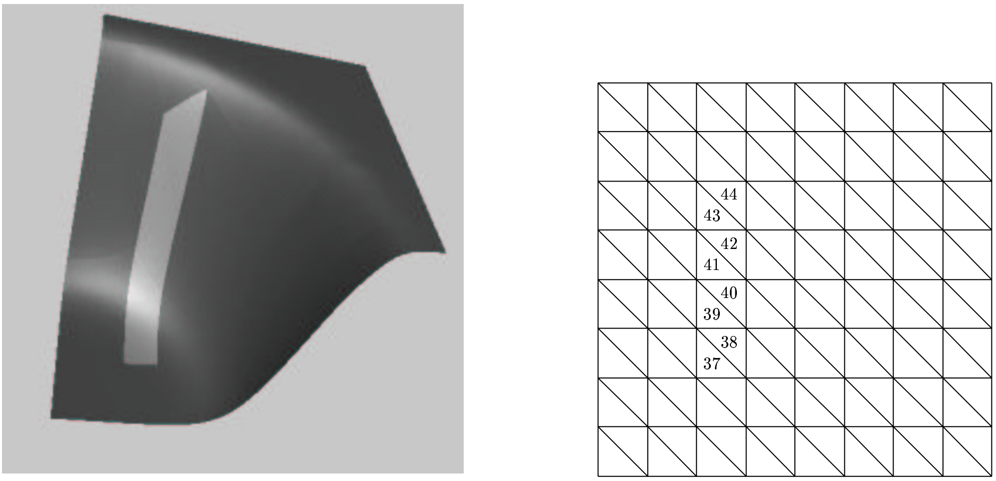

Fig. 12. Filling the hole in Example 11.2.

result of filling the hole. The extended spline $\tilde{s}_f$ is almost as good an approximation of f as s_f, and the two splines are virtually indistinguishable to the eye. Indeed, $\|f - \tilde{s}_f\|_{\widetilde{\Omega}} = 1.2900\,(-2)$ and $\|f - \tilde{s}_f\|_1 = 7.5396\,(-4)$. $\square$

Example 11.2. *Let H be the 10–sided polygonal hole in the interior of $\widetilde{\Omega}$ obtained by removing the eight triangles marked in Fig. 12 (right). Let f be the same test function as in Example 11.1.*

Discussion: Using Algorithm 6.1, we process the triangles in order from 37 to 44. The spline filling the hole H is shown in Fig. 12 (left). Two-sided macro elements are used for all triangles except for the last, which uses the standard three-sided element. Here $\|f - \tilde{s}_f\|_{\widetilde{\Omega}} = 1.2900\,(-2)$ while $\|f - \tilde{s}_f\|_1 = 7.5122\,(-4)$. $\square$

It is clear that any error introduced in the extension into the first triangle processed in Example 11.2 (number 37 in Fig. 12 (right)) will propagate into the next triangle and so on until we get to the last triangle (cf. the discussion of error bounds in the previous section). This did not cause any problem in this example, but could lead to oscillations for very long thin holes. This can be avoided by choosing the order in which the triangles are processed to stop the propagation. We now illustrate this with the T-shaped hole shown in Fig. 13.

Example 11.3. *Let H be the 18–sided T-shaped hole shown in Fig. 13, and let f be as in the above two examples.*

Discussion: Algorithm 6.1 suggests that we process the triangles in the order 37–41, 46–42, 90–57. The last triangle to be done is number 57, where a three-sided macro-element is used. In this case the extended spline, shown in Fig. 14 (left), is close to the spline s_f except in the triangles numbered 42 and 57. Here $\|f - \tilde{s}_f\|_{\widetilde{\Omega}} = 6.8531\,(-2)$ while $\|f - \tilde{s}_f\|_1 = 1.0174\,(-3)$. $\square$

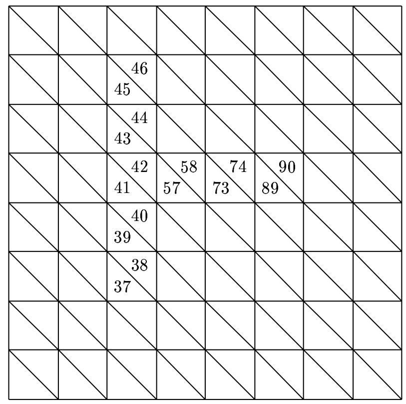

Fig. 13. The hole in Examples 11.3 and 11.4.

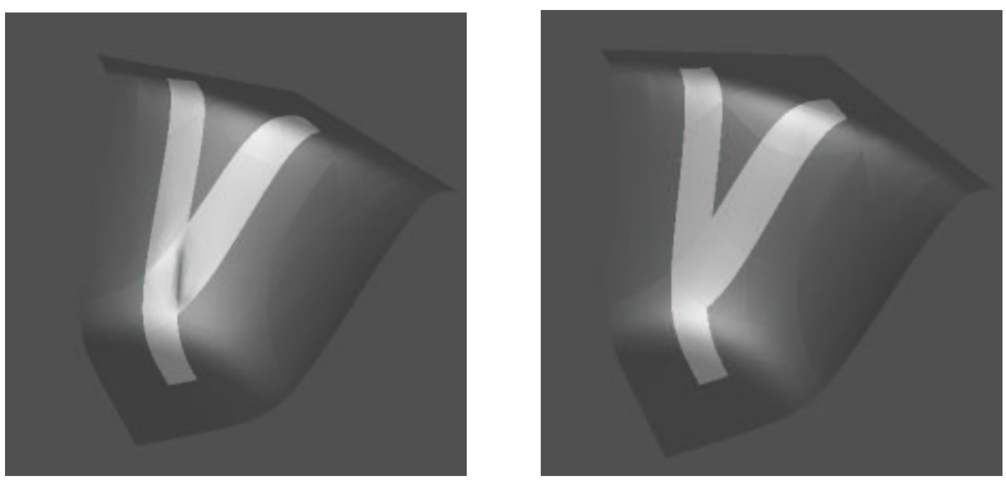

Fig. 14. The extensions in Examples 11.3 and 11.4.

In this example $\tilde{s}$ clearly exhibits some oscillations, and does not fit f nearly as well as s_f. We now show how this problem can be overcome by processing the triangles in a different order.

Example 11.4. *Let H and f be as in Example 11.3, but now suppose we process the triangles in the order 41–42, 46–43, 37–40, 57–58, and 90–73.*

Discussion: In this case triangles 41 and 42 employ the one-sided macro-element. Three-sided macro-elements are used in triangles 40, 43, and 73. In this case the extended spline, see Fig. 14 (right) is much closer to the spline s_f, and in fact now $\|f - \tilde{s}_f\|_{\widetilde{\Omega}} = 1.2900\,(-2)$ while $\|f - \tilde{s}_f\|_1 = 7.8046\,(-4)$. $\square$

We now give an example where the hole has one edge on the boundary of $\widetilde{\Omega}$. Here we take $f(x, y) = \sin(2(x - y))$ as the test function. In this case $\|f - s_f\|_{\widetilde{\Omega}} = 1.5932\,(-4)$ while $\|f - s_f\|_1 = 2.3449\,(-5)$.

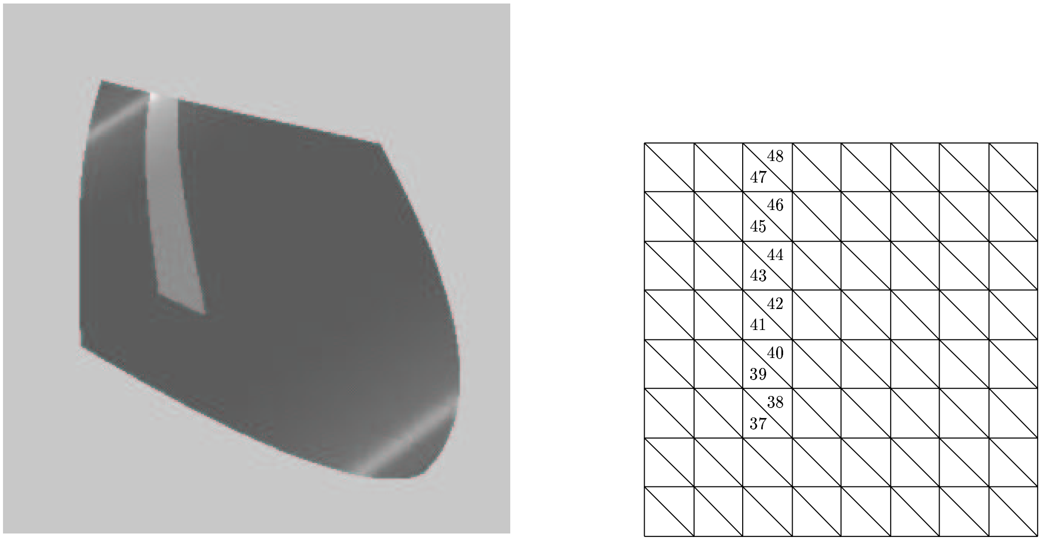

Fig. 15. Filling the hole in Example 11.5.

Example 11.5. *Let H be the 14–sided polygonal hole in $\widetilde{\Omega}$ obtained by removing 12 triangles from $\widetilde{\triangle}$ as shown in Fig. 15 (right).*

Discussion: The spline filling the hole is shown in Fig. 15 (left). Here $\|f - \tilde{s}_f\|_{\widetilde{\Omega}} = 4.9114\,(-4)$ while $\|f - \tilde{s}_f\|_1 = 2.8034\,(-5)$. $\square$

Our final example shows that the method also works for parametric surfaces. Here we take $x(u,v) = 2u$, $y(u,v) = \sin(4u)$, and $z(u,v) = \cos(4v)$ which produces the partial cylinder shown in Fig. 16.

Example 11.6. *Let H be the 18–sided T-shaped hole in $\widetilde{\Omega}$ obtained by removing the 16 triangles marked in Fig. 13.*

Discussion: The spline filling the hole in the cylinder is shown in Fig. 16. $\square$

§12. Remarks

Remark 12.1. In developing the macro-elements presented here, we have made extensive use of Peter Alfeld's java code for examining determining sets of superspline spaces. The code is described in [1], and can be used or downloaded from `http://www.math.utah.edu/~alfeld`.

Remark 12.2. The construction described here is not unique in the sense that there are other choices of the smoothness conditions and degrees of freedom which produce macro-element spaces on the same triangulations.

Remark 12.3. For the case $r = 2$, it is possible to build n-sided macro-element spaces using quintic splines instead of the degree 6 splines used here, provided that the Clough-Tocher split is be replaced by the more complicated split used in [13].

Fig. 16. Filling the hole in Example 11.6.

Remark 12.4. If the initial triangulation $\triangle^{(0)}$ of the hole H contains vertices inside H, then it is easy to design macro-element spaces which allow Hermite interpolation at those interior vertices. In this case we could treat the case $\kappa_e = 5$ in Definition 6.5 in the same way as $\kappa_e = 4$ and add the disks $D_\rho(v)$ to the minimal determining set for all interior vertices v.

Remark 12.5. Another way to gain additional control over the shape of a macro-element surface is to use the degrees of freedom of the element to minimize an energy functional instead of setting them using information across the boundary edges. Filling holes using C^1 minimal energy splines was investigated in [4,5]. Here we can work with macro-elements with arbitrary smoothness.

Remark 12.6. Theorems 9.1 and 10.1 give error bounds in the uniform norm. Analogous results hold for the p-norms.

Remark 12.7. The problem of filling holes in a suface is of major importance in CAGD, and a variety of methods have been proposed. See [4,10] for references to some of the literature.

Remark 12.8. Our thanks to Frank Zeilfelder for his very careful reading of the manuscript, and for suggesting several improvements.

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