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FITTING MONOTONE SURFACES TO SCATTERED DATA USING C^1 PIECEWISE CUBICS *

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Abstract. We derive sufficient conditions on the Bézier net of a Bernstein-Bézier polynomial defined on a triangle in the plane to insure that the corresponding surface is monotone. We then apply these conditions to construct a new algorithm for fitting a monotone surface to gridded data. The method uses C^1 cubic splines defined on the triangulation obtained by drawing both diagonals of each subrectangle. In addition, we present an algorithm for the monotone scattered data interpolation problem which is based on a method for creating gridded data from the scattered data. Numerical results for several test examples are presented.

Key words. splines, monotone surfaces, scattered data

AMS subject classifications. 41A15, 65D05, 65D07

1. Introduction.

The problem of constructing a surface which interpolates given data and which preserves the shape of the underlying function generating the data has been studied in many papers, see e.g. [1–9,12–17,20–22]. Many of these papers are concerned with preserving convexity in some sense. In this paper we are interested in preserving monotonicity. We begin by defining what we mean by a monotone surface.

DEFINITION 1. *Let $z = z(\xi, \eta)$ be a continuous function defined on $\Omega \subset \mathbb{R}^2$. We call z a monotone increasing function on Ω provided that $z(\xi_2, \eta_2) \ge z(\xi_1, \eta_1)$ for all points (ξ_1, η_1) and (ξ_2, η_2) in Ω such that $\xi_2 \ge \xi_1$ and $\eta_2 \ge \eta_1$.*

Before stating the problem of interest, we also need to define what we mean by a monotone data set.

DEFINITION 2. *Let $D = \{(\xi_i, \eta_i, \gamma_i)\}_{i=1}^N \subset \mathbb{R}^3$ be a finite data set. We say that D is a monotone increasing data set provided that $\gamma_j \ge \gamma_i$ for all points (ξ_i, η_i) and (ξ_j, η_j) such that $\xi_j \ge \xi_i$ and $\eta_j \ge \eta_i$.*

Note that in this definition, the points (ξ_i, η_i) are not required to lie on a grid; they can be scattered throughout a general domain Ω.

PROBLEM 3. Given a monotone increasing data set $D = \{(\xi_i, \eta_i, \gamma_i)\}_{i=1}^N$, construct a surface $s \in C^1(\Omega)$ which is monotone increasing and interpolates the data in the sense that

$$s(\xi_i, \eta_i) = \gamma_i, \qquad i = 1, \ldots, N.$$

At present, there are two main approaches to solving Problem 3. One approach (cf. [21,22]) involves minimizing some measure of smoothness over a convex cone of smooth functions which are monotone. The drawback of this approach is that the solutions must be found by solving constrained minimization problems, and are generally somewhat complicated, non-local and non-piecewise-polynomial functions.

The second approach to solving Problem 3 is to try to use a space of piecewise polynomials defined over a partition of the set Ω, usually into triangles. Up until now,

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this approach has been studied only in the case where the data is given on a grid (cf. [1–6,8,9,12,13,15]).

In this paper we present new methods for solving the monotone interpolation problem for both gridded and scattered data, based on C^1 cubic splines on an appropriate triangulation. The paper is organized as follows. In Section 2 we present a sufficient condition for a polynomial patch defined on a triangle to be monotone. As a corollary we show that a polynomial patch is monotone whenever its corresponding Bernstein-Bézier control net is monotone. This is the direct analog of similar results for positivity and convexity (cf. [7,16] and references therein).

In Section 3 we present a new method for gridded data which is a natural complement to existing methods of Asaturyan & Unsworth [1], Beatson & Ziegler [2], and Carlson & Fritsch [3]. Our method is based on the Sibson split of each rectangle into four triangles, and as with the methods just mentioned, the idea is to first compute gradient values at each grid point, and then adjust these values to assure monotonicity. These steps are discussed in Section 4. Error bounds for the method are derived in Section 5, and numerical examples are presented in Section 6.

The problem of scattered data is attacked in Section 7 where we discuss a way to reduce the scattered data to gridded data. While this approach is not suitable for large numbers of data points, as we see in the numerical examples of Section 8, it performs very well for moderate amounts of data.

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2. Sufficient conditions for monotonicity of a patch. Let T be a nondegenerate triangle in the plane with vertices $V_\nu = (x_\nu, y_\nu)$, $\nu = 1,2,3$. Given a positive integer n and real numbers $f_{i,j,k}$ for $i+j+k = n$ and $i,j,k \geq 0$, the associated *Bernstein-Bézier polynomial* of degree n is defined by

$$f(P) = \sum_{i+j+k=n} f_{i,j,k} B^n_{i,j,k}(P) \tag{1}$$

where $B^n_{i,j,k}(P) = \frac{n!}{i!j!k!} r^i s^j t^k$ are the Bernstein basis polynomials of degree n, and (r,s,t) are the barycentric coordinates of P with respect to V_1, V_2, V_3.

The following theorem is the main result of this section. It gives sufficient conditions for the monotonicity of a Bernstein-Bézier polynomial defined on a triangle.

THEOREM 4. *The polynomial f in (1) is monotone on T provided*

$$(y_2 - y_3) f_{i+1,j,k} + (y_3 - y_1) f_{i,j+1,k} + (y_1 - y_2) f_{i,j,k+1} \geq 0 \tag{2}$$

$$(x_3 - x_2) f_{i+1,j,k} + (x_1 - x_3) f_{i,j+1,k} + (x_2 - x_1) f_{i,j,k+1} \geq 0, \tag{3}$$

for all $i+j+k = n-1$.

Proof. Let $P \in T$. Then its barycentric coordinates satisfy $r,s,t \geq 0$. Now

$$\frac{\partial f(P)}{\partial x} = \frac{\partial f(P)}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f(P)}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial f(P)}{\partial t}\frac{\partial t}{\partial x},$$

where

$$\frac{\partial r}{\partial x} = \frac{(y_2 - y_3)}{2A}, \qquad \frac{\partial r}{\partial y} = \frac{(x_3 - x_2)}{2A}, \tag{4}$$

Fig. 1. The Bézier Net for $n = 3$.

$$\frac{\partial s}{\partial x} = \frac{(y_3 - y_1)}{2A}, \qquad \frac{\partial s}{\partial y} = \frac{(x_1 - x_3)}{2A}, \tag{5}$$

$$\frac{\partial t}{\partial x} = \frac{(y_1 - y_2)}{2A}, \qquad \frac{\partial t}{\partial y} = \frac{(x_2 - x_1)}{2A}, \tag{6}$$

and A is the area of T. Moreover,

$$\frac{\partial f(P)}{\partial r} = n(rE_1 + sE_2 + tE_3)^{n-1} E_1 f_{0,0,0}, \tag{7}$$

$$\frac{\partial f(P)}{\partial s} = n(rE_1 + sE_2 + tE_3)^{n-1} E_2 f_{0,0,0}, \tag{8}$$

$$\frac{\partial f(P)}{\partial t} = n(rE_1 + sE_2 + tE_3)^{n-1} E_3 f_{0,0,0}, \tag{9}$$

where the E_i are the formal partial shift operators

$$E_1 f_{i,j,k} = f_{i+1,j,k}, \quad E_2 f_{i,j,k} = f_{i,j+1,k}, \quad E_3 f_{i,j,k} = f_{i,j,k+1}$$

introduced by Chang and Davis [7]. Thus

$$\begin{aligned} \frac{\partial f(P)}{\partial x} = & \frac{n}{2A}\Big[(y_2 - y_3)(rE_1 + sE_2 + tE_3)^{n-1} E_1 f_{0,0,0} \\ & + (y_3 - y_1)(rE_1 + sE_2 + tE_3)^{n-1} E_2 f_{0,0,0} \\ & + (y_1 - y_2)(rE_1 + sE_2 + tE_3)^{n-1} E_3 f_{0,0,0}\Big]. \end{aligned} \tag{10}$$

Expanding the above expression, we see that the coefficient for the term $r^i s^j t^k$ with $i + j + k = n - 1$ is

$$\frac{n!}{2A\, i!j!k!}[(y_2 - y_3) f_{i+1,j,k} + (y_3 - y_1) f_{i,j+1,k} + (y_1 - y_2) f_{i,j,k+1}].$$

Thus (2) implies $\partial f(P)/\partial x \geq 0$ for all $P \in T$. Similarly, (3) implies $\partial f(P)/\partial y \geq 0$ for all $P \in T$. □

It was pointed out to us by Dietrich Braess that Theorem 4 has a simple interpretation in terms of the control surface associated with the Bernstein-Bézier polynomial f. (Recall that the *control surface* g associated with f is the C^0 piecewise linear surface which satisfies $g(\xi_{i,j,k}) = c_{i,j,k}$, where $\xi_{i,j,k} = (iV_1 + jV_2 + kV_3)/n$ are the *domain points* associated with the ordinates $f_{i,j,k}$, for all $i + j + k = n$).

COROLLARY 5. *The Bernstein-Bézier polynomial polynomial f is monotone on T whenever its control surface g is monotone.*

Proof. It is easy to check that the conditions of Theorem 4 are equivalent to the monotonicity of g. This result can also be established directly by observing that the quantities in (2) are just the coefficients of the Bernstein-Bézier polynomial representation of $D_x f$ in terms of the Bernstein basis polynomials $B^{n-1}_{i,j,k}$ of degree $n-1$, while those in (3) are just the coefficients of $D_y f$. Thus both $D_x f$ and $D_y f$ will be nonnegative whenever (2) and (3) hold. □

In connection with Theorem 4, we remark that

1) There are a total of $n(n+1)/2$ pairs of inequalities in (2) and (3). For example, when $n=3$, there are 12 inequalities. Each pair of inequalities is associated with one of the non-shaded triangles shown in Figure 1.
2) When $n=2$, the conditions (2) – (3) are also necessary. To see this, simply take $r=1$, $s=1$, and $t=1$, respectively, in the expansion (10). (This fact was also observed by Chui, Chui & He [8]).
3) Necessary and sufficient conditions on the $f_{i,j,k}$ guaranteeing the convexity of $f(P)$ have been developed by several authors; see [7].

3. A new method for interpolating gridded data. In this section we present a new method for interpolating gridded data which uses C^1 cubic splines defined on the triangulation obtained by dividing each subrectangle of the grid into four subtriangles. Let H be a rectangle which has been divided into subrectangles

$$H_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}], \tag{11}$$

for $i = 1, \ldots, nx-1$ and $j = 1, \ldots, ny-1$, by the grid lines corresponding to $x_1 < x_2 < \cdots < x_{nx}$ and $y_1 < y_2 < \cdots < y_{ny}$.

Suppose that for each $i = 1, \ldots, nx$ and $j = 1, \ldots, ny$, we are given position and derivative values $z_{ij}, z^x_{ij}, z^y_{ij}$. Our aim is to construct a monotone surface in $C^1(H)$ which interpolates this data in the sense that

$$s(x_i, y_j) = z_{ij}, \quad s_x(x_i, y_j) = z^x_{ij}, \quad s_y(x_i, y_j) = z^y_{ij}, \tag{12}$$

for $i = 1, \ldots, nx$ and $j = 1, \ldots, ny$.

This problem has been solved in various ways (cf. [1-6,8,12,13,15]). Here we will use C^1 piecewise cubic polynomials defined on the triangulation Δ of H obtained by drawing in both diagonals in each subrectangle H_{ij}. Figure 2 shows one such subrectangle. This subdivision is called the *Sibson split* of H_{ij}. Let $S(\Delta)$ be the linear space of C^1 cubic splines defined on Δ whose normal derivatives along the edges are linear (rather than quadratic).

For each $1 \le i \le nx-1$ and $1 \le j \le ny-1$, let

$$s_{ij} = s\big|_{H_{ij}}.$$

It is known (cf. [11]) that s_{ij} is uniquely determined by the 12 pieces of data giving the values and gradient at each of the four corners of H_{ij}. Indeed, if we write s_{ij} in Bernstein-Bézier form, then its 25 coefficients (see Figure 2) can be given explicitly:

Fig. 2. The Sibson split of H_{ij}.

THEOREM 6. *Let $h_i^x = x_{i+1} - x_i$ and $h_j^y = y_{j+1} - y_j$. Then the Bézier coefficients (numbered as in Figure 2) of the unique s_{ij} solving the interpolation problem on H_{ij} are given by*

$$
\begin{array}{ll}
c_1 = z_{ij} & c_2 = z_{i+1,j} \\
c_3 = z_{i+1,j+1} & c_4 = z_{i,j+1} \\
c_5 = z_{ij} + h_i^x z_{ij}^x/3 & c_6 = z_{i+1,j} - h_i^x z_{i+1,j}^x/3 \\
c_7 = z_{i+1,j} + h_j^y z_{i+1,j}^y/3 & c_8 = z_{i+1,j+1} - h_j^y z_{i+1,j+1}^y/3 \\
c_9 = z_{i+1,j+1} - h_i^x z_{i+1,j+1}^x/3 & c_{10} = z_{i,j+1} + h_i^x z_{i,j+1}^x/3 \\
c_{11} = z_{i,j+1} - h_j^y z_{i,j+1}^y/3 & c_{12} = z_{ij} + h_j^y z_{ij}^y/3 \\
c_{13} = (c_5 + c_{12})/2 & c_{14} = (c_6 + c_7)/2 \\
c_{15} = (c_8 + c_9)/2 & c_{16} = (c_{10} + c_{11})/2 \\
c_{17} = (2c_{13} + 2c_{14} + c_5 + c_6 - c_1 - c_2)/4 & \\
c_{18} = (2c_{14} + 2c_{15} + c_7 + c_8 - c_2 - c_3)/4 & \\
c_{19} = (2c_{15} + 2c_{16} + c_9 + c_{10} - c_3 - c_4)/4 & \\
c_{20} = (2c_{16} + 2c_{13} + c_{11} + c_{12} - c_4 - c_1)/4 & \\
c_{21} = (c_{17} + c_{20})/2 & c_{22} = (c_{17} + c_{18})/2 \\
c_{23} = (c_{18} + c_{19})/2 & c_{24} = (c_{19} + c_{20})/2 \\
c_{25} = (c_{21} + c_{23})/2 = (c_{22} + c_{24})/2. &
\end{array}
$$

Proof. The interpolation conditions force the values of coefficients $c_1, \ldots, c_{12}$ to be as stated above. The C^1 continuity conditions force the values of $c_{13}, \ldots, c_{16}$. Now we claim that the values of $c_{17}, \ldots, c_{20}$ are determined by the requirement of linear cross boundary derivatives. We discuss c_{17}.

Consider the y derivative along the edge from vertex $V_1 = (x_i, y_j)$ to vertex $V_2 = (x_{i+1}, y_j)$. In the triangle containing this edge, s_{ij} has the form

$$
\begin{aligned}
s_{ij}(r,s,t) = & r^3c_1 + 3r^2sc_5 + 3r^2tc_{13} + 3rs^2c_6 + 6rstc_{17} + 3rt^2c_{21} \\
& + s^3c_2 + 3s^2tc_{14} + 3st^2c_{22} + t^3c_{25},
\end{aligned}
$$

where (r, s, t) are the barycentric coordinates with respect to the triangle T_1 with vertices at V_1, V_2, and V_5, where V_5 is the point of intersection of the two diagonals. The direction normal to the edge is the y-direction, and so the cross boundary derivative along the edge $y=0$ (where $t=0$ and $r=1-s$) is given by

$$\begin{aligned}\frac{\partial s_{ij}}{\partial y}\bigg|_{y=0} =&\Big(\frac{\partial s_{ij}}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial s_{ij}}{\partial s}\frac{\partial s}{\partial y}+\frac{\partial s_{ij}}{\partial t}\frac{\partial t}{\partial y}\Big)\Big|_{t=0}=\beta\Big(-\frac{\partial s_{ij}}{\partial r}-\frac{\partial s_{ij}}{\partial s}+2\frac{\partial s_{ij}}{\partial t}\Big)\bigg|_{t=0}\\ =&\beta\,[-3(1-s)^2c_1-6(1-s)sc_5-3s^2c_6-3s^2c_2-3(1-s)^2c_5\\ &-6(1-s)sc_6+12(1-s)sc_{17}+6s^2c_{14}+6(1-s)^2c_{13}],\end{aligned}$$

where $\beta=1/2(y_5-y_1)$. This is a linear polynomial in s if and only if the coefficient of s^2 is zero. This gives the formula for c_{17}. A similar discussion applies to the coefficients c_{18}, c_{19} and c_{20}. The remaining coefficients are determined by C^1 continuity. □

Starting with a set of monotone data (x_i, y_j, z_{ij}), the spline s_{ij} constructed in Theorem 6 will not be monotone on H_{ij} for arbitrary choices of the gradients. It is easy to see that if we set all gradients to zero, then by Corollary 5, s_{ij} is monotone. However, in general this is not a good choice for the gradients as it produces a surface which is very flat at each of the four corners of H_{ij}.

In the remainder of this section we derive a set of *sufficient conditions* on the gradients which guarantee that s_{ij} is monotone on H_{ij}.

THEOREM 7. *Suppose*

$$z^x_{ij}\ge 0,\quad z^x_{i+1,j}\ge 0,\quad z^x_{i,j+1}\ge 0,\quad z^x_{i+1,j+1}\ge 0, \tag{13}$$

and

$$z^y_{ij}\ge 0,\quad z^y_{i+1,j}\ge 0,\quad z^y_{i,j+1}\ge 0,\quad z^y_{i+1,j+1}\ge 0, \tag{14}$$

are nonnegative real numbers such that

$$z^x_{ij}+z^x_{i+1,j}\le 3(z_{i+1,j}-z_{ij})/h^x_i \qquad z^x_{i,j+1}+z^x_{i+1,j+1}\le 3(z_{i+1,j+1}-z_{i,j+1})/h^x_i \tag{15}$$

$$z^y_{ij}+z^y_{i,j+1}\le 3(z_{i,j+1}-z_{ij})/h^y_j \qquad z^y_{i+1,j}+z^y_{i+1,j+1}\le 3(z_{i+1,j+1}-z_{i+1,j})/h^y_j \tag{16}$$

$$z^x_{ij}-z^x_{i,j+1}\le [6(z_{i,j+1}-z_{ij})-2h^y_j z^y_{i,j+1}]/h^x_i \tag{17}$$

$$z^x_{ij}-z^x_{i,j+1}\le [6(z_{i,j+1}-z_{ij})-2h^y_j z^y_{ij}]/h^x_i \tag{18}$$

$$z^x_{i+1,j+1}-z^x_{i+1,j}\le [6(z_{i+1,j+1}-z_{i+1,j})-2h^y_j z^y_{i+1,j+1}]/h^x_i \tag{19}$$

$$z^x_{i+1,j+1}-z^x_{i+1,j}\le [6(z_{i+1,j+1}-z_{i+1,j})-2h^y_j z^y_{i+1,j}]/h^x_i \tag{20}$$

$$z^y_{ij}-z^y_{i+1,j}\le [6(z_{i+1,j}-z_{ij})-2h^x_i z^x_{i+1,j}]/h^y_j \tag{21}$$

$$z^y_{ij}-z^y_{i+1,j}\le [6(z_{i+1,j}-z_{ij})-2h^x_i z^x_{ij}]/h^y_j \tag{22}$$

$$(23) \qquad z^y_{i+1,j+1} - z^y_{i,j+1} \le [6(z_{i+1,j+1} - z_{i,j+1}) - 2h^x_i z^x_{i+1,j+1}/h^y_j$$

$$(24) \qquad z^y_{i+1,j+1} - z^y_{i,j+1} \le [6(z_{i+1,j+1} - z_{i,j+1}) - 2h^x_i z^x_{i,j+1}]/h^y_j$$

$$(25) \qquad z^x_{ij} + z^x_{i+1,j+1} \le z^x_{i+1,j} + z^x_{i,j+1} + 3(z_{i,j+1} - z_{ij} + z_{i+1,j+1} - z_{i+1,j})/2h^x_i$$

$$(26) \qquad z^y_{ij} + z^y_{i+1,j+1} \le z^y_{i+1,j} + z^y_{i,j+1} + 3(z_{i+1,j} - z_{ij} + z_{i+1,j+1} - z_{i,j+1})/2h^y_j.$$

Then the spline s_{ij} is monotone on the rectangle H_{ij}.

Proof. We apply Corollary 5 to each of the four subtriangles of H_{ij}. This gives 48 inequalities, since there are 12 conditions for each triangle, but some of them are redundant. Clearly, the condition $c_1 \le c_5$ is equivalent to $z^x_{ij} \ge 0$, with similar equivalences for the other coefficients on the boundary of H_{ij}. The conditions $c_5 \le c_6$, $c_{10} \le c_9$, $c_{12} \le c_{11}$ and $c_7 \le c_8$ are equivalent to (15) and (16). The conditions $c_{13} \le c_{20}$, $c_{20} \le c_{16}$, $c_{14} \le c_{18}$, $c_{18} \le c_{15}$, $c_{13} \le c_{17}$, $c_{17} \le c_{14}$, $c_{16} \le c_{19}$, and $c_{19} \le c_{15}$ are equivalent to (17) – (24). The condition that $c_{21} \le c_{22}$ is equivalent to

$$h^x_i(z^x_{ij} + z^x_{i+1,j} + z^x_{i,j+1} + z^x_{i+1,j+1}) + 2h^y_j(z^y_{ij} - z^y_{i+1,j} + z^y_{i+1,j+1} - z^y_{i,j+1}) \\ \le 6(-z_{ij} + z_{i+1,j} + z_{i+1,j+1} - z_{i,j+1}).$$

This is implied by (26) combined with (15). Similarly, $c_{21} \le c_{24}$ is implied by (25) combined with (16). All other conditions of the corollary follow from the C^1 continuity of the spline (which insures, for example, that all of the points on the control net associated with $c_{21}, c_{22}, c_{23}, c_{24}, c_{25}$ are collinear). □

The conditions given in Theorem 7 are rather complicated. Our next theorem gives a simplified set of sufficient conditions.

THEOREM 8. *Suppose (13) – (14) hold, and that*

$$(27) \quad z^x_{ij} + z^x_{i+1,j} \le 5(z_{i+1,j} - z_{ij})/2h^x_i, \qquad z^x_{i,j+1} + z^x_{i+1,j+1} \le 5(z_{i+1,j+1} - z_{i,j+1})/2h^x_i$$

$$(28) \qquad z^x_{ij} \le z^x_{i,j+1} + A_{ij}/h^x_i$$

$$(29) \qquad z^x_{i+1,j+1} \le z^x_{i+1,j} + A_{i+1,j}/h^x_i$$

where in general,

$$(30) \qquad A_{ij} := \min\left\{3(z_{i,j+1} - z_{ij})/2, 6(z_{i,j+1} - z_{ij}) - 2h^y_j \max[z^y_{ij}, z^y_{i,j+1}]\right\}.$$

In addition, suppose that similar conditions hold for the y partial derivatives. Then the spline s_{ij} which interpolates as in (12) is monotone on H_{ij}.

Proof. We discuss only the x conditions as the y conditions are similar. We prove that these conditions imply (15), (17) – (20), (25), and then the result follows by Theorem 7. Clearly, (27) implies (15). By (28) we have

$$z^x_{ij} \le z^x_{i,j+1} + \frac{6}{h^x_i}(z_{i,j+1} - z_{ij}) - \frac{2h^y_j}{h^x_i} \max[z^y_{ij}, z^y_{i,j+1}].$$

This implies (17) and (18). Similarly, (19) and (20) follow from (29). By (28) and (29) we have

$$z_{ij}^x \le z_{i,j+1}^x + \frac{3}{2h_i^x}(z_{i,j+1} - z_{ij}), \quad z_{i+1,j+1}^x \le z_{i+1,j}^x + \frac{3}{2h_i^x}(z_{i+1,j+1} - z_{i+1,j}).$$

Adding the above two inequalities, we get (25). □

4. Estimating the gradients. In this section we show how to choose gradients associated with a given set of monotone data (x_i, y_j, z_{ij}), $1 \le i \le nx$ and $1 \le j \le ny$, so that the sufficient conditions of Theorem 8 are satisfied for each subrectangle, and thus the associated interpolating spline s is monotone. One way to do this is to take all z_{ij}^x and z_{ij}^y to be zero. This does not lead to good fits, however. To get better gradients, we proceed in two steps:

Step 1): Choose some initial set of nonnegative z_{ij}^x and z_{ij}^y

Step 2): Adjust these values to make sure that the conditions of Theorem 8 are satisfied for each subrectangle.

Step 1 can be accomplished in several ways. For example, to compute values for the z_{ij}^x, we can use quadrature rules which are exact for cubic polynomials. Alternatively, Step 1 can be accomplished by using any of several available univariate monotone interpolation methods (see e.g. Fritsch & Carlson [14], DeVore & Yan [10] and Schumaker [20]). For Step 2, we use the following algorithm (which we state only for the z_{ij}^x).

ALGORITHM 9. [Adjusting the z_{ij}^x].

1. Input $z_{ij}^x \ge 0 \quad i = 1, \ldots, nx, \quad j = 1, \ldots, ny.$

2. Adjust z_{ij}^x **to satisfy** (27)
  for $j = 1$ **to** ny
    for $i = 1$ **to** $nx - 1$
    if $z_{ij}^x + z_{i+1,j}^x > r_{ij} := 5(z_{i+1,j} - z_{ij})/2(x_{i+1} - x_i)$
      then set

$$z_{ij}^x = \frac{r_{ij} z_{ij}^x}{z_{i+1,j}^x + z_{ij}^x}$$

$$z_{i+1,j}^x = \frac{r_{i,j} z_{i+1,j}^x}{z_{i+1,j}^x + z_{ij}^x}$$

      endif

3. Adjust z_{ij}^x **to satisfy** (28)
  for $i = 1$ **to** $nx - 1$
    for $j = ny - 1$ **to** 1 **step** -1
    if $z_{ij}^x > z_{i,j+1}^x + A_{ij}/h_i^x$
    (where A_{ij} is as in (30)
      then set

$$z_{ij}^x = z_{i,j+1}^x + A_{ij}/h_i^x$$

    endif

4. Adjust z_{ij}^x **to satisfy** (29)
  for $i = 2$ **to** nx
    for $j = 1$ **to** $ny - 1$
    if $z_{i,j+1}^x > z_{ij}^x + A_{ij}/h_{i-1}^x$
      then set

Fig. 3. Adjustment of Step 2.

```
        z^x_{i,j+1} = z^x_{ij} + A_{ij}/h^x_{i-1}
    endif
end
```

All of these adjustments involve reducing the size of gradient values. The adjustment made in Step 2 is illustrated in Figure 3 where the acceptable region defined in Theorem 8 for the pair $(z^x_{ij}, z^x_{i+1,j})$ is cross-hatched. If we start with values which are not in this region, then we project them radially into the region. Clearly, the z^x_{ij} values can change in each of the steps of the algorithm, and the final values after Step 4 satisfy conditions (29). We now have to show that these final values also satisfy the conditions (27) and (28). Let $z^{x,\nu}_{ij}$ denote the value of z^x_{ij} after the ν^{th} step of Algorithm 9.

THEOREM 10. *The values z^x_{ij} produced by Algorithm 9 satisfy (27) – (29) for all $1 \le i \le nx-1$ and $1 \le j \le ny-1$.*

Proof. Since in Steps 3 and 4 we only reduce the values of the z^x_{ij}, it is clear that condition (27) remains satisfied. Thus we need only prove that (28) is satisfied. Now

$$z^{x,4}_{i,j+1} = \min\{z^{x,3}_{i,j+1}, z^{x,4}_{ij} + A_{ij}/h^x_{i-1}\}.$$

Using the fact that the $A_{ij} \ge 0$ by (16), we have

$$\begin{aligned} z^{x,4}_{i,j+1} + A_{ij}/h^x_i &= \min\{z^{x,3}_{i,j+1} + A_{ij}/h^x_i, \quad z^{x,4}_{ij} + A_{ij}/h^x_{i-1} + A_{ij}/h^x_i\} \\ &\ge \min\{z^{x,3}_{ij}, z^{x,4}_{ij}\} \\ &\ge \min\{z^{x,4}_{ij}, z^{x,4}_{ij}\} \\ &= z^{x,4}_{ij}, \end{aligned}$$

which is (28), and the theorem is established. □

5. Error analysis. In this section we will give error bounds for the method described in Sections 4 and 5. Our analysis follows Beatson & Ziegler [2]. The main result is Theorem 14 below. Throughout the rest of this section, C_1, C_2, etc., denote some constants which do not depend on (x_i, y_j) or any functions. First we need three lemmas.

LEMMA 11. *Given $z(x,y) \in C^3(H)$, where H is a rectangle, let s be the interpolant described in Theorem 6 satisfying*

$$s(x_i, y_j) = z(x_i, y_j), \quad s_x(x_i, y_j) = z^x_{ij}, \quad s_y(x_i, y_j) = z^y_{ij}$$

for $1 \le i \le nx$, $1 \le j \le ny$. *Then*

$$\|z - s\|_\infty \le C_1 h \max_{ij}\{|z_x(x_i, y_j) - z^x_{ij}|, |z_y(x_i, y_j) - z^y_{ij}|\} + C_2 h^3 \|D^3 z\|_\infty, \tag{31}$$

where $h = max_{ij}\{(x_{i+1} - x_i), (y_{j+1} - y_j)\}$, *and*

$$\|D^3 z\|_\infty = \max_{\nu+\mu=3} \left\| \frac{\partial^3 z}{\partial x^\nu \partial y^\mu} \right\|_\infty.$$

Proof. We only need to consider a single rectangle H_{ij} as defined in (11). Let $S(\Delta_{ij})$ be the linear space of functions in $S(\Delta)$ restricted to H_{ij}. Let P be the projection from $C^1(H_{ij})$ onto $S(\Delta_{ij})$ defined by interpolation to function values and gradients at the four corner points of H_{ij}. Referring to Theorem 6, it is clear that

$$\|Pz\|_\infty \le C_3 \max(\|z\|_C, h^x_i \|z_x\|_C, h^y_j \|z_y\|_C),$$

where $\| \, \|_C$ denote the maximum absolute value at the four corners. We note that $Pg = g$ for any polynomial g of total degree 2, since such polynomials are contained in $S(H_{ij})$.

Next, we claim that for any $f \in C^3(H_{ij})$,

$$\|f - Pf\|_\infty \le C_4 h^3 \|D^3 f\|_\infty.$$

Indeed, if Tf is the Taylor expansion of total degree 2 of f about the point (x_i, y_j), then

$$\|f - Pf\|_\infty \le \|f - Tf\|_\infty + \|Pf - PTf\|_\infty,$$

and the claim follows.

Now noting that $(s - Pz)(x, y)$ vanishes at the four corners of H_{ij}, the result follows from the above estimates and the fact that

$$\|z - s\|_\infty \le \|z - Pz\|_\infty + \|P^2 z - Ps\|_\infty.$$

□

We also need the following lemma (cf. [2]):

LEMMA 12. *Let* $f \in C^3[0, h]$ *be a monotone increasing function with* $[f(h) - f(0)]/h = \Delta$, f'_0, f'_h *and* $\tilde f'_0, \tilde f'_h$ *be nonnegative. Suppose*

a) $\max\{|f'(0) - f'_0|, |f'(h) - f'_h|\} \le C_5 h^2$

b) $\tilde f'_0 \le f'_0, \tilde f'_h \le f'_h$.

c) If $(f'_0, f'_h) \ne (\tilde f'_0, \tilde f'_h)$, *then* $\tilde f'_0 + \tilde f'_h \ge 2\Delta$.

Then

$$\max\{|f'(0) - \tilde f'_0|, |f'(h) - \tilde f'_h|\} \le [C_5 + \|f^{(3)}\|_\infty/6]h^2.$$

LEMMA 13. *Suppose* $z \in C^3([0, 1] \times [0, 1])$ *and*

$$|z_x(x_i, y_j) - z^{x,1}_{ij}| < C_6 h^2$$

for $1 \le i \le nx$ *and* $1 \le j \le ny$. *Then*

$$|z_x(x_i, y_j) - z^{x,2}_{ij}| < h^2(C_6 + \|D^3 z\|_\infty),$$

for $1 \le i \le nx$ *and* $1 \le j \le ny$*, where* $h = \max_i(x_{i+1} - x_i)$*.*

Proof. The result is trivial if no adjustment is made in Step 2. Otherwise, $z_{ij}^{x,2} + z_{i+1,j}^{x,2} = 5(z(x_{i+1}, y_j) - z(x_i, y_j))/2(x_{i+1} - x_i)$, and Lemma 12 applies. □

We now prove the main result of this section, giving an error bound in the case where the knot spacing is *uniform* in both the x and y variables with $h_i^x = h_j^y$.

THEOREM 14. *Let* $z \in C^3([0,1] \times [0,1])$ *be a monotone increasing function. Let* $h = 1/(n-1)$*, and suppose* $x_i = (i-1)h$*,* $y_i = (i-1)h$*,* $1 \le i \le n$*. Suppose we are given gradients satisfying*

$$|z_x(x_i, y_j) - z_{ij}^{x,1}| < C_7 h^2, \qquad |z_y(x_i, y_j) - z_{ij}^{y,1}| < C_7 h^2$$

for $1 \le i \le nx$ *and* $1 \le j \le ny$*, and suppose these gradients are then adjusted using Algorithm 9. Then the monotone spline* s *which interpolates the resulting gradients and the values* $z_{ij} = z(x_i, y_j)$ *as in (12) satisfies*

$$\begin{aligned} \|z - s\|_\infty &\le C_8 h^2 \|D^3 z\|_\infty \\ \|z_x - s_x\|_\infty &\le C_9 h \|D^3 z\|_\infty \\ \|z_y - s_y\|_\infty &\le C_9 h \|D^3 z\|_\infty . \end{aligned} \tag{32}$$

Proof. By Taylor's expansion,

$$\begin{aligned} z(x_{i+1}, y_{j+1}) - z(x_{i+1}, j) &= z(x_i, y_{j+1}) - z(x_i, y_j) \\ &+ h[z_x(x_i, y_{j+1}) - z_x(x_i, y_j)] + \epsilon_{ij}^{(1)}, \end{aligned} \tag{33}$$

where $|\epsilon_{ij}^{(1)}| \le C_{10} h^3 \|D^3 z\|_\infty$. From Lemma 13 and (33) we have

$$0 \le [y_j, y_{j+1}] z(x_i, \cdot) + [z_{i,j+1}^{x,2} - z_{ij}^{x,2}] + \epsilon_{ij}^{(2)}, \tag{34}$$

where $|\epsilon_{ij}^{(2)}| \le C_{11} h^2 \|D^3 z\|_\infty$.

If an adjustment is needed in Step 3 of Algorithm 9, then by (28) it follows that

$$z_{ij}^{x,3} - z_{i,j+1}^{x,3} \ge \min\left\{1.5[y_j, y_{j+1}] z(x_i, \cdot), 6[y_j, y_{j+1}] z(x_i, \cdot) - 2\max[z_{ij}^{y,2}, z_{i,j+1}^{y,2}]\right\}.$$

If the first term inside the min is smaller than the second term, we see that

$$z_{ij}^{x,3} - z_{i,j+1}^{x,3} \ge [y_j, y_{j+1}] z(x_i, \cdot). \tag{35}$$

If the second term is smaller, we may assume without loss of generality that $z_{ij}^{y,2} \ge z_{i,j+1}^{y,2}$. (If $z_{ij}^{y,2} < z_{i,j+1}^{y,2}$, the proof is similar.) Then

$$\begin{aligned} z_{ij}^{x,3} - z_{i,j+1}^{x,3} &\ge 6[y_j, y_{j+1}] z(x_i, \cdot) - 2 z_{ij}^{y,2} \\ &\ge 6[y_j, y_{j+1}] z(x_i, \cdot) - 2\{2.5[y_j, y_{j+1}] z(x_i, \cdot) - z_{i,j+1}^{y,2}\} \\ &\ge [y_j, y_{j+1}] z(x_i, \cdot). \end{aligned}$$

Thus, (35) is also true in this case, which implies that

$$[z_{i,j+1}^{x,2} - z_{i,j+1}^{x,3}] + C_{11} h^2 \|D^3 z\|_\infty \ge z_{ij}^{x,2} - z_{ij}^{x,3}.$$

It follows that

$$z_{ij}^{x,2} - z_{ij}^{x,3} \le (n-j)C_{11}h^2 \|D^3 z\|_\infty \le C_{12}h\|D^3 z\|_\infty, \quad 0 \le j \le n. \tag{36}$$

Similarly, we can show that

$$z_{ij}^{x,2} - z_{ij}^{x,4} \le C_{13}h\|D^3 z\|_\infty. \tag{37}$$

Combining (36) and (37) with Lemma 12 completes the proof. □

By construction, our monotone interpolant is exact for quadratics. In many cases, the constants C_{12} and C_{13} in (36) and (37) will be zero, in which case the approximation order is 3, i.e., $\|z - s\|_\infty = \mathcal{O}(h^3)$. Since we have forced the cross-boundary derivatives to be linear, we cannot expect the method to reproduce cubics, and so an error bound of order 4 is certainly not possible.

6. Numerical tests for gridded data. In this section we present the results of several numerical experiments with gridded data. The idea is to compare the performance of our method with those of Carlson & Fritsch [3–6,15] and Beatson & Ziegler [2]. As test data, we take the values of the following four functions on square grids defined on the unit square H:

(F1) Sigmoidal function:

$$f_1(x,y) = (1 + 2e^{-3(9r-6.7)})^{-\frac{1}{2}}, \quad r = \sqrt{x^2+y^2}.$$

(F2) Bilinear function:

$$f_2(x,y) = \begin{cases} |8x-4|(8y-4)/32 + 0.5, & \text{if } (x-0.5)(y-0.5) \ge 0 \\ 0.5, & \text{otherwise} \end{cases}$$

(F3)

$$f_3(x,y) = (\sqrt{x^2+y^2} - 0.6)_+^4.$$

(F4)

$$f_4(x,y) = \begin{cases} e^{-(r-0.6)^{-2}}, & \text{if } r > 0.6 \\ 0, & \text{otherwise} \end{cases} \quad r = \sqrt{x^2+y^2}.$$

For each of these functions we generated the data $z_{ij} = f((i-1)/(n-1), (j-1)/(n-1))$ for $1 \le i \le n$ and $1 \le j \le n$ for values of $n = 5, 9, 17, 33, 65$, which correspond to grids with spacing .25, .125, .0625, .03125, and .015625. For each surface s, we computed the discrete uniform error norm

$$E = \| f - s \|_{l_\infty(G)},$$

where G is a 99×99 uniform grid on H.

Tables 1 – 4 show the errors obtained for functions F1 – F4 for the three methods. The first column (labelled BZ) shows the results for the C^1 quadratic method of Beatson & Ziegler, based on dividing each rectangle in the grid into 16 subtriangles. Column two (labelled CF) shows the results for the C^1 bicubic method of Carlson & Fritsch, where each surface patch is defined on an undivided rectangle. Finally, column three (labelled HS) shows the results of our method, where the gradients

n	BZ	CF	HS
5	0.2314260	0.1943635	0.1920871
9	6.3867509E-02	4.3835938E-02	4.5126766E-02
17	8.0936253E-03	6.8327785E-03	6.8091750E-03
33	9.2029572E-04	8.9526176E-04	4.4894218E-04
65	1.2481213E-04	1.3566017E-04	3.5762787E-05

Table 1. Comparison of methods for function F1.

n	BZ	CF	HS
5	4.1649312E-02	3.7025332E-02	4.0314794E-02
9	2.0824671E-02	1.8512666E-02	2.0008683E-02
17	1.0412335E-02	9.2563331E-03	1.0004342E-02
33	5.2061677E-03	4.6281815E-03	5.0021708E-03
65	9.4044209E-04	1.2282729E-03	1.6135573E-03

Table 2. Comparison of methods for function F2.

n	BZ	CF	HS
5	1.2486354E-02	1.0424078E-02	3.7271231E-03
9	2.2790730E-03	1.5496612E-03	4.2398274E-04
17	3.2073259E-04	2.0542741E-04	3.8892031E-05
33	3.8892031E-05	2.6762486E-05	3.8444996E-06
65	4.6491623E-06	2.6412308E-06	5.9604645E-07

Table 3. Comparison of methods for function F3.

n	BZ	CF	HS
5	9.0837870E-03	6.9294348E-03	6.8800766E-03
9	2.1321434E-03	1.2686048E-03	1.0934900E-03
17	2.5695749E-04	1.7740973E-04	9.5663592E-05
33	3.8124621E-05	2.3546047E-05	7.2778203E-06
65	4.0924642E-06	2.6002526E-06	4.5681372E-07

Table 4. Comparison of methods for function F4.

were constructed using standard four-point quadrature formulae, and adjusted using Algorithm 9.

All three methods do a good job of producing smooth interpolating surfaces which are monotone. In most of the cases shown in the tables, our method was as accurate or more accurate than the other two. Ideally, we should also have included a comparison with the method of Asaturyan & Unsworth [1], but unfortunately, we did not have access to running code. Their method uses C^1 biquadratic patches defined on subrectangles obtained by dividing each original rectangle into 4 subrectangles.

7. Reducing scattered data to gridded data. For scattered data, it is difficult to solve Problem 3 using splines defined on a triangulation with vertices at the

Fig. 4. Adjusting the z_{ij}.

data points. For example, even the C^0 piecewise linear surface corresponding to a triangulation of monotone data is not in general monotone. Thus, we do not try to solve Problem 3 using piecewise polynomials on a triangulation using the data points as vertices. Instead, we reduce it to a gridded data problem.

Given a monotone scattered data set $\{(\xi_i,\eta_i,\gamma_i)\}_{i=1}^N$, let $D=\{(\xi_i,\eta_i)\}_{i=1}^N$. A rectangular grid can be created by drawing horizontal and vertical lines passing through all points $(\xi_i,\eta_i)\in D$. Suppose there are nx vertical lines and ny horizontal lines. We denote the corresponding grid points by $G=\{(x_i,y_j)\}_{i=1,j=1}^{nx,ny}$.

We now show how to construct data values z_{ij}^M for $1\le i\le nx$ and $1\le j\le ny$ which are monotone and are consistent with the original data, i.e.,

$$z_{ij}^M=\gamma_\nu, \quad \text{where } \nu \text{ is such that } x_i=\xi_\nu \text{ and } y_j=\eta_\nu, \tag{38}$$

for all $(i,j)\in I$, where

$$I=\{(i,j):\ (x_i,y_j)\in D\}.$$

To construct z_{ij}^M, we begin with an arbitrary set of consistent grid values z_{ij}. These can be obtained by setting

$$z_{ij}=Q(x_i,y_j) \qquad \text{for all } (i,j), \tag{39}$$

where Q is any interpolant satisfying $Q(\xi_\nu,\eta_\nu)=\gamma_\nu$ for $\nu=1,\dots,N$ (see [18] for a survey of scattered data interpolation methods).

We now show how to adjust the z_{ij} to produce a monotone gridded data set which remains consistent with the original data. The idea is to adjust the z_{ij} values for $(i,j)\notin I$, starting in the upper-right corner, and working down and to the left towards the lower-left corner. The order in which we make the adjustments is indicated by the numbering of the vertices shown in Figure 4.

Suppose that for some $(i,j)\notin I$, we want to adjust the value of z_{ij} corresponding to (x_i,y_j), having done all the previous points. Let

$$I_{ij}^+=\{(l,k):\ l\ge i,\ k\ge j\}$$

$$I_{ij}^-=\{(l,k)\in I:\ l\le i,\ k\le j\}$$

$$m_{ij}^-=\max\{z_{lk}:\ (l,k)\in I_{ij}^-\}$$

$$m_{ij}^{+} = \min\{z_{lk} : (l,k) \in I_{ij}^{+}\}.$$

The sets of points $R_{ij}^{+} = \{(x_l, y_k) : (l,k) \in I_{ij}^{+}\}$ and $R_{ij}^{-} = \{(x_l, y_k) : (l,k) \in I_{ij}^{-}\}$ are contained in the rectangles with dark outlines shown in Figure 4. Note that I_{ij}^{+} and I_{ij}^{-} are defined differently — I_{ij}^{+} includes all points to the right and above (i,j), while I_{ij}^{-} only includes points to the left and below (i,j) *which lie in* I. Since the original data is monotone, and the adjustment process maintains monotonicity, at every step of the process, $m_{ij}^{-} \le m_{ij}^{+}$. Now we define

$$z_{ij}^{M} = \begin{cases} m_{ij}^{+}, & \text{if } z_{ij} > m_{ij}^{+} \\ m_{ij}^{-}, & \text{if } z_{ij} < m_{ij}^{-} \\ z_{ij}, & \text{otherwise.} \end{cases}$$

Note that this adjustment process does not change the z_{ij} for points $i, j \in I$, and so the final values are still consistent with the given data. We can summarize this construction as

ALGORITHM 15. [Construction of monotone gridded data]

for $k = nx + ny$ **to** 2 **step** -1
  let $l = nx + ny - k$
  for $m = 0$ **to** l **step** 1
    let $i = nx - l + m$ **and** $j = ny - m$
    if $i > 0$ **and** $j > 0$
      if $(i,j) \in I$, **then**
          $z_{ij}^{M} = z_{ij}$
      else
        compute m_{ij}^{-} and m_{ij}^{+}
        if $z_{ij} > m_{ij}^{+}$, **then**
          $z_{ij}^{M} = m_{ij}^{+}$
        else if $z_{ij} < m_{ij}^{-}$, **then**
          $z_{ij}^{M} = m_{ij}^{-}$
        else
          $z_{ij}^{M} = z_{ij}$
        endif
      endif
    endif
end

The main drawback of the method proposed in this section for reducing monotone scattered data to montone gridded data is that given N scattered data points, we may end up with a grid consisting of order N^2 rectangles, and some of the rectangles may be very small in one or both directions. The total number of rectangles can be reduced (while eliminating bad ones) by adjusting the location of the data points slightly so that all points whose x-coordinates are essentially the same (within some tolerance ϵ) are moved so that they have the same x-coordinate (with a similar adjustment on the y-coordinates). We do not claim that this is an ideal algorithm, but as the example in the following section shows, it does perform quite well on moderate-sized problems.

8. Numerical tests for scattered data. In this section we illustrate the performance of our method for scattered data. As a test, we take the values of F1 at 34

Fig. 5. Hardy MQ interpolant of F1 at 34 random points.

Fig. 6. Monotone interpolant of F1 at 34 random points.

randomly generated points in the unit square, including the four corners. We use the method of Sect. 3, choosing Q to be the Hardy multiquadric interpolant

$$Q(x,y) = \sum_{i=1}^{n} c_i \phi_i(x,y)$$

corresponding to the radial basis functions

$$\phi_i(x,y) = \sqrt{(x-x_i)^2 + (y-y_i)^2 + R}, \qquad i = 1, \ldots, n,$$

where R is a fixed parameter.

Figure 5 shows the multiquadric interpolant corresponding to $R = 0.01$. It is clearly not monotone. Figure 6 shows the result of creating monotone gridded data using Algorithm 15, and then interpolating this gridded data using the interpolation method described in Sect. 3.

For comparison, we computed the discrete uniform error on a 99×99 uniform grid: the error for the Hardy MQ surface was 0.2868, and for our C^1 cubic monotone surface, 0.2752. Thus, we not only managed to create a monotone surface from the scattered data, but also got a smaller error.

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