
ELSEVIER
Volume 158 Number 1 May 2009 ISSN 0021-9045
Special Issue in memory of Professor George G. Lorentz (1910–2006)
JOURNAL OF
Approximation
Theory
Editors-in-Chief: Paul Nevai and Amos Ron
Book Review Editor: Andrei Martínez-Finkelshtein
Editorial Board
Heinz H. Bauschke
Ranko Bojanic
Borislav Bojanov
Carl de Boor
Peter B. Borwein
Aldric L. Brown
Martin Buhmann
E. W. Cheney
Charles K. Chui
Patrick Louis Combettes
Wolfgang Dahmen
Ingrid Daubechies
Serguei Denissov
Frank Deutsch
Ronald A. DeVore
Zeev Ditzian
Nira Dyn
Tamás Erdélyi
Hans G. Feichtinger
Leonid Golinskii
Manfred v. Golitschek
Mourad Ismail
Rong-Qing Jia
Kirill Kopotun
András Kroó
Sergey Krushchev
Arno B. J. Kuijlaars
Dany Leviatan
Guillermo López Lagomasino
Doron S. Lubinsky
Alphonse P. Magnus
Francisco Marcellán
Hrushikesh N. Mhaskar
Charles A. Micchelli
Rolf J. Nessel
Günther Nürnberger
Peter Oswald
Franz Peherstorfer
Allan Pinkus
Ulrich Reif
Edward B. Saff
Robert Schaback
Zuowei Shen
Barry Simon
József Szabados
Vladimir N. Temlyakov
Vladimir M. Tikhomirov
Vilmos Totik
Walter Van Assche
Hans Wallin
Joseph D. Ward
Przemyslaw Wojtaszczyk
Roderick Wong
Yuan Xu
Founding Editor: Oved Shisha
Editor 1990–2000: Allan Pinkus
Available online at
ScienceDirect
www.sciencedirect.com

ELSEVIER

Available online at www.sciencedirect.com
ScienceDirect

Journal of Approximation Theory 158 (2009) 126–142

Journal of Approximation Theory

www.elsevier.com/locate/jat

A C^1 quadratic trivariate macro-element space defined over arbitrary tetrahedral partitions

Larry L. Schumaker[a,*], Tatyana Sorokina[b], Andrew J. Worsey[c]

[a] *Department of Mathematics, Vanderbilt University, Nashville, TN 37240, United States*
[b] *Department of Mathematics, Towson University, Towson, MD 21252, United States*
[c] *Department of Mathematical Sciences, Middle Tennessee State University, Murfreesboro, TN 37132, United States*

Received 4 October 2007; received in revised form 24 January 2008; accepted 18 April 2008
Available online 31 October 2008

Communicated by C.K. Chui and H.N. Mhaskar

Dedicated to the memory of G.G. Lorentz

Abstract

In 1988, Worsey and Piper constructed a trivariate macro-element based on C^1 quadratic splines defined over a split of a tetrahedron into 24 subtetrahedra. However, this local element can only be used to construct a corresponding macro-element spline space over tetrahedral partitions that satisfy some very restrictive geometric constraints. We show that by further refining their split, it is possible to construct a macro-element also based on C^1 quadratic splines that can be used with arbitrary tetrahedral partitions. The resulting macro-element space is stable and provides full approximation power.

Keywords: Trivariate splines; Macro-elements

1. Introduction

Because of their usefulness for numerical computations (in particular for scattered data fitting and the numerical solution of boundary-value problems for PDE's), considerable effort has gone into the development of macro-element spaces based on piecewise polynomial (spline) spaces. The theory is especially well-developed in the bivariate setting, where the splines are defined on

* Corresponding author.
E-mail addresses: larry.schumaker@vanderbilt.edu (L.L. Schumaker), tsorokina@towson.edu (T. Sorokina), ajworsey@mtsu.edu (A.J. Worsey).

0021-9045/$ - see front matter © 2008 Elsevier Inc. All rights reserved.
doi:10.1016/j.jat.2008.04.014

triangulations. For a comprehensive treatment, a historical discussion, and extensive references, see [4]. Much less has been done in the trivariate setting, which is based on tetrahedral partitions, see [1,3,7–10], and the recent book [4].

The starting point for this paper is the construction by Worsey and Piper [8] of a trivariate macro-element based on C^1 quadratic splines. It is defined by splitting each tetrahedron in a given tetrahedral partition $\triangle$ into 24 subtetrahedra, and has $4n_V$ degrees of freedom, where n_V is the number of vertices of $\triangle$, see also Section. 18.5 of [4]. Unfortunately, their macro-element can only be used with a highly restrictive class of initial tetrahedral partitions, the so-called proper Worsey–Piper partitions, see Section 16.7.3 of [4]. It has been an open question for several years whether there exists a trivariate macro-element based on C^1 quadratic splines which can be used with arbitrary initial tetrahedral partitions. The purpose of this paper is to create such an element.

The key to removing the restrictions in [8] is to create a more complicated refinement of each tetrahedron in $\triangle$. This leads to a significant increase in the local complexity of the macro-elements, but the corresponding macro-element space still has a modest $4n_V+2n_E+4n_F$ degrees of freedom, where n_V, n_E, n_F are the numbers of vertices, edges, and faces in $\triangle$.

Removing the restrictions is of more than academic interest, since for an arbitrary tetrahedral partition, there is no known algorithm for creating a proper Worsey–Piper refinement. In contrast, our new macro-element space can be constructed over arbitrary tetrahedral partitions. It also has a stable local basis, and provides full approximation power of smooth functions. The space can be used to create C^1 quadratic interpolating splines for trivariate scattered data. Such splines are useful for contouring purposes in volume visualization, see [5].

The paper is organized as follows. In Section 2 we recall some useful concepts and notation from the Bernstein–Bézier theory of trivariate splines. Section 3 contains an algorithm to describe the split used to construct our macro-element. In Section 4 we construct a minimal determining set and compute the dimension of our basic macro-element space defined on the split of a single tetrahedron. In Section 5 we extend these results to a macro-element space defined on an appropriate refinement of an arbitrary initial tetrahedral partition. We also construct a stable minimal determining set, compute the dimension of the macro-element space, and show that the space has full approximation power. In Section 6 we use our macro-element space to solve a Hermite interpolation problem, and show that the resulting interpolant approximates smooth functions to optimal order. We conclude the paper with remarks and references.

2. Preliminaries

Let $\mathcal{P}_2$ be the ten-dimensional space of all trivariate polynomials of degree at most two, and let $\triangle_M$ be a tetrahedral partition of a polyhedral domain Ω in $\mathbb{R}^3$. Then we define

$$\mathcal{S}_2^1(\triangle_M) := \{s \in C^1(\Omega) : s|_T \in \mathcal{P}_2,\ \text{all } T \in \triangle_M\}. \tag{2.1}$$

In this paper we will work with partitions $\triangle_M$ that are obtained from an arbitrary tetrahedral partition by an appropriate refinement procedure to be described in the following section. To analyze this space of trivariate splines, we will use standard Bernstein–Bézier techniques as explained in detail in [4].

For convenience, in this section we recall a few basic ideas and some notation. Let $\mathcal{D}_{2,\triangle_M}$ be the set of all vertices of $\triangle_M$ together with the set of all midpoints of edges of $\triangle_M$. These are the so-called *domain points*. Then for any $s \in \mathcal{S}_2^1(\triangle_M)$ and any $T \in \triangle_M$, the polynomial $s|_T$ is uniquely determined by the set of ten B-coefficients associated with the domain points in $\mathcal{D}_{2,\triangle_M} \cap T$. If ξ is a domain point, we write $c(\xi)$ and $\mathcal{C}(\xi) := (\xi, c(\xi))$ for the corresponding coefficient and control point. In addition, if v_1, v_2 are two neighboring vertices of $\triangle_M$, we write $\xi(v_1, v_2) := (v_1+v_2)/2$, $c(v_1, v_2)$, and $\mathcal{C}(v_1, v_2)$ for the corresponding domain point, coefficient, and control point. Note that in this notation v_1 and v_2 are interchangeable.

It is known that the dimension of $\mathcal{S}_2^0(\triangle_M)$ is equal to the cardinality of $\mathcal{D}_{2,\triangle_M}$. To find the dimension of $\mathcal{S}_2^1(\triangle_M)$, we will construct a *minimal determining set*, i.e., a subset $\mathcal{M} \subset \mathcal{D}_{2,\triangle_M}$ such that if we set the B-coefficients of $s \in \mathcal{S}_2^1(\triangle_M)$ corresponding to every domain point in $\mathcal{M}$, then all remaining coefficients are uniquely determined in such a way that the corresponding spline belongs to $C^1(\Omega)$. Throughout the paper we will make heavy use of the fact that C^1 smoothness between two polynomial pieces of s is described by simple linear conditions on its B-coefficients, see [2] or Section 17.2 of [4]. Recall that if v is a vertex of $\triangle_M$, then the *ball* $B(v)$ (of radius 1) is defined to be the the set consisting of v together with the domain points located at the midpoints of all edges attached to v. We will also construct a *minimal nodal determining set* $\mathcal{N}$ for $\mathcal{S}_2^1(\triangle_M)$. It is defined in terms of linear functionals based on derivatives. For any multi-index $\alpha := (\alpha_1, \alpha_2, \alpha_3)$, we write $D^\alpha := D_x^{\alpha_1} D_y^{\alpha_2} D_z^{\alpha_3}$. Now suppose $\mathcal{N} = \{\lambda_i\}_{i=1}^n$ is a set of linear functionals of the form

$$\lambda_i := \varepsilon_{\xi_i} \sum_{|\alpha|\le 1} a_i^\alpha D^\alpha, \quad i = 1, \ldots, n,$$

where ε_{ξ_i} is point evaluation at the point ξ_i. Then λ_i is called a *nodal functional*, and ξ_i is called its *carrier*.

A set $\mathcal{N}$ of nodal functionals defined on $\mathcal{S}_2^1(\triangle_M)$ is called a *nodal determining set for* $\mathcal{S}_2^1(\triangle_M)$ provided that if $s \in \mathcal{S}_2^1(\triangle_M)$ and $\lambda s = 0$ for all $\lambda \in \mathcal{N}$, then $s \equiv 0$, see [4]. If there is no smaller nodal determining set for $\mathcal{S}_2^1(\triangle_M)$, then $\mathcal{N}$ is called a *nodal minimal determining set* for $\mathcal{S}_2^1(\triangle_M)$, and the dimension of $\mathcal{S}_2^1(\triangle_M)$ is given by the cardinality of $\mathcal{N}$. The linear functionals in $\mathcal{N}$ are called the *nodal degrees of freedom* of $\mathcal{S}$. If $\mathcal{N}$ is a nodal determining set for $\mathcal{S}_2^1(\triangle_M)$ such that for each tetrahedron $T \in \triangle$, the data $\{\lambda s\}_{\lambda\in\mathcal{N}_T}$ uniquely determine $s|_T$, where $\mathcal{N}_T := \{\lambda \in \mathcal{N} :$ the carrier of λ is contained in $T\}$, then $\mathcal{S}_2^1(\triangle_M)$ is called a *macro-element space*, see [4].

3. The split

In this section we present an algorithm for splitting a tetrahedron T into subtetrahedra in a way that allows the construction of our macro-element. First we need to introduce some additional points in T. To help understand their locations, see Figs. 1–8. Suppose $T := \langle v_1, v_2, v_3, v_4\rangle$ is a tetrahedron. For all distinct $i, j, k, l \in \mathbb{Z}_4 := \{1, 2, 3, 4\}$, let

$$\begin{aligned} w &:= (v_1 + v_2 + v_3 + v_4)/4,\\ u_i &:= (v_j + v_k + v_l)/3,\\ v_{ij} &:= (v_i + v_j)/2,\\ u_{ij} &:= (u_i + u_j)/2. \end{aligned} \tag{3.1}$$

Let Q^i be the convex hull of the points $\langle v_i, v_{ij}, v_{ik}, v_{il}, u_j, u_k, u_l, w\rangle$, see Fig. 1 (left). It is easy to see that $T = Q^1 \cup Q^2 \cup Q^3 \cup Q^4$. Let

$$\begin{aligned} p_i &:= (v_{ij} + v_{ik} + v_{il})/3,\\ q_i &:= (2p_i + u_j + u_k + u_l)/5,\\ r_i &:= (u_j + u_k + u_l)/3, \end{aligned} \tag{3.2}$$

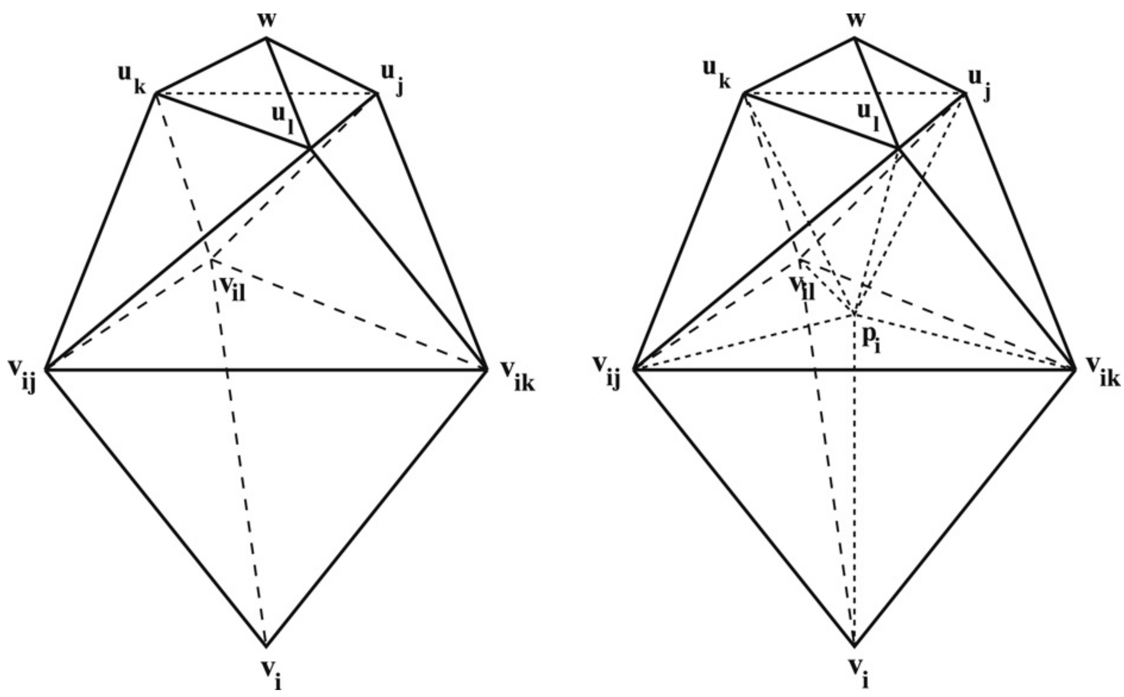

Fig. 1. The set Q^i and some tetrahedral subsets.

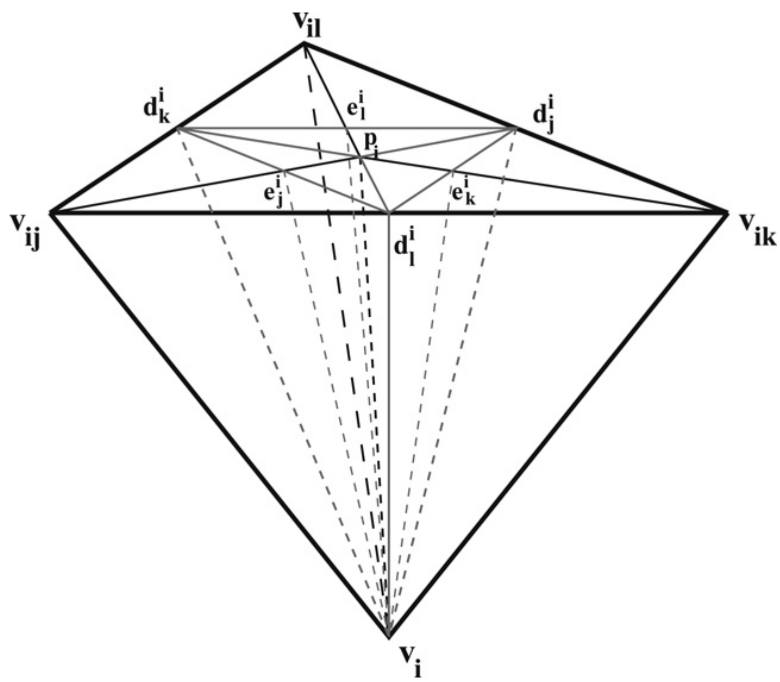

Fig. 2. The split of the tetrahedron T_1^i.

where j, k, l are the remaining integers in $\mathbb{Z}_4$. It is a simple exercise in algebra to show that p_i, q_i, r_i, w lie in the interior of the line segment $\langle v_i, u_i \rangle$, and

$$\begin{aligned} p_i &= (5q_i + v_i)/6, \qquad & q_i &= (2p_i + 3r_i)/5, \\ r_i &= (4w + 5q_i)/9, \qquad & w &= (u_i + 3r_i)/4. \end{aligned} \tag{3.3}$$

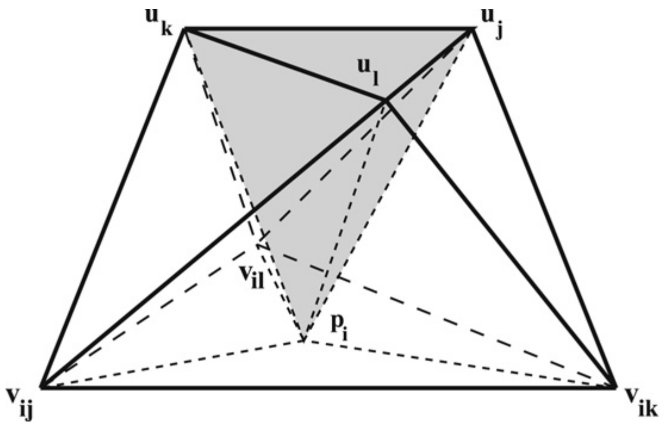

Fig. 3. The tetrahedron T_2^i (shaded).

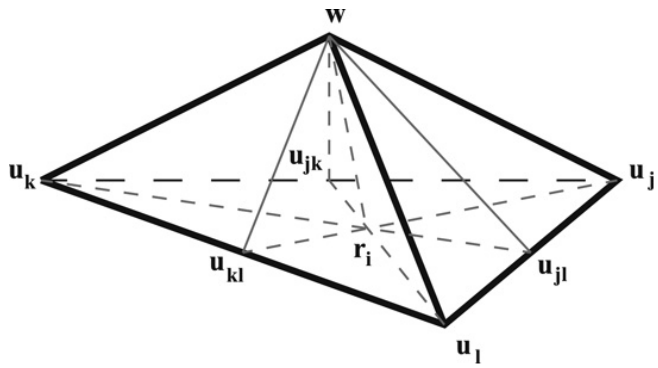

Fig. 4. The split of the tetrahedron T_3^i.

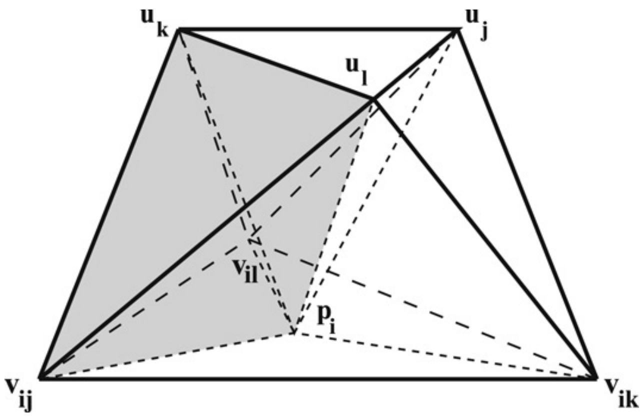

Fig. 5. The tetrahedron $T_{4,j}^i$ (shaded).

For any distinct $i, j, k, l \in \mathbb{Z}_4$, let

$$\begin{aligned}
d_l^i &:= (v_{ij} + v_{ik})/2 = (v_i + 3u_l)/4, \\
e_j^i &:= (d_k^i + d_l^i)/2 = (v_{ij} + 3p_i)/4, \\
b_l^i &:= (2p_i + u_l)/3 = (5q_i + 4d_l^i)/9, \\
z_j^i &:= (p_i + u_{kl})/2 = (5q_i + v_{ij})/6, \\
t_j^i &:= (4e_j^i + 3u_{kl})/7 = (6z_j^i + v_{ij})/7, \\
x_{j,l}^i &:= (3u_l + 8e_j^i)/11 = (9b_l^i + 2v_{ij})/11.
\end{aligned} \tag{3.4}$$

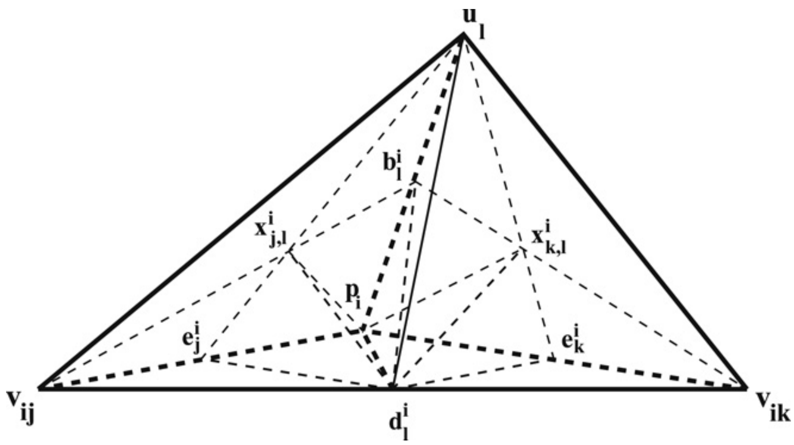

Fig. 6. The split of the tetrahedron $T^i_{5,l}$.

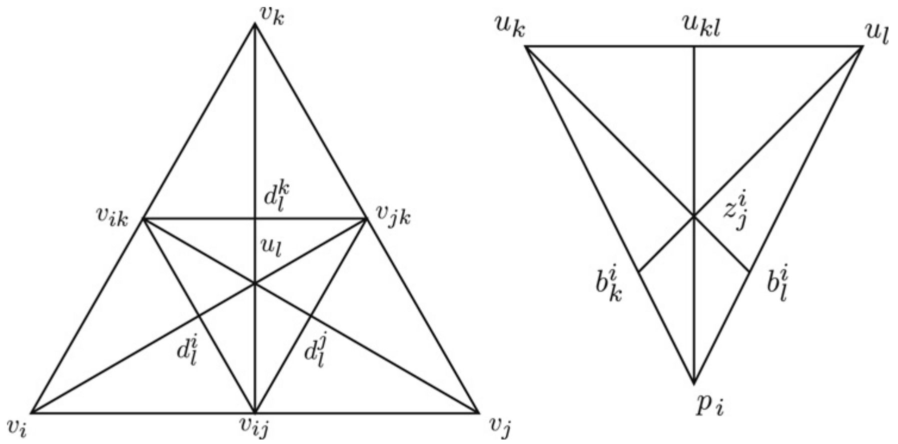

Fig. 7. Typical faces of T and $T^i_{4,j}$.

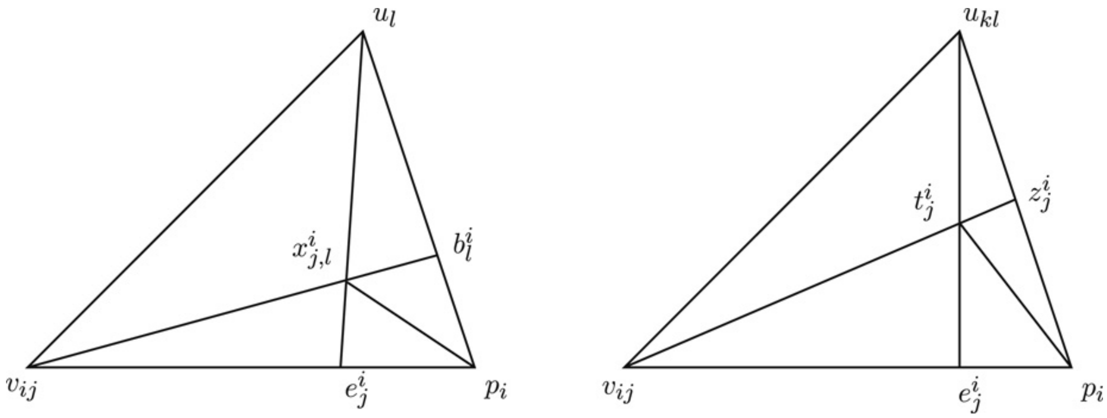

Fig. 8. A face of $T^i_{4,j}$ and a slice through t^i_j

Here we have written a comma in the subscript of $x^i_{j,l}$ to indicate that it depends on the order j, l. This is in contrast to points of the form v_{ij} and u_{ij} which are the same as v_{ji} and u_{ji}, respectively. Here the superscript i refers to the quarter Q^i in which these points lie. It is easy to see that the points q_i, t^i_j, z^i_j all lie on the line segment $\langle u_i, v_{ij}\rangle$, while the point $x^i_{j,l}$ lies on the line segment $\langle d^i_l, t^i_j\rangle$. In fact

$$z^i_j = (7t^i_j + 5q_i)/12, \qquad x^i_{j,l} = (4d^i_l + 7t^i_j)/11. \tag{3.5}$$

Finally, in preparation for describing our splitting process, we need some notation from bivariate spline theory. Suppose $F := \langle v_1, v_2, v_3 \rangle$ is a triangle, and that u_1, u_2, u_3 are points on its edges, where u_i is on the edge opposite v_i for $i = 1, 2, 3$. Given a point v in the interior of F, suppose we connect v to each of the points v_i and u_i. Then F is partitioned into six subtriangles, called the Powell–Sabin-6 (PS6) split of F, see Fig. 7 (right). If we add the lines connecting the u_i to each other, we get a partition of F into twelve triangles, called the Powell–Sabin-12 (PS12) split of F, see Fig. 7 (left). For more on these splits, see [4,6].

Algorithm 3.1. Given a tetrahedron T, for each $1 \le i \le 4$, perform the following steps:

(1) Consider the tetrahedron $T_1^i := \langle v_i, v_{ij}, v_{ik}, v_{il} \rangle$ shown in Fig. 2. Construct a PS12 split on $\langle v_{ij}, v_{ik}, v_{il} \rangle$ using the edge points d_j^i, d_k^i, d_l^i and the centroid p_i. The lines connecting the edge points intersect at the points e_j^i, e_k^i, e_l^i. Then connect v_i to p_i and to each of the d's and e's to split T_1^i into twelve subtetrahedra.

(2) Consider the tetrahedron $T_2^i := \langle p_i, u_j, u_k, u_l \rangle$ shown in Fig. 3. Construct a PS6 split on each of its faces using the edge points u_{jk}, u_{jl}, u_{kl} and b_j^i, b_k^i, b_l^i, and the centers z_j^i, z_k^i, z_l^i, r_i. Then connect q_i to all points in the faces of T_2^i to split T_2^i into 24 tetrahedra.

(3) Consider the tetrahedron $T_3^i := \langle w, u_j, u_k, u_l \rangle$ shown in Fig. 4. The face $\langle u_j, u_k, u_l \rangle$ has already been split into 6 triangles forming a PS6 split using the edge points u_{jk}, u_{jl}, u_{kl} and the centroid r_i. Now split T_3^i into six subtetrahedra by connecting w to r_i and to the three edge points.

(4) Consider the tetrahedron $T_{4,j}^i := \langle p_i, v_{ij}, u_k, u_l \rangle$ shown in Fig. 5. The face $\langle p_i, u_k, u_l \rangle$ has already been split into six triangles by connecting z_j^i to $p_i, b_l^i, u_l, u_{kl}, b_k^i$, see Fig. 7 (right). Now split the face $\langle p_i, u_l, v_{ij} \rangle$ into five triangles by connecting $x_{j,l}^i$ to the points $p_i, b_l^i, u_l, v_{ij}, e_j^i$, see Fig. 8 (left). Split the face $\langle p_i, v_{ij}, u_k \rangle$ into five triangles in a similar way. Split the face $\langle v_{ij}, u_k, u_l \rangle$ into two triangles by connecting v_{ij} to u_{kl}. Finally, connect the point t_j^i (which lies in the interior of $T_{4,j}^i$) to all points on the faces of $T_{4,j}$ to split it into 18 tetrahedra. Do the same for $T_{4,k}^i$ and $T_{4,l}^i$.

(5) Consider the tetrahedron $T_{5,l}^i := \langle u_l, v_{ij}, v_{ik}, p_i \rangle$ shown in Fig. 6. All of its faces are already split. In particular, the face $\langle p_i, v_{ij}, v_{ik} \rangle$ was split into four triangles by connecting d_l^i to e_k^i, p_i, e_j^i. The face $\langle p_i, v_{ij}, u_l \rangle$ was split into 5 triangles by connecting the point $x_{j,l}^i$ to $p_i, b_l^i, u_l, v_{ij}, e_j^i$. The face $\langle p_i, v_{ij}, u_l \rangle$ was split into 5 triangles using $x_{k,l}^i$. The face $\langle u_l, v_{ij}, v_{ik} \rangle$ was split into two triangles by connecting d_l^i to u_l. Now connect d_l^i to $x_{j,l}^i, b_l^i, x_{k,l}^i$ to partition $T_{5,l}^i$ into 10 subtetrahedra, all with a vertex at d_j^i. Do the same for $T_{5,j}^i$ and $T_{5,k}^i$.

Lemma 3.2. *The tetrahedral partition T_M of T produced by Algorithm 3.1 contains 26 boundary vertices, 91 interior vertices, 72 boundary edges, 572 interior edges, 48 boundary faces, 984 interior faces, and 504 tetrahedra.*

Proof. A simple count shows that steps (1)–(5) split each Q^i into $6+12+24+3\cdot 18+3\cdot 10 = 126$ tetrahedra, which shows that T is split into 504 tetrahedra. The other assertions can easily be checked by using the Java program described in Remark 1. The Euler relations connecting these numbers are described in Remark 2. □

We conclude this section by observing that Algorithm 3.1 produces a split with very special geometry:

(G1) For each edge of the form $e := \langle u_k, u_l\rangle$, there exists a plane containing all edges attached to u_{kl} not collinear with e. In particular, $v_{ij}, t^i_j, e^i_j, z^i_j, p_i, q_i, r_i$ and w are all in this plane.
(G2) For each edge of the form $e := \langle p_i, u_l\rangle$, there exists a plane containing all edges attached to b^i_l not collinear with e. In particular, $z^i_j, t^i_j, x^i_{j,l}, d^i_l, x^i_{k,l}, z^i_k, t^i_k$ and q_i are all in this plane.

4. The Macro-element

Given an arbitrary tetrahedron $T := \langle v_1, v_2, v_3, v_4\rangle$, let T_M be the tetrahedral partition of T in Lemma 3.2. Let $\mathcal{S}^1_2(T_M)$ be the corresponding trivariate spline space defined in (2.1). We now describe a minimal determining set for $\mathcal{S}^1_2(T_M)$. Throughout the paper, whenever we work with integers i, j, k, l in $\mathbb{Z}_4$, we assume they are distinct. Let

$$\mathcal{M}_V := \bigcup_{i=1}^{4}\{\xi(v_i), \xi(v_i, v_{ij}), \xi(v_i, v_{ik}), \xi(v_i, v_{il}) : j, k, l \in \mathbb{Z}_4\},$$

$$\mathcal{M}_E := \bigcup_{i=1}^{3}\bigcup_{j=i+1}^{4}\{\xi(v_{ij}, u_k), \xi(v_{ij}, u_l) : k, l \in \mathbb{Z}_4\},$$

$$\mathcal{M}_F := \bigcup_{i=1}^{4}\{\xi(u_i, w), \xi(d^j_i, p_j), \xi(d^k_i, p_k), \xi(d^l_i, p_l) : j, k, l \in \mathbb{Z}_4\}.$$

Theorem 4.1. *The set $\mathcal{M} := \mathcal{M}_V \cup \mathcal{M}_E \cup \mathcal{M}_F$ is a minimal determining set for $\mathcal{S}^1_2(T_M)$, and* $\dim \mathcal{S}^1_2(T_M) = 44$.

Proof. To show that $\mathcal{M}$ is a minimal determining set, suppose we fix the coefficients of a quadratic spline s defined on T_M corresponding to the domain points in $\mathcal{M}$. We then divide the vertices of T_M into 14 types, and in a series of 14 steps examine all vertices of each given type. At each step we deal with the balls (of radius 1) around vertices of the given type. For each such vertex v, we choose a set $\mathcal{B}(v)$ of four control points that have already been set or determined in earlier steps, and whose domain points lie in $B(v)$. If additional control points with domain points in $B(v)$ have been set or determined in earlier steps, we explicitly verify that they lie in the three-dimensional affine subspace in $\mathbb{R}^4$ spanned by $\mathcal{B}(v)$. Finally, we give explicit formulae for the coefficients of s associated with the remaining domain points in the ball $B(v)$, and show that the corresponding control points lie in the three-dimensional affine subspace in $\mathbb{R}^4$ spanned by $\mathcal{B}(v)$. These formulae come directly from smoothness conditions, making use of the geometry of the partition. For all $i, j, k, l \in \mathbb{Z}_4$, carry out the following steps:
Step 1: For each v_i, let

$$\mathcal{B}(v_i) := \{\mathcal{C}(v_i), \mathcal{C}(v_i, v_{ij}), \mathcal{C}(v_i, v_{ik}), \mathcal{C}(v_i, v_{il})\}.$$

These control points correspond to domain points in $\mathcal{M}$, and thus have already been fixed. By the geometry (see Fig. 2) and the formulae in (3.2) and (3.4),

$$\begin{aligned}\xi(v_i, d^i_l) &= [\xi(v_i, v_{ij}) + \xi(v_i, v_{ik})]/2,\\ \xi(v_i, e^i_j) &= [\xi(v_i, d^i_k) + \xi(v_i, d^i_l)]/2, \qquad (4.1)\\ \xi(v_i, p_i) &= [\xi(v_i, v_{ij}) + \xi(v_i, v_{ik}) + \xi(v_i, v_{il})]/3.\end{aligned}$$

We now define the coefficients of s associated with the domain points on the left in (4.1) by simply replacing ξ by c in these formulae. Thus, for example, we take $c(v_i, d_l^i) := [c(v_i, v_{ij}) + c(v_i, v_{ik})]/2$. This insures that the corresponding control points lie in the span of $\mathcal{B}(v_i)$, and so we have C^1 smoothness at v_i.

Step 2. For each v_{ij}, let

$$\mathcal{B}(v_{ij}) := \{\mathcal{C}(v_{ij}, v_i), \mathcal{C}(v_{ij}, v_j), \mathcal{C}(v_{ij}, u_k), \mathcal{C}(v_{ij}, u_l)\}.$$

These control points have been fixed at the outset. By the geometry (see Figs. 2, 7 and 8) and the formulae in (3.1) and (3.4),

$$\begin{aligned}
\xi(v_{ij}) &= [\xi(v_{ij}, v_i) + \xi(v_{ij}, v_j)]/2,\\
\xi(v_{ij}, d_l^i) &= [\xi(v_{ij}, v_i) + 3\xi(v_{ij}, u_l)]/4,\\
\xi(v_{ij}, e_j^i) &= [\xi(v_{ij}, d_k^i) + \xi(v_{ij}, d_l^i)]/2,\\
\xi(v_{ij}, x_{j,l}^i) &= [3\xi(v_{ij}, u_l) + 8\xi(v_{ij}, e_j^i)]/11,\\
\xi(v_{ij}, u_{kl}) &= [\xi(v_{ij}, u_k) + \xi(v_{ij}, u_l)]/2,\\
\xi(v_{ij}, t_j^i) &= [4\xi(v_{ij}, e_j^i) + 3\xi(v_{ij}, u_{kl})]/7.
\end{aligned} \tag{4.2}$$

We now define coefficients corresponding to the domain points on the left of these formulae by replacing ξ with c. This gives control points that lie in the span of $\mathcal{B}(v_{ij})$, and we have C^1 smoothness at v_{ij}.

Step 3. For each d_l^i, let

$$\mathcal{B}(d_l^i) := \{\mathcal{C}(d_l^i, v_i), \mathcal{C}(d_l^i, v_{ij}), \mathcal{C}(d_l^i, v_{ik}), \mathcal{C}(d_l^i, p_i)\}.$$

The first of these control points was determined in Step 1, the second and third in Step 2, and the last was fixed. By the geometry (see Figs. 2 and 6) and the formulae in (3.4),

$$\begin{aligned}
\xi(d_l^i) &= [\xi(d_l^i, v_{ij}) + \xi(d_l^i, v_{ik})]/2,\\
\xi(d_l^i, e_j^i) &= [\xi(d_l^i, v_{ij}) + 3\xi(d_l^i, p_i)]/4,\\
\xi(d_l^i, u_l) &= [4\xi(d_l^i) - \xi(d_l^i, v_i)]/3,\\
\xi(d_l^i, b_l^i) &= [2\xi(d_l^i, p_i) + \xi(d_l^i, u_l)]/3,\\
\xi(d_l^i, x_{j,l}^i) &= [9\xi(d_l^i, b_l^i) + 2\xi(d_l^i, v_{ij})]/11.
\end{aligned} \tag{4.3}$$

Defining the corresponding coefficients by the analogous formula with ξ replaced by c, we see that the corresponding control points lie in the span of $\mathcal{B}(d_l^i)$, and we have C^1 smoothness at d_l^i.

Step 4. For each e_j^i, let

$$\mathcal{B}(e_j^i) := \{\mathcal{C}(e_j^i, v_i), \mathcal{C}(e_j^i, v_{ij}), \mathcal{C}(e_j^i, d_k^i), \mathcal{C}(e_j^i, d_l^i)\}.$$

These control points were computed in Steps 1, 2, and 3. By the geometry (see Figs. 2 and 6) and the formulae in (3.4),

$$\begin{aligned}
\xi(e_j^i) &= [\xi(e_j^i, d_k^i) + \xi(e_j^i, d_l^i)]/2,\\
\xi(e_j^i, p_i) &= [4\xi(e_j^i) - \xi(e_j^i, v_{ij})]/3,\\
\xi(e_j^i, x_{j,l}^i) &= [8\xi(e_j^i) + 4\xi(e_j^i, d_l^i) - \xi(e_j^i, v_i)]/11,\\
\xi(e_j^i, t_j^i) &= [8\xi(e_j^i) - \xi(e_j^i, v_i)]/7.
\end{aligned} \tag{4.4}$$

It follows that the control points corresponding to coefficients satisfying the analogous formulae with ξ replaced by c lie in the span of $\mathcal{B}(e^i_j)$, and we have C^1 smoothness at e^i_j.

Step 5. For each p_i, let

$$\mathcal{B}(p_i) := \{\mathcal{C}(p_i, v_i), \mathcal{C}(p_i, d^i_j), \mathcal{C}(p_i, d^i_k), \mathcal{C}(p_i, d^i_l)\}.$$

The first of these was computed in Step 1, while the others were set. In this case not all coefficients corresponding to the remaining domain points in $B(p_i)$ are free. In particular, the coefficients $c(p_i, e^i_j), c(p_i, e^i_k), c(p_i, e^i_l)$ were already computed in Step 4. Thus, we have to verify that the corresponding control points lie in the span of $\mathcal{B}(p_i)$ in order to be sure there is no inconsistency. We check one. Taking appropriate combinations of previously defined control points gives

$$\mathcal{C}(p_i, e^i_j) = [\mathcal{C}(p_i, d^i_k) + \mathcal{C}(p_i, d^i_l)]/2,$$

which shows that $\mathcal{C}(e^i_j, p_i)$ lies in the span of $\mathcal{B}(p_i)$. By the geometry (see Figs. 6–8),

$$\begin{aligned}
\xi(p_i) &= [\xi(p_i, d^i_j) + \xi(p_i, d^i_k) + \xi(p_i, d^i_l)]/3,\\
\xi(p_i, b^i_l) &= [4\xi(p_i, d^i_l) + 6\xi(p_i) - \xi(p_i, v_i)]/9,\\
\xi(p_i, x^i_{j,l}) &= [8\xi(p_i, e^i_j) + 9\xi(p_i, b^i_l) - 6\xi(p_i)]/11,\\
\xi(p_i, z^i_j) &= [3\xi(p_i, b^i_k) + 3\xi(p_i, b^i_l) - 2\xi(p_i)]/4,\\
\xi(p_i, q_i) &= [6\xi(p_i) - \xi(p_i, v_i)]/5,\\
\xi(p_i, t^i_j) &= [12\xi(p_i, z^i_j) - 5\xi(p_i, q_i)]/7.
\end{aligned} \tag{4.5}$$

It follows that the control points associated with coefficients satisfying the analogous formulae lie in the span of $\mathcal{B}(p_i)$, and we have C^1 smoothness at p_i.

Step 6. For each u_l, let

$$\mathcal{B}(u_l) := \{\mathcal{C}(u_l, v_{ij}), \mathcal{C}(u_l, v_{ik}), \mathcal{C}(u_l, v_{jk}), \mathcal{C}(u_l, w)\}.$$

The first three of these control points were computed in Step 2, while the last was fixed at the outset. The coefficients $c(u_l, d^i_l), c(u_l, d^j_l), c(u_l, d^k_l)$ were computed in Step 3. We must verify that the corresponding control points lie in the span of $\mathcal{B}(u_l)$. We check just one. Taking appropriate combinations of previously defined control points gives

$$\mathcal{C}(u_l, d^i_l) = [\mathcal{C}(u_l, v_{ij}) + \mathcal{C}(u_l, v_{ik})]/2.$$

By the geometry (see Figs. 6–8),

$$\begin{aligned}
\xi(u_l) &= [\xi(u_l, v_{ij}) + \xi(u_l, v_{jk}) + \xi(u_l, v_{ik})]/3,\\
\xi(u_l, u_{kl}) &= [2\xi(u_l, w) + \xi(u_l, v_{ij})]/3,\\
\xi(u_l, r_i) &= [2\xi(u_l, u_{jl}) + 2\xi(u_l, u_{kl}) - \xi(u_l)]/3,\\
\xi(u_l, q_i) &= [9\xi(u_l, r_i) - 4\xi(u_l, w)]/5,\\
\xi(u_l, b^i_l) &= [5\xi(u_l, q_i) + 4\xi(u_l, d^i_l)]/9,\\
\xi(u_l, z^i_j) &= [9\xi(u_l, b^i_l) + 6\xi(u_l, u_{kl}) - 3\xi(u_l)]/12,\\
\xi(u_l, t^i_j) &= [6\xi(u_l, z^i_j) + \xi(u_l, v_{ij})]/7,\\
\xi(u_l, x^i_{j,l}) &= [7\xi(u_l, t^i_j) + 4\xi(u_l, d^i_l)]/11.
\end{aligned} \tag{4.6}$$

Using these formulae to define coefficients associated with the domain points on the left in (4.6), we get control points that lie in the span of $\mathcal{B}(u_l)$, and we have C^1 smoothness at u_l.

Step 7. For w, let

$$\mathcal{B}(w) := \{\mathcal{C}(w, u_1), \mathcal{C}(w, u_2), \mathcal{C}(w, u_3), \mathcal{C}(w, u_4)\}.$$

These control points were fixed at the outset. By the geometry (see Fig. 4),

$$\begin{aligned}
\xi(w) &= [\xi(w, u_1) + \xi(w, u_2) + \xi(w, u_3) + \xi(w, u_4)]/4,\\
\xi(w, r_i) &= [\xi(w, u_j) + \xi(w, u_k) + \xi(w, u_l)]/3,\\
\xi(w, u_{kl}) &= [\xi(w, u_k) + \xi(w, u_l)]/2.
\end{aligned} \tag{4.7}$$

Defining the corresponding coefficients by the analogous formula with ξ replaced by c, it follows that the associated control points lie in the span of $\mathcal{B}(w)$, and we have C^1 smoothness at w.

Step 8. For each r_i, let

$$\mathcal{B}(r_i) := \{\mathcal{C}(r_i, u_j), \mathcal{C}(r_i, u_k), \mathcal{C}(r_i, u_l), \mathcal{C}(r_i, w)\}.$$

These control points were determined in Steps 6 and 7. By the geometry (see Fig. 4),

$$\begin{aligned}
\xi(r_i) &= [\xi(r_i, u_j) + \xi(r_i, u_k) + \xi(r_i, u_l)]/3,\\
\xi(r_i, u_{kl}) &= [\xi(r_i, u_k) + \xi(r_i, u_l)]/2,\\
\xi(r_i, q_i) &= [9\xi(r_i) - 4\xi(r_i, w)]/5.
\end{aligned} \tag{4.8}$$

We now choose coefficients associated with the domain points on the left in these formulae by replacing ξ by c. By construction the corresponding control points lie in the span of $\mathcal{B}(r_i)$, and we have C^1 smoothness at r_i.

Step 9. For each u_{kl}, let

$$\mathcal{B}(u_{kl}) := \{\mathcal{C}(u_{kl}, u_k), \mathcal{C}(u_{kl}, u_l), \mathcal{C}(u_{kl}, r_i), \mathcal{C}(u_{kl}, r_j)\}.$$

The first two of these control points were computed in Step 6, while the last two were determined in Step 8. By the geometry,

$$\xi(u_{kl}) = [\xi(u_{kl}, u_k) + \xi(u_{kl}, u_l)]/2, \tag{4.9}$$

so we can set $c(u_{kl}) := [c(u_{kl}, u_k) + c(u_{kl}, u_l)]/2$. The coefficient $c(u_{kl}, w)$ was determined in Step 7. However, taking appropriate combinations of previously defined control points gives

$$\mathcal{C}(u_{kl}, w) = [3\mathcal{C}(u_{kl}, r_i) + 3\mathcal{C}(u_{kl}, r_j) - 2\mathcal{C}(u_{kl})]/4,$$

and so there is no inconsistency. Similarly, the coefficient $c(u_{kl}, v_{ij})$ was determined in Step 2, but there is no inconsistency since

$$\mathcal{C}(u_{kl}, v_{ij}) = 3\mathcal{C}(u_{kl}) - 2\mathcal{C}(u_{kl}, w).$$

By the geometry (see Fig. 8) and the formulae in (3.3),

$$\begin{aligned}
\xi(u_{kl}, q_i) &= [9\xi(u_{kl}, r_i) - 4\xi(u_{kl}, w)]/5,\\
\xi(u_{kl}, z_j^i) &= [5\xi(u_{kl}, q_i) - 3\xi(u_{kl}, r_i) + 2\xi(u_{kl})]/4,\\
\xi(u_{kl}, t_j^i) &= [6\xi(u_{kl}, z_j^i) + \xi(u_{kl}, v_{ij})]/7.
\end{aligned} \tag{4.10}$$

Replacing the ξ by c in these formulae, we see that the associated control points lie in the span of $\mathcal{B}(u_{kl})$, and we have C^1 smoothness at u_{kl}.

Step 10. For each q_i, let

$$\mathcal{B}(q_i) := \{\mathcal{C}(q_i, u_j), \mathcal{C}(q_i, u_k), \mathcal{C}(q_i, u_l), \mathcal{C}(q_i, p_i)\}.$$

These were determined in Steps 5 and 6. Two other coefficients associated with domain points in $B(q_i)$ were already determined, namely $c(q_i, r_i)$ in Step 8, and $c(q_i, u_{kl})$ in Step 9.
Taking appropriate combinations of previously defined control points gives

$$\mathcal{C}(q_i, r_i) = [\mathcal{C}(q_i, u_j) + \mathcal{C}(q_i, u_k) + \mathcal{C}(q_i, u_l)]/3,$$

and

$$\mathcal{C}(q_i, u_{kl}) = [\mathcal{C}(q_i, u_k) + \mathcal{C}(q_i, u_l)]/2,$$

which insure that there are no inconsistencies. By the geometry (see Fig. 8) and the formulae in (3.2),

$$\begin{aligned} \xi(q_i) &= [2\xi(q_i, p_i) + \xi(q_i, u_j) + \xi(q_i, u_k) + \xi(q_i, u_l)]/5, \\ \xi(q_i, b_l^i) &= [2\xi(q_i, p_i) + \xi(q_i, u_l)]/3, \\ \xi(q_i, z_j^i) &= [2\xi(q_i, p_i) + \xi(q_i, u_k) + \xi(q_i, u_l)]/4. \end{aligned} \tag{4.11}$$

It follows that the control points corresponding to coefficients satisfying the analogous formulae with ξ replaced by c lie in the span of $\mathcal{B}(q_i)$, and we have C^1 smoothness at q_i.
Step 11. For each t_j^i, let

$$\mathcal{B}(t_j^i) := \{\mathcal{C}(t_j^i, v_{ij}), \mathcal{C}(t_j^i, p_i), \mathcal{C}(t_j^i, u_k), \mathcal{C}(t_j^i, u_l)\}.$$

These were determined in Steps 2, 5, and 6. The coefficients $c(t_j^i, e_j^i)$ and $c(t_j^i, u_{kl})$ were previously computed in Steps 4 and 9. But there is no inconsistency since

$$\mathcal{C}(t_j^i, e_j^i) = [3\mathcal{C}(t_j^i, p_i) + \mathcal{C}(t_j^i, v_{ij})]/4,$$

and

$$\mathcal{C}(t_j^i, u_{kl}) = [\mathcal{C}(t_j^i, u_k) + \mathcal{C}(t_j^i, u_l)]/2.$$

By the geometry (see Figs. 7 and 8),

$$\begin{aligned} \xi(t_j^i) &= [4\xi(t_j^i, e_j^i) + 3\xi(t_j^i, u_{kl})]/7, \\ \xi(t_j^i, z_j^i) &= [2\xi(t_j^i, p_i) + \xi(t_j^i, u_k) + \xi(t_j^i, u_l)]/4, \\ \xi(t_j^i, b_l^i) &= [2\xi(t_j^i, p_i) + \xi(t_j^i, u_l)]/3, \\ \xi(t_j^i, x_{j,l}^i) &= [9\xi(t_j^i, b_l^i) + 2\xi(t_j^i, v_{ij})]/11. \end{aligned} \tag{4.12}$$

Defining the corresponding coefficients by the analogous formula with ξ replaced by c, it follows that the associated control points lie in the span of $\mathcal{B}(t_j^i)$, and we have C^1 smoothness at t_j^i.
Step 12. For each z_j^i, let

$$\mathcal{B}(z_j^i) := \{\mathcal{C}(z_j^i, q_i), \mathcal{C}(z_j^i, p_i), \mathcal{C}(z_j^i, u_k), \mathcal{C}(z_j^i, u_l)\}.$$

These were determined in Steps 5, 6, and 10. The coefficients $c(z_j^i, u_{kl})$ and $c(z_j^i, t_j^i)$ were previously determined in Steps 9 and 11. But there is no inconsistency, since

$$\mathcal{C}(z_j^i, u_{kl}) = [\mathcal{C}(z_j^i, u_k) + \mathcal{C}(z_j^i, u_l)]/2,$$

and

$$\mathcal{C}(z_j^i, t_j^i) = [-5\mathcal{C}(z_j^i, q_i) + 6\mathcal{C}(z_j^i, p_i) + 3\mathcal{C}(z_j^i, u_k) + 3\mathcal{C}(z_j^i, u_l)]/7.$$

By the geometry (see Figs. 7 and 8),

$$\begin{aligned} \xi(z_j^i) &= [2\xi(z_j^i, p_i) + \xi(z_j^i, u_k) + \xi(z_j^i, u_l)]/4, \\ \xi(z_j^i, b_l^i) &= [2\xi(z_j^i, p_i) + \xi(z_j^i, u_l)]/3. \end{aligned} \tag{4.13}$$

For each domain point on the left in (4.13), we now define the corresponding coefficient by the analogous formula with ξ replaced by c, and it follows that the associated control points lie in the span of $\mathcal{B}(t_j^i)$, and we have C^1 smoothness at t_j^i.

Step 13. For each $x_{j,l}^i$, let

$$\mathcal{B}(x_{j,l}^i) := \{\mathcal{C}(x_{j,l}^i, v_{ij}), \mathcal{C}(x_{j,l}^i, p_i), \mathcal{C}(x_{j,l}^i, u_l), \mathcal{C}(x_{j,l}^i, t_j^i)\}.$$

These were determined in Steps 2, 5, 6, and 11. The coefficient $c(x_{j,l}^i, e_j^i)$ was computed in Step 4, but there is no inconsistency since

$$\mathcal{C}(x_{j,l}^i, e_j^i) = [\mathcal{C}(x_{j,l}^i, v_{ij}) + 3\mathcal{C}(x_{j,l}^i, p_i)]/4.$$

Now by the geometry,

$$\begin{aligned} \xi(x_{j,l}^i) &= [3\xi(x_{j,l}^i, u_l) + 8\xi(x_{j,l}^i, e_j^i)]/11, \\ \xi(x_{j,l}^i, b_l^i) &= [2\xi(x_{j,l}^i, p_i) + \xi(x_{j,l}^i, u_l)]/3, \end{aligned} \tag{4.14}$$

and we can compute the corresponding coefficients with the same formulae. The coefficient $c(x_{j,l}^i, d_l^i)$ was computed in Step 3, but there is no inconsistency since

$$\mathcal{C}(x_{j,l}^i, d_l^i) = [11\mathcal{C}(x_{j,l}^i) - 7\mathcal{C}(x_{j,l}^i, t_j^i)]/4.$$

We have shown that s is C^1 at $x_{j,l}^i$.

Step 14. For each b_l^i, let

$$\mathcal{B}(b_l^i) := \{\mathcal{C}(b_l^i, p_i), \mathcal{C}(b_l^i, u_l), \mathcal{C}(b_l^i, z_j^i), \mathcal{C}(b_l^i, z_k^i)\}.$$

These were determined in Steps 5, 6, and 12. By the geometry,

$$\xi(b_l^i) = [2\xi(b_l^i, p_i) + \xi(b_l^i, u_l)]/3,$$

and we can set $c(b_l^i) := [2c(b_l^i, p_i) + c(b_l^i, u_l)]/3$. Note that the coefficients $c(b_l^i, d_l^i)$, $c(b_l^i, q_i)$, $c(b_l^i, t_j^i)$, $c(b_l^i, t_k^i)$, $c(b_l^i, x_{j,k}^i)$ and $c(b_l^i, x_{j,l}^i)$ were computed in Steps 3, 10, 11, and 13, respectively. But these lead to no inconsistencies, since

$$\begin{aligned} \mathcal{C}(b_l^i, d_l^i) &= [9\mathcal{C}(b_l^i) - 3\mathcal{C}(b_l^i, z_j^i) - 3\mathcal{C}(b_l^i, z_k^i)]/3, \\ \mathcal{C}(b_l^i, q_i) &= [-9\mathcal{C}(b_l^i) + 12\mathcal{C}(b_l^i, z_j^i) + 12\mathcal{C}(b_l^i, z_k^i)]/15, \\ \mathcal{C}(b_l^i, t_j^i) &= [-5\mathcal{C}(b_l^i, q_i) + 12\mathcal{C}(b_l^i, z_j^i)]/7, \\ \mathcal{C}(b_l^i, x_{j,l}^i) &= [7\mathcal{C}(b_l^i, t_j^i) + 4\mathcal{C}(b_l^i, d_l^i)]/11. \end{aligned}$$

We have shown that s is C^1 at b_l^i.

We have now computed all coefficients of s, and have shown that s is C^1 at every vertex of T_M. It follows that $\mathcal{M}$ is a minimal determining set for $\mathcal{S}_2^1(T_M)$, and the dimension of $\mathcal{S}_2^1(T_M)$ is just the cardinality of $\mathcal{M}$, which is easily seen to be 44. □

We now show how to use the space $\mathcal{S}_2^1(T_M)$ to solve a simple Hermite interpolation problem involving 44 pieces of data at vertices, at midpoints of edges, and at four points on each face of T. Given $f \in C^1(T)$, let $D_{v,u} f$ be the directional derivative of f in the direction from v to u.

Theorem 4.2. *Suppose $f \in C^1(T)$. Then there exists a unique $s \in \mathcal{S}_2^1(T_M)$ such that for all distinct $i, j, k, l \in \mathbb{Z}_4$,*

(1) $s(v_i) = f(v_i)$,
(2) $D_{v_i,v_{ij}} s(v_i) = D_{v_i,v_{ij}} f(v_i)$,
(3) $D_{v_{ij},u_k} s(v_{ij}) = D_{v_{ij},u_k} f(v_{ij})$,
(4) $D_{d_j^i,p_i} s(d_j^i) = D_{d_j^i,p_i} f(d_j^i)$,
(5) $D_{u_i,w} s(u_i) = D_{u_i,w} f(u_i)$.

Proof. First we set $c(v_i) := f(v_i)$ and

$$c(v_i, v_{ij}) := f(v_i) + D_{v_i,v_{ij}} f(v_i)/2, \quad j \in \mathbb{Z}_4 \setminus \{i\}.$$

Next for each vertex v_{ij}, we use the formula in Step 2 of the proof of Theorem 4.1 to compute $c(v_{ij})$, and set

$$c(v_{ij}, u_k) := c(v_{ij}) + D_{v_{ij},u_k} f(v_{ij})/2, \quad k \in \mathbb{Z}_4 \setminus \{i, j\}.$$

For each vertex d_j^i, we use the formula in Step 3 of the proof of Theorem 4.1 to compute $c(d_j^i)$ and set

$$c(d_j^i, p_i) := c(d_j^i) + D_{d_j^i,p_i} f(d_j^i)/2.$$

Finally, for each vertex u_i, we use the formula in Step 6 of the proof of Theorem 4.1 to compute $c(u_i)$, and set

$$c(u_i, w) := c(u_i) + D_{u_i,w} f(u_i)/2.$$

At this point we have computed all coefficients of s corresponding to the minimal determining set $\mathcal{M}$ of Theorem 4.1 (and a few others using the C^1 smoothness conditions). But then by the theorem all other coefficients are determined. □

5. The macro-element space

Given an arbitrary tetrahedral partition $\triangle$ of a polyhedral domain Ω, let $\triangle_M$ be the result of applying the splitting process of Section 3 to each tetrahedron T of $\triangle$. Let $\mathcal{S}_2^1(\triangle_M)$ be the corresponding trivariate spline space defined in (2.1). In this section we construct a minimal determining set for $\mathcal{S}_2^1(\triangle_M)$ and use it to show that the space has full approximation power.

For each vertex v of $\triangle$, choose a tetrahedron $T_v := \langle v_1, v_2, v_3, v_4\rangle$ in $\triangle$ such that $v = v_1$, and let $\mathcal{M}_v := \{\xi(v_1), \xi(v_1, v_{12}), \xi(v_1, v_{13}), \xi(v_1, v_{14})\}$. For each edge e of $\triangle$, choose a tetrahedron $T_e := \langle v_1, v_2, v_3, v_4\rangle$ in $\triangle$ such that $e = \langle v_1, v_2\rangle$, and let $\mathcal{M}_e := \{\xi(v_{12}, u_3), \xi(v_{12}, u_4)\}$. Finally, for each face F of $\triangle$, choose a tetrahedron $T_F := \langle v_1, v_2, v_3, v_4\rangle$ in $\triangle$ with $F := \langle v_1, v_2, v_3\rangle$ and let $\mathcal{M}_F := \{\xi(u_4, w), \xi(d_4^1, p_1), \xi(d_4^2, p_2), \xi(d_4^3, p_3)\}$. Let $\mathcal{V}, \mathcal{E}, \mathcal{F}$ be the sets of all vertices, edges, and faces of $\triangle$. Let n_V, n_E, n_F be the cardinalities of these sets.

Theorem 5.1. *The set*

$$\mathcal{M} := \bigcup_{v\in\mathcal{V}} \mathcal{M}_v \cup \bigcup_{e\in\mathcal{E}} \mathcal{M}_e \cup \bigcup_{F\in\mathcal{F}} \mathcal{M}_F$$

is a minimal determining set for $\mathcal{S}_2^1(\triangle_M)$. Moreover,

$$\dim \mathcal{S}_2^1(\triangle_M) = 4n_V + 2n_E + 4n_F. \tag{5.1}$$

Proof. First we show that $\mathcal{M}$ is a determining set. Suppose we set the coefficients of $s \in \mathcal{S}_2^1(\triangle_M)$ corresponding to all domain points in $\mathcal{M}$. For each vertex v, this amounts to setting four coefficients associated with domain points in the ball $B(v)$. But then C^1 smoothness conditions can be used to compute all other coefficients of s corresponding to domain points in the ball $B(v)$. Next consider an edge $e := \langle v_1, v_2\rangle$ of $\triangle$ with midpoint v_{12}. We already know the coefficients $c(v_1, v_{12})$ and $c(v_2, v_{12})$, and by C^1 smoothness, we can now compute $c(v_{12}) = (c(v_1, v_{12}) + c(v_2, v_{12}))/2$. By the definition of $\mathcal{M}$, we have set $c(v_{12}, u_3)$ and $c(v_{12}, u_4)$ for some tetrahedron $T_e := \langle v_1, v_2, v_3, v_4\rangle$. But then using the C^1 smoothness conditions we can compute all other coefficients of s corresponding to domain points in the ball $B(v_{12})$. We repeat this process for every edge of $\triangle$.

Now let $F := \langle v_1, v_2, v_3\rangle$ be a face of $\triangle$, and let T_F be the tetrahedron appearing in the definition of $\mathcal{M}_F$. Consider the three vertices of the form d_j^i that lie in F. Using the first formula in Step 6 of the proof of Theorem 4.1, we can compute $c(u_4)$, where u_4 is the centroid of the face F. We have already computed the coefficients $c(u_4, v_{12})$ and $c(u_4, v_{13})$ in the previous step. By the definition of $\mathcal{M}_F$, we have set $c(u_4, w_F)$, where w_F is the centroid of the tetrahedron T_F. From these four coefficients, we can now use C^1 smoothness to compute all coefficients of s corresponding to the remaining domain points in $B(u_4)$. Similarly, for each of the points d_4^1, d_4^2, d_4^3 on F we can compute all coefficients in the balls $B(d_4^i)$ for $i = 1, 2, 3$.

At this point we have computed enough coefficients to be able to apply Theorem 4.1 to each tetrahedron in $\triangle$, and by the theorem all remaining coefficients of s are determined. This proves that $\mathcal{M}$ is a determining set. To show that is a minimal determining set, we need only show that s is C^1 at every vertex of $\triangle_M$. We have already shown this for vertices of $\triangle$, the midpoints of edges of $\triangle$, the centroids u_F of each face, and the points d_4^1, d_4^2, d_4^3 on each face. Any other vertex v of $\triangle_M$ is inside a tetrahedron $T \in \triangle$, and Theorem 4.1 guarantees that s is C^1 at v.

To establish (5.1), we use the fact (see [4]) that the dimension of $\mathcal{S}_2^1(\triangle_M)$ is equal to the cardinality of the MDS $\mathcal{M}$, which is easily seen to be given by the stated formula. □

6. Hermite interpolation

As in the previous section, let $\triangle_M$ be the result of applying the splitting process of Section 3 to each tetrahedron T of an arbitrary tetrahedral partition $\triangle$ of a polyhedral domain Ω. In Theorem 4.2 we showed that there is a natural Hermite interpolant associated with the macro-element $\mathcal{S}_2^1(T_M)$. We now establish the analogous result for the full macro-element space $\mathcal{S}_2^1(\triangle_M)$. For each edge e of $\triangle$, let $D_{e,1}$ and $D_{e,2}$ be the directional derivatives associated with two orthogonal unit vectors lying in a plane perpendicular to e. Let m_e be the midpoint of the edge e. For each face F of $\triangle$, let D_F be the directional derivative corresponding to a unit vector perpendicular to F. Let $d_{F,1}, d_{F,2}, d_{F,3}$ and u_F be the vertices of $\triangle_M$ of the form d_j^i and u_k that lie on F.

Theorem 6.1. *Suppose* $f \in C^1(\triangle)$*. Then there exists a unique* $s_f \in \mathcal{S}_2^1(\triangle_M)$ *such that*

(1) $D^\alpha s_f(v) = D^\alpha f(v)$*, for all* $0 \le |\alpha| \le 1$ *and* $v \in \mathcal{V}$*,*
(2) $D_{e,j} s_f(m_e) = D_{e,j} f(m_e)$*, for* $j = 1, 2$ *and all* $e \in \mathcal{E}$*,*
(3) $D_F s_f(d_{F,j}) = D_F f(d_{F,j})$*, for* $j = 1, 2, 3$ *and all* $F \in \mathcal{F}$*,*
(4) $D_F s_f(u_F) = D_F f(u_F)$*, all* $F \in \mathcal{F}$*.*

Proof. It is easy to check that the number of interpolation conditions equals the dimension of $\mathcal{S}_2^1(\triangle_M)$. For each tetrahedron T of $\triangle$, we can use the data given here to compute the Hermite data needed to apply Theorem 4.2. □

Theorem 6.1 implicitly defines a nodal minimal determining set for $\mathcal{S}_2^1(\triangle_M)$. The construction also insures that this nodal minimal determining set is local and stable in the sense of Definition 17.21 of [4]. But then the results of Section 17.7 of [4] give us the following error bound for the Hermite interpolant of Theorem 6.1, where as before $|\triangle|$ is the mesh size of $\triangle$.

Theorem 6.2. *There exists a constant* K *such that for every* $f \in C^{m+1}(\Omega)$ *with* $0 \le m \le 2$*,*

$$\|D^\alpha(f - s_f)\|_\Omega \le K|\triangle|^{m+1-|\alpha|}|f|_{m+1,\Omega},$$

for all $|\alpha| \le m$*. If* Ω *is convex, then the constant* K *depends only on the smallest solid and faces angles in* $\triangle$*, while if it is nonconvex, then* K *may also depend on the Lipschitz constant of the boundary of* Ω*.*

7. Remarks

Remark 1. Peter Alfeld has written a Java program which is extremely useful for experimenting with trivariate spline spaces. Using exact arithmetic, it can compute the dimension of trivariate spline spaces and help find minimal determining sets. The code can be downloaded from www.math.utah.edu/˜pa. We have used the program to help design our element, and also to verify its correctness.

Remark 2. The well-known Euler relations (see [4]) for a tetrahedral partition state that $n_T = n_{E_I} + n_{V_B} - n_{V_I} - 3 = n_{F_I}/2 + n_{F_B}/4$, $n_{E_B} = 3n_{V_B} - 6$, and $n_{F_B} = 2n_{E_B}/3$, where n_T is the number of tetrahedra in the partition, and $n_{V_B}, n_{V_I}, n_{E_B}, n_{E_I}, n_{F_B}, n_{F_I}$ are the numbers of boundary and interior vertices, edges, and faces, respectively. These formulae can be used to check the counts given in Lemma 3.2.

Remark 3. The split of a tetrahedron T into 24 subtetrahedron obtained by performing a Powell–Sabin-6 split of each face and then connecting all resulting vertices on the faces to the centroid of T was introduced in [8], where it was called a Powell–Sabin split. Here we follow [4] where it is renamed the Worsey–Piper split.

Remark 4. It remains open whether it is possible to construct a comparable C^1 quadratic spline macro-element defined on arbitrary tetrahedral partitions with fewer tetrahedra. However, after an extensive study of various possibilities, we conjecture this is impossible.

Remark 5. A close examination of the proofs shows that the minimal determining set $\mathcal{M}$ in Theorem 4.1 is local and stable in the sense of Section 17.3 of [4]. But then the results of Section 17.4 of [4] can be applied to obtain the L_q approximation power of the space $\mathcal{S}_2^1(\triangle_M)$. Moreover, the $\mathcal{M}$-basis introduced in Theorem 17.16 of [4] provides a stable 1-local basis for $\mathcal{S}_2^1(\triangle_M)$.

Acknowledgments

We would like to thank Peter Alfeld for not only making his Java program available, but for modifying it to help with our research. We also want to thank him for numerous helpful discussions related to this work.

References

[1] P. Alfeld, A trivariate Clough–Tocher scheme for tetrahedral data, Comput. Aided Geom. Design 1 (1984) 169–181.
[2] C. de Boor, B-form basics, in: G.E. Farin (Ed.), Geometric Modeling: Algorithms and New Trends, SIAM, Philadelphia, 1987, pp. 131–148.
[3] M.J. Lai, L.L. Schumaker, Trivariate C^r polynomial macro-elements, Constr. Approx. 26 (2007) 11–28.
[4] M.J. Lai, L.L. Schumaker, Spline Functions on Triangulations, Cambridge University Press, Cambridge, 2007.
[5] G. Nurnberger, C. Rössl, H.P. Seidel, F. Zeilfelder, Quasi-interpolation by quadratic piecewise polynomials in three variables, Comput. Aided Geom. Design 22 (2005) 221–249.
[6] M.J.D. Powell, M.A. Sabin, Piecewise quadratic approximations on triangles, ACM Trans. Math. Software 3 (1977) 316–325.
[7] A.J. Worsey, G. Farin, An n-dimensional Clough–Tocher interpolant, Constr. Approx. 3 (1987) 99–110.
[8] A.J. Worsey, B. Piper, A trivariate Powell–Sabin interpolant, Comput. Aided Geom. Design 5 (1988) 177–186.
[9] Alexander Ženišek, Polynomial approximation on tetrahedrons in the finite element method, J. Approx. Theory 7 (1973) 334–351.
[10] Alexander Ženišek, Hermite interpolation on simplexes and the finite element method, in: Proc. Equadiff III, Brno, 1973, pp. 271–277.