On the Approximation Power of Bivariate Splines

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Abstract. We show how to construct stable quasi-interpolation schemes in the bivariate spline spaces $\mathcal{S}_d^r(\triangle)$ with $d \geq 3r+2$ which achieve optimal approximation order. In addition to treating the usual max norm, we also give results in the L_p norms, and show that the methods also approximate derivatives to optimal order. We pay special attention to the approximation constants, and show that they depend only on the the smallest angle in the underlying triangulation and the nature of the boundary of the domain.

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§1. Introduction

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$. Given a finite triangulation $\triangle$ of Ω, we are interested in *spaces of splines of smoothness r and degree d* of the form

$$S_d^r(\triangle) := \{s \in C^r(\Omega) : s|_T \in \mathcal{P}_d, \text{ for all } T \in \triangle\},$$

where $\mathcal{P}_d$ denotes the space of polynomials of total degree at most d.

The main result of this paper is the following theorem which states the existence of a quasi-interpolation operator Q_m which maps $L_1(\Omega)$ into the spline space $\mathcal{S}_d^r(\triangle)$ in such a way that if f lies in a Sobolev space $W_p^{m+1}(\Omega)$ with $0 \leq m \leq d$, then $Q_m f$ approximates f and its derivatives to optimal order.

Theorem 1.1. *Fix $d \geq 3r+2$ and $0 \leq m \leq d$. Then there exists a linear quasi-interpolation operator Q_m mapping $L_1(\Omega)$ into $S_d^r(\triangle)$ and a constant C such that if f is in the Sobolev space $W_p^{m+1}(\Omega)$ with $1 \leq p \leq \infty$,*

$$\|D_x^\alpha D_y^\beta (f - Q_m f)\|_{p,\Omega} \leq C\ |\triangle|^{m+1-\alpha-\beta}\ |f|_{m+1,p,\Omega}, \tag{1.1}$$

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for all $0 \le \alpha+\beta \le m$. Here $|\triangle|$ is the maximum of the diameters of the triangles in $\triangle$. If Ω is convex, then the constant C depends only on d, p, m, and on the smallest angle $\theta_\triangle$ in $\triangle$. If Ω is nonconvex, C also depends on the Lipschitz constant $L_{\partial\Omega}$ associated with the boundary of Ω.

Error bounds as in (1.1) are well-known in the finite element literature for $d \ge 4r+1$. The first attempt to establish (1.1) for the range $d \ge 3r+2$ appears in de Boor & Höllig [5], where the authors dealt with the case $p = \infty$, $\alpha = \beta = 0$, and $m = d$. Later Chui & Lai [8] examined the same case for $d = 3r+2$. Unfortunately, both "proofs" were defective in that they involved a "constant" C which was not shown to be bounded, and in fact becomes arbitrarily large for triangulations which contain near-singular vertices (see Sect. 7 below for a precise definition of such a vertex). Recently, Chui, Hong, & Jia [7] gave what appears to be a correct proof of (1.1) for $p = \infty$, $\alpha+\beta = 0$, and $m = d$. Their argument involves constructing a quasi-interpolant in a certain super-spline subspace of $\mathcal{S}_d^r(\triangle)$.

In addition to providing what we believe is a simpler construction than in [7], the purpose of this paper is to extend the earlier results by establishing (1.1) for

1) general $1 \le p \le \infty$,
2) all choices of $0 \le m \le d$,
3) general $0 \le \alpha+\beta \le m$,
4) general (not necessarily convex) domains Ω.

The key to our approach is to work with a suitable super-spline subspace of $\mathcal{S}_d^r(\triangle)$ which is different than that in [7], and involves basis splines with smaller supports (see Remark 1).

The outline of the paper is as follows. Sect. 2 is devoted to some preliminaries. In Sect. 3 we develop some useful properties of triangulations. We establish a number of properties of polynomials in Sect. 4. While some of these are well-known, to make this paper as self-contained as possible, we present full proofs of most of them. We develop a general framework for establishing error bounds for spline quasi-interpolants in Sect. 5, and discuss domain points and smoothness conditions in Sect. 6. Near-degenerate edges and near-singular vertices are discussed in Sect. 7, and the phenomenon of propogation is explained in Sect. 8. In Sect. 9 we introduce the super-spline spaces of interest here, and in Sect. 10 we use them to establish our main result. We conclude the paper with several remarks.

§2. Preliminaries

In this paper Ω is assumed to be the union of a set of triangles. This means that the boundary $\partial\Omega$ is piecewise linear, and thus is Lipschitz with a constant $L_{\partial\Omega}$ which depends on the size of the angles between the edges of $\partial\Omega$. The error bound (1.1) is expressed in terms of the *mesh-dependent* L_p norm

$$\|D_x^\alpha D_y^\beta (f - Q_m f)\|_{p,\Omega}^p := \sum_{T\in\triangle} \|D_x^\alpha D_y^\beta (f - Q_m f)\|_{p,T}^p.$$

typically used in the finite-element literature. The expression on the right-hand side of (1.1) involves the usual Sobolev semi-norms

$$|f|_{k,p,\Omega} := \begin{cases} \left(\sum_{\nu+\mu=k} \|D_x^\nu D_y^\mu f\|_{p,\Omega}^p \right)^{1/p}, & 1 \le p < \infty \\ \sum_{\nu+\mu=k} \|D_x^\nu D_y^\mu f\|_{\infty,\Omega}, & p = \infty. \end{cases}$$

We shall make use of the following extension theorem of Stein [15], p. 181.

Lemma 2.1. *Let Ω be a bounded domain whose boundary consists of piecewise linear segments. Then there exists a linear extension operator E extending functions from Ω to $\mathbb{R}^2$ so that*

(a) $E(f)|_\Omega = f$,

(b) $\|D_x^\alpha D_y^\beta E(f)\|_{p,\mathbb{R}^2} \le K_1 \ \|D_x^\alpha D_y^\beta f\|_{p,\Omega}$, *for all* $f \in W_p^{m+1}(\Omega)$ *and all* $1 \le p \le \infty$ *and* $0 \le \alpha + \beta \le m+1$, *where the constant* K_1 *is dependent on* p, m, *and the Lipschitz constant* $L_{\partial\Omega}$ *of the boundary* $\partial\Omega$.

§3. Properties of Triangulations

In this section we introduce some useful notation, and collect several results needed later. Suppose T is a triangle. Then

$$|T| := \text{the diameter of the smallest disk containing } T, \tag{3.1}$$

$$\rho_T := \text{the radius of the largest disk contained in } T, \tag{3.2}$$

$$A_T := \text{the area of the triangle } T, \tag{3.3}$$

$$\theta_T := \text{the smallest angle in the triangle } T. \tag{3.4}$$

By simple trigonometry, it is easy to see that

$$\frac{|T|}{\rho_T} \le \frac{2}{\tan(\theta_T/2)}. \tag{3.5}$$

Given a triangulation $\triangle = \{T_i\}_{i=1}^N$ of a set Ω, at times we shall work with a subset $\mathcal{T}$ of $\triangle$ consisting of a cluster of several triangles. We define

$$\#\mathcal{T} := \text{the number of triangles in } \mathcal{T},$$

$$\rho_{\mathcal{T}} := \min_{T\in\mathcal{T}} \rho_T,$$

$$\theta_{\mathcal{T}} := \min_{T\in\mathcal{T}} \theta_T,$$

$$U_{\mathcal{T}} := \bigcup_{T_i\in\mathcal{T}} T_i,$$

$$|U_{\mathcal{T}}| := \text{diameter of the smallest disk containing } U_{\mathcal{T}}.$$

For later use we need some estimates on these quantities, assuming that the triangles of $\mathcal{T}$ are fairly closely clustered. To make this concept more precise, suppose v is a vertex of a triangle in $\triangle$. Then the *star of* v is the union of all triangles which share the vertex v. We denote it by $\text{star}^1(v) := \text{star}(v)$. Similarly, we define the *star of order* ℓ recursively by

$$\text{star}^\ell(v) := \{\cup T : \ T \text{ shares a vertex with a triangle in } \text{star}^{\ell-1}(v)\}.$$

Lemma 3.1. *Suppose $\mathcal{T}$ is a collection of triangles such that $U_\mathcal{T} \subset \text{star}^\ell(v)$. Then*

$$\#\mathcal{T} \le K_2 := \begin{cases} \sum_{\nu=0}^k a^{2\nu+1}, & \ell = 2k+1, \\ \sum_{\nu=1}^k a^{2\nu}, & \ell = 2k, \end{cases} \tag{3.6}$$

where $a := 2\pi/\theta_\mathcal{T}$.

Proof: We first consider the case where $U_\mathcal{T} = \text{star}(v)$. Suppose that there are N vertices attached to v. Then clearly $N\theta_\mathcal{T} \le 2\pi$, and so $N \le 2\pi/\theta_\mathcal{T}$. Since N is also the number of triangles surrounding v, this establishes (3.6) for $\ell = 1$.

We say that a vertex w is at *level* j with respect to v if we have to follow at most j edges to get from w to v. If $U_\mathcal{T} = \text{star}^\ell(v)$, then there are vertices at each of the levels $0, \ldots, \ell$. Moreover, by the above observation, the number of vertices at level j is bounded by a^j, and the total number of triangles surrounding vertices at level j is at most a^{j+1}.

To get a bound on the number of triangles in $\text{star}^\ell(v)$ in the case where $\ell = 2k+1$, it suffices to count the number of triangles surrounding vertices at levels $0, 2, \ldots, 2k$. This is at most $a + a^3 + \cdots + a^{2k+1}$, which establishes (3.6) for ℓ odd. When $\ell = 2k$, we only have to count the triangles surrounding vertices at levels $1, 3, \ldots, 2k-1$. $\square$

Lemma 3.2. *Suppose $\mathcal{T}$ is a set of triangles such that $U_\mathcal{T}$ is a connected subset of $\text{star}^\ell(v)$ for some vertex v. Then*

$$\frac{|U_\mathcal{T}|}{\rho_\mathcal{T}} \le 2\ell K_3, \tag{3.7}$$

where $K_3 := 1/[\tan(\theta_\mathcal{T}/2)(\sin(\theta_\mathcal{T}))^n]$ with $n = 2(2\ell-1)\pi/\theta_\mathcal{T}$. Moreover, for any two triangles $T, \widetilde{T}$ in $U_\mathcal{T}$,

$$\frac{A_T}{A_{\widetilde{T}}} \le K_3^2. \tag{3.8}$$

Proof: First we note that if e and $\tilde{e}$ are any two edges of a triangle T, then

$$|e| \ \le \ b|\tilde{e}|, \tag{3.9}$$

where $b = 1/\sin(\theta_\mathcal{T})$. Now any two triangles T and $\widetilde{T}$ in $\mathcal{T}$ are connected by a path of edges which pass through at most $2\ell - 1$ vertices. Since at most $2\pi/\theta_\mathcal{T}$ triangles can touch any given vertex, this means that we can get from one edge of $\widetilde{T}$ to an edge of T by crossing over at most n edges. Each time we cross an edge, the size of the next edge to be crossed is at most b larger. Combining this with (3.5), we see that $|e_{max}|/\rho_{\mathcal{T}} \le K_3$, where e_{max} is the longest edge in $\mathcal{T}$.

Now to prove (3.7), we observe that if x and y are two points in $U_{\mathcal{T}}$ at a maximal distance apart, then x and y must be vertices of triangles in $\mathcal{T}$. Thus there is a path of edges $e_1, \ldots, e_k$ from x to y going through v and involving at most 2ℓ edges. Thus $|U_{\mathcal{T}}| \le 2\ell |e_{max}|$, and (3.7) follows.

To prove (3.8), we simply note that for any $T, \widetilde{T} \in \mathcal{T}$, $A_T \le \pi |e_{max}|^2$ while $A_T \ge \pi \rho_{\mathcal{T}}^2$. □

§4. Polynomial Approximation

Suppose that T is a given triangle with vertices $v_i = (x_i, y_i)$, $i = 1, 2, 3$. Let $B^d_{ijk}(v)$ be the usual Bernstein polynomials of degree d associated with T for $i + j + k = d$. It is well known that these polynomials form a basis for $\mathcal{P}_d$, so that every polynomial $P \in \mathcal{P}_d$ can be written uniquely in the form

$$P(v) = \sum_{i+j+k=d} c_{ijk} B^d_{ijk}(v), \tag{4.1}$$

and that $\sum_{i+j+k=d} B^d_{ijk}(v) \equiv 1$. The representation (4.1) is called the *Bernstein-Bézier representation* or B-form of P. It is common practice to associate the coefficients c_{ijk} with the set of *domain points*

$$\mathcal{D}_T := \left\{ \xi^T_{ijk} = \frac{(iv_1 + jv_2 + kv_3)}{d} \right\}_{i+j+k=d}. \tag{4.2}$$

Our first lemma shows that the B^d_{ijk} form a *stable* basis for $\mathcal{P}_d$.

Lemma 4.1. *Fix $1 \le p \le \infty$. Then there exists a constant K_4 dependent only on d such that for any polynomial $P \in \mathcal{P}_d$,*

$$\frac{\|c\|_p}{K_4} \le \frac{1}{A_T^{1/p}} \, \|P\|_{p,T} \le \|c\|_p. \tag{4.3}$$

Here c is the vector of coefficients of P in lexicographical order, and

$$\|c\|_p = \left(\sum_{i+j+k=d} |c_{ijk}|^p \right)^{1/p}, \qquad 1 \le p < \infty, \tag{4.4}$$

$$\|c\|_\infty = \max_{i+j+k=d} |c_{ijk}|, \qquad p = \infty.$$

Proof: First we establish the inequality on the right of (4.3). For $p = \infty$ it follows from the fact that the B^d_{ijk} are nonnegative and sum to 1. We now prove it for $1 \le p < \infty$. Let $1/p + 1/q = 1$. Then writing P in B-form, we have

$$\begin{aligned} \|P\|^p_{p,T} &\le \int_T \Big(\sum_{i+j+k=d} |c_{ijk}|^p \Big) \Big(\sum_{i+j+k=d} |B^d_{ijk}(x,y)|^q \Big)^{p/q} dx dy \\ &\le \sum_{i+j+k=d} |c_{ijk}|^p \int_T \Big(\sum_{i+j+k=d} B^d_{ijk}(x,y) \Big)^{p/q} dx dy \\ &= \sum_{i+j+k=d} |c_{ijk}|^p A_T. \end{aligned}$$

This establishes the right-hand side of (4.3).

We now establish the left-hand side of (4.3) for $p = \infty$. Note that $Ac = r$ with $A = (\phi_m(\eta_n))$ and $r = P(\eta_n)$, where $\{\phi_m\}$ are the basis functions B_{ijk} and $\{\eta_n\}$ are the domain points $\{\xi^T_{ijk}\}$ in the same lexicographical order as the coefficients in c. Note that the entries of the matrix A depend only on d. Since interpolation at the ξ^T_{ijk} by polynomials in $\mathcal{P}_d$ is unique, A is invertible, and we get $\|c\|_\infty \le \|A^{-1}\|_\infty \|r\|_\infty \le \|A^{-1}\|_\infty \|P\|_{\infty,T}$. This gives the left-hand side of (4.3) for $p = \infty$ with $K_4 := \|A^{-1}\|_\infty$.

By mapping T to the standard simplex $T_s = \{(x,y), 0 \le x, y \le 1, x + y \le 1\}$, and using the fact that all norms on the finite dimensional space of polynomials are equivalent, i.e., $\|P\|_{\infty,T_s} \le K\|P\|_{p,T_s}$, it is easy to see that $\|P\|_{\infty,T} \le K\|P\|_{p,T}/A_T^{1/p}$. Now the result for general p follows since $\|c\|^p_p \le \binom{d+2}{2} \|c\|^p_\infty$. □

Our next lemma is a form of *Markov inequality* for polynomials in $\mathcal{P}_d$.

Lemma 4.2. *Let $1 \le p \le \infty$. Then there exists a constant K_5 dependent only on d such that for all polynomials $P \in \mathcal{P}_d$,*

$$\|D^\alpha_x D^\beta_y P\|_{p,T} \le \frac{K_5}{\rho_T^{\alpha+\beta}} \|P\|_{p,T}, \quad 0 \le \alpha + \beta \le d. \tag{4.5}$$

Proof: We consider only the case $1 \le p < \infty$. The case $p = \infty$ is similar, and simpler. Let $u = v_2 - v_1 = (x_2 - x_1, y_2 - y_1)$ and $v = v_3 - v_1 = (x_3 - x_1, y_3 - y_1)$. Then the directional derivatives of P are given by

$$\begin{aligned} D_u P &= (x_2 - x_1) D_x P + (y_2 - y_1) D_y P \\ D_v P &= (x_3 - x_1) D_x P + (y_3 - y_1) D_y P. \end{aligned}$$

It follows that

$$\begin{aligned} D_x P &= \frac{(y_3 - y_1) D_u P - (y_2 - y_1) D_v P}{2A_T}, \\ D_y P &= \frac{(x_2 - x_1) D_v P - (x_3 - x_1) D_u P}{2A_T}. \end{aligned}$$

Now clearly,

$$\rho_T|y_3 - y_1| \le A_T, \qquad \rho_T|y_2 - y_1| \le A_T.$$

Combining these inequalities, we have

$$\begin{aligned} |D_x P(x,y)| \le & \frac{|y_3 - y_1|}{2A_T}|D_u P(x,y)| + \frac{|y_2 - y_1|}{2A_T}|D_v P(x,y)| \\ \le & \frac{1}{2\rho_T}\left(|D_u P(x,y)| + |D_v P(x,y)|\right). \end{aligned}$$

The analogous estimate for $|D_y P|$ can be established in the same way.

It is well-known that

$$D_u P(v) = d \sum_{i+j+k=d-1} (c_{i,j+1,k} - c_{i+1,j,k}) B_{ijk}^{d-1}(v),$$

where B_{ijk}^{d-1} are the Bernstein basis polynomials of degree $d-1$ relative to T. Using Lemma 4.1 first on $D_u P$ and then on P, we now have

$$\begin{aligned} \|D_u P\|_{p,T} &\le d\gamma \left(A_T \sum_{i+j+k=d-1} |(c_{i,j+1,k} - c_{i+1,j,k})|^p \right)^{1/p} \\ &\le 2d\gamma A_T^{1/p} \|c\|_p \le 2d\gamma K_4 \; \|P\|_{p,T}, \end{aligned}$$

where $\gamma = [\binom{d+1}{2}]^{1-1/p}$. The analogous estimate for $\|D_v P\|_{p,T}$ can be established in the same way. Combining these, we have

$$\|D_x P\|_{p,T} \le \frac{1}{\rho_T}\Big(\|D_u P\|_{p,T} + \|D_v P\|_{p,T} \Big) \le \frac{4d\gamma K_2}{\rho_T} \; \|P\|_{p,T}.$$

This establishes (4.5) for $\alpha = 1$ and $\beta = 0$. The proof for $\alpha = 0$ and $\beta = 1$ is similar. The result for general α and β then follows by applying the D_x and D_y derivatives repeatedly. □

Next we introduce the so-called averaged Taylor polynomials (cf. [6], p. 91ff). Let $B(x_0, y_0, \rho) = \{(x,y) \in \mathbb{R}^2 : ((x-x_0)^2 + (y-y_0)^2)^{1/2} < \rho\}$ be the disk centered about (x_0, y_0) with radius ρ. For simplicity, we write $B := B(x_0, y_0, \rho)$. Let

$$g_B(x,y) = \begin{cases} c\exp(-1/(1 - ((x-x_0)^2 + (y-y_0)^2)/\rho^2), & \text{if } (x,y) \in B(x_0,y_0,\rho) \\ 0, & \text{otherwise} \end{cases}$$

be a *mollifier* or *cut-off function* such that $\int_{\mathbb{R}^2} g_B(x,y)\,dx\,dy = 1$.

For any function $f \in C^m(\mathbb{R}^2)$, let

$$T_{m,(u,v)}f(x,y) = \sum_{\alpha+\beta\le m} \frac{D_u^\alpha D_u^\beta f(u,v)}{\alpha!\beta!}(x-u)^\alpha (y-v)^\beta$$

be the Taylor polynomial of degree m of f at (u,v). Then the *averaged Taylor polynomial of degree m over $B(x_0,y_0,\rho)$* is defined as

$$F_{m,B}f(x,y) = \int_{B(x_0,y_0,\rho)} T_{m,(u,v)}f(x,y)\, g_B(u,v)\, du\, dv. \tag{4.6}$$

Integrating by parts, we have the equivalent formula

$$\begin{aligned} &F_{m,B}f(x,y) \\ &= \sum_{\alpha+\beta\le m} \frac{1}{\alpha!\beta!}\int_{B(x_0,y_0,\rho)} D_u^\alpha D_v^\beta f(u,v)(x-u)^\alpha (y-v)^\beta g_B(u,v)\, du\, dv \\ &= \sum_{\alpha+\beta\le m} \frac{(-1)^{\alpha+\beta}}{\alpha!\beta!}\int_{B(x_0,y_0,\rho)} f(u,v) D_u^\alpha D_v^\beta \left[(x-u)^\alpha (y-v)^\beta g_B(u,v)\right] du\, dv, \end{aligned}$$

which shows that the averaged Taylor polynomial is well-defined for any integrable function $f \in L_1(B(x_0,y_0,\rho))$. Clearly, $F_{m,B}f$ is a polynomial of degree $\le m$. It is also known (cf. [6]) that

Lemma 4.3. *For any $0 \le \alpha+\beta \le m$ and $f \in W_1^{\alpha+\beta}(B(x_0,y_0,\rho))$,*

$$D_x^\alpha D_y^\beta F_{m,B}f = F_{m-\alpha-\beta,B}(D_x^\alpha D_y^\beta f).$$

We recall the following formula for the exact remainder of the classical Taylor polynomial:

$$\begin{aligned} &f(x,y) - T_{m,(u,v)}f(x,y) \\ &= (m+1)\sum_{\alpha+\beta=m+1} \frac{(x-u)^\alpha (y-v)^\beta}{\alpha!\beta!}\int_0^1 D_1^\alpha D_2^\beta f((x,y)+t(u-x,v-y))t^m\, dt. \end{aligned}$$

Here the differential operators D_1 and D_2 denote differentiation with respect to the first and second variables, respectively. This implies that

$$\begin{aligned} &f(x,y) - F_{m,B}f(x,y) \\ &= \int_{B(x_0,y_0,\rho)} f(x,y) g_B(u,v)\, du\, dv - \int_{B(x_0,y_0,\rho)} T_{m,(u,v)}f(x,y) g_B(u,v)\, du\, dv \\ &= \sum_{\alpha+\beta=m+1} \frac{m+1}{\alpha!\beta!}\int_{B(x_0,y_0,\rho)}\int_0^1 g_B(u,v)(x-u)^\alpha (y-v)^\beta \times \\ &\qquad D_1^\alpha D_2^\beta f((x,y)+t(u-x,v-y))t^m\, dt\, du\, dv, \end{aligned} \tag{4.7}$$

and we immediately have

Lemma 4.4. *For any polynomial $f \in \mathcal{P}_m$, $f = F_{m,B} f$.*

Given a triangle $T \in \triangle$, let $B_T := B(x_T, y_T, \rho_T) \subset T$ be the largest disk contained in T. We now estimate the norm of the operator F_{m,B_T}.

Lemma 4.5. *For any $f \in L_p(T)$ with $1 \le p \le \infty$,*

$$||F_{m,B_T} f||_{p,T} \le K_6 ||f||_{p,T}.$$

Here K_6 is a constant dependent only on θ_T.

Proof: We first note that

$$\|D_u^\alpha D_v^\beta g_{B_T}\|_{L_\infty(\mathbb{R}^2)} \le \frac{C_1}{\rho_T^{\alpha+\beta+2}}, \quad \text{for all nonnegative integers } \alpha, \beta.$$

Then for fixed $(x, y) \in T$, by the Leibniz formula and (3.5)

$$\begin{aligned}
&|D_u^\alpha D_v^\beta (x-u)^\alpha (y-v)^\beta g_{B_T}(u,v)| \\
&\quad \le \sum_{\substack{\alpha_1 \le \alpha \\ \beta_1 \le \beta}} \binom{\alpha}{\alpha_1} \binom{\beta}{\beta_1} |(x-u)^{\alpha-\alpha_1} (y-v)^{\beta-\beta_1} D_u^{\alpha-\alpha_1} D_v^{\beta-\beta_1} g_{B_T}(u,v)| \\
&\quad \le \sum_{\substack{\alpha_1 \le \alpha \\ \beta_1 \le \beta}} \binom{\alpha}{\alpha_1} \binom{\beta}{\beta_1} |T|^{\alpha-\alpha_1+\beta-\beta_1} \frac{C_1}{\rho_T^{\alpha-\alpha_1+\beta-\beta_1+2}} \ \le \ C_2/\rho_T^2,
\end{aligned}$$

for any $(u, v) \in \mathbb{R}^2$. Using (3.5), we see that C_2 is a constant dependent only on the smallest angle θ_T of T. Given $1 \le p \le \infty$, let $1/p + 1/q = 1$. Then for all $f \in L_p(T)$, using (3.5) again, we have

$$\begin{aligned}
&\|F_{m,B_T} f\|_{p,T} \\
&\quad \le \sum_{\alpha+\beta \le m} \frac{1}{\alpha!\beta!} \left\| \int_{B_T} f(u,v) D_u^\alpha D_v^\beta \left[(x-u)^\alpha (y-v)^\beta g_{B_T}(u,v) \right] du\, dv \right\|_{p,T} \\
&\quad \le \sum_{\alpha+\beta \le m} \frac{1}{\alpha!\beta!} \left\| \left(\int_{B_T} |f(u,v)|^p \, du\, dv \right)^{1/p} \times \right. \\
&\quad\quad \left. \left(\int_{B_T} |D_u^\alpha D_v^\beta (x-u)^\alpha (y-v)^\beta g_{B_T}(u,v)|^q \, du\, dv \right)^{1/q} \right\|_{p,T} \\
&\quad \le \sum_{\alpha+\beta \le m} \frac{1}{\alpha!\beta!} ||f||_{p,T} \left(\int_T \left(\int_{B_T} \left(C_2 \frac{1}{\rho_T^2} \right)^q du\, dv \right)^{p/q} dx\, dy \right)^{1/p} \\
&\quad \le C_3 ||f||_{p,T} \left(\left(\rho_T^{-2q} \pi \rho_T^2 \right)^{p/q} |T|^2 \right)^{1/p} \\
&\quad \le C_4 ||f||_{p,T}.
\end{aligned}$$

Since C_4 depends only on θ_T, this completes the proof. $\square$

Our aim now is to give an error bound for how well the polynomial $F_{m,B_T}f$ approximates the function f, assuming that f lies in a Sobolev space. We need a bound not only on a single triangle T, but also on the union $U_{\mathcal{T}}$ of a set $\mathcal{T}$ of triangles in the triangulation $\triangle$ of Ω.

Lemma 4.6. *Fix $1 \le p \le \infty$ and $m \ge 0$. Let $U_{\mathcal{T}}$ be a polygonal domain consisting of the union of a set $\mathcal{T}$ of triangles lying in $\mathrm{star}^{\ell}(v)$ for some vertex v. Let T be an arbitrary triangle in $\mathcal{T}$. Then there exists a positive constant K_7 depending only on m, ℓ, $\theta_{\mathcal{T}}$, and the Lipschitz constant of $\partial\Omega$ such that for all $f \in W_p^{m+1}(U_{\mathcal{T}})$,*

$$\|D_x^{\alpha} D_y^{\beta} (f - F_{m,B_T} f) \|_{p,U_{\mathcal{T}}} \le K_7 |U_{\mathcal{T}}|^{m+1-\alpha-\beta} |f|_{m+1,p,U_{\mathcal{T}}}.$$

Proof: We need only prove

$$||f - F_{m,B_T} f||_{p,U_{\mathcal{T}}} \le K |U_{\mathcal{T}}|^{m+1} |f|_{m+1,p,U_{\mathcal{T}}}, \tag{4.8}$$

since then Lemma 4.3 implies

$$\begin{aligned}
&||D_x^{\alpha} D_y^{\beta} (f - F_{m,B_T} f) ||_{p,U_{\mathcal{T}}} \\
&\quad = ||D_x^{\alpha} D_y^{\beta} f - F_{m-\alpha-\beta,B_T}(D_x^{\alpha} D_y^{\beta} f)||_{p,U_{\mathcal{T}}} \\
&\quad \le K |U_{\mathcal{T}}|^{m+1-\alpha-\beta} |D_x^{\alpha} D_y^{\beta} f|_{m+1-\alpha-\beta,p,U_{\mathcal{T}}} \\
&\quad \le K |U_{\mathcal{T}}|^{m+1-\alpha-\beta} |f|_{m+1,p,U_{\mathcal{T}}}.
\end{aligned}$$

To establish (4.8), we first use the Stein extension Theorem 2.1 to extend f to the convex hull $\widehat{U}_{\mathcal{T}}$ of $U_{\mathcal{T}}$. We continue to write f for the extended function. Then

$$\|f\|_{m+1,p,\widehat{U}_{\mathcal{T}}} \le K_1 \|f\|_{m+1,p,U_{\mathcal{T}}}$$

for any $f \in W_p^{m+1}(U_{\mathcal{T}})$. Since $U_{\mathcal{T}}$ is a polygonal domain, the constant K_1 depends on the Lipschitz constant of the boundary of $U_{\mathcal{T}}$, which in turn depends on the smallest angle $\theta_{\mathcal{T}}$ and may also depend on the Lipschitz constant $L_{\partial\Omega}$ if the boundary of $U_{\mathcal{T}}$ contains a part of $\partial\Omega$. In view of (4.7), we need an estimate for

$$\int_{B_T} \int_0^1 g_{B_T}(u,v)(x-u)^{\alpha}(y-v)^{\beta} D_1^{\alpha} D_2^{\beta} f((x,y) + t(u-x, v-y))\, t^m \, dt\, du\, dv.$$

Let $(\mu,\nu) = (x,y) + t(u-x, v-y)$. Then $d\mu\, d\nu\, dt = t^2\, du\, dv\, dt$. Let

$$D := \{(u,v,t) : t \in (0,1] \text{ and } \left| \frac{(\mu,\nu)-(x,y)}{t} + (x - x_0, y - y_0) \right| < \rho_T\},$$

where (x_0, y_0) is the center of the disk B_T. Then for $(u, v, t) \in B_T \times (0, 1]$, $(\mu, \nu, t) \in D$. Since

$$\sqrt{(\mu - x)^2 + (\nu - y)^2}/t < \rho_T + \sqrt{(x - x_0)^2 + (y - y_0)^2},$$

we have

$$t_0(\mu, \nu) := \frac{\sqrt{(\mu - x)^2 + (\nu - y)^2}}{\rho_T + \sqrt{(x - x_0)^2 + y - y_0)}} < t.$$

Thus, letting χ_D be the characteristic function of D, we have

$$\begin{aligned}
&\int_{B_T} \int_0^1 g_{B_T}(u, v)(x - u)^\alpha (y - v)^\beta D_1^\alpha D_2^\beta f((x, y) + t(u - x, v - y)) t^m \, du \, dv dt \\
&= \int_D g_{B_T}\Big(\frac{(\mu - x, \nu - y)}{t} + (x, y)\Big)(x - \mu)^\alpha (y - \nu)^\beta D_1^\alpha D_2^\beta f(\mu, \nu) t^{-3} \, d\mu \, d\nu \, dt \\
&= \int_{\langle (x,y), B_T \rangle} (x - \mu)^\alpha (y - \nu)^\beta D_\mu^\alpha D_\nu^\beta f(\mu, \nu) \times \\
&\quad \int_0^1 \chi_D(\mu, \nu, t) g_{B_T}((x, y) + (\mu - x, \nu - y)/t) t^{-3} \, dt \, d\mu \, d\nu,
\end{aligned}$$

where $\langle (x, y), B_T \rangle$ denotes the convex hull of (x, y) and B_T. Note that

$$\begin{aligned}
&\Big| \int_0^1 \chi_D(\mu, \nu, t) g_{B_T}((x, y) + (\mu - x, \nu - y)/t) t^{-3} \, dt \Big| \\
&\qquad \le \frac{C_1}{\rho_T^2} \int_{t_0(\mu,\nu)}^1 t^{-3} dt \\
&\qquad = \frac{C_1}{2\rho_T^2} \left(\frac{\left(\rho_T + \sqrt{(x - x_0)^2 + (y - y_0)^2}\right)^2}{(\mu - x)^2 + (\nu - y)^2} - 1 \right) \\
&\qquad \le C_1 \left(1 + \frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{\rho_T} \right)^2 ((\mu - x)^2 + (\nu - y)^2)^{-1}.
\end{aligned}$$

By Lemma 3.2, we have

$$\frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{\rho_T} \le \frac{|U_T|}{\rho_T} \le C_2 := 2\ell K_3,$$

and letting q be such that $1/p + 1/q = 1$, we have

$$\|f - F_{m,B_{\mathcal{T}}} f\|_{p,U_{\mathcal{T}}} \le \sum_{\alpha+\beta=m+1} \frac{(m+1)}{\alpha!\beta!} \times$$

$$\Big\| \int_{\langle (x,y), B_{\mathcal{T}}\rangle} |D_\mu^\alpha D_\nu^\beta f(\mu,\nu)|((x-\mu)^2 + (y-\nu)^2))^{(m-1)/2} \Big\|_{p,U_{\mathcal{T}}} C_1(1+C_2)^2$$

$$\le C_1(1+C_2)^2 \sum_{\alpha+\beta=m+1} \frac{(m+1)}{\alpha!\beta!} \times$$

$$\left[\int_{U_{\mathcal{T}}} \left(\int_{\widehat{U}_{\mathcal{T}}} |D_\mu^\alpha D_\nu^\beta f(\mu,\nu)|((x-\mu)^2 + (y-\nu)^2))^{(m-1)/2} d\mu d\nu \right)^p dx dy \right]^{1/p}$$

$$\le C_3 \sum_{\alpha+\beta=m+1} \left[\int_{U_{\mathcal{T}}} ||D_\mu^\alpha D_\nu^\beta f||^p_{p,\widehat{U}_{\mathcal{T}}} \left(\int_{\widehat{U}_{\mathcal{T}}} |\widehat{U}_{\mathcal{T}}|^{(m-1)q} d\mu d\nu \right)^{p/q} dx dy \right]^{1/p}$$

$$= C_3 |f|_{m+1,p,\widehat{U}_{\mathcal{T}}} \left[\left(|U_{\mathcal{T}}|^{(m-1)q+2} \right)^{p/q} |U_{\mathcal{T}}|^2 \right]^{1/p}$$

$$= C_3 |U_{\mathcal{T}}|^{m+1} |f|_{m+1,p,U_{\mathcal{T}}}.$$

Here, the constant C_3 is dependent on the smallest angle $\theta_{\mathcal{T}}$. This completes the proof. □

We remark that the proof of Lemma 4.6 is just a modification of Lemma (4.3.8) in [6], p. 100.

§5. An Error Bound for Spline Quasi-interpolation

Let $\triangle$ be a triangulation of a bounded polygonal domain Ω. In this section we investigate the approximation power of certain *quasi-interpolation operators* mapping functions in $L_1(\Omega)$ into splines defined over $\triangle$.

Theorem 5.1. *Fix $0 \le m \le d$. Suppose Γ is some finite index set, and let $\{\phi_\xi\}_{\xi\in\Gamma}$ be a set of splines in $\mathcal{S}_d^0(\triangle)$ such that*

- H1) *there exists an integer ℓ such that for each ξ, the support of ϕ_ξ is contained in star$^\ell(v_\xi)$ for some vertex $v_\xi \in \triangle$;*
- H2) $K_8 := \max_\xi \ \|\phi_\xi\|_{\infty,\Omega} < \infty$;
- H3) $K_9 := \max_T \#(\Sigma_T) < \infty$, *where* $\Sigma_T := \{\xi : \ T \subset \sigma(\phi_\xi)\}$ *and* $\sigma(\phi_\xi)$ *denotes the support of* ϕ_ξ.

Suppose in addition that there exists a set of linear functionals $\{\lambda_{\xi,m}\}_{\xi\in\Gamma}$ defined on $L_1(\Omega)$ with the property that for all $\xi \in \Gamma$, there is a triangle T_ξ contained in the support of ϕ_ξ with

$$|\lambda_{\xi,m} f| \le \frac{K_{10}}{A_{T_\xi}^{1/p}} \|f\|_{p,T_\xi} \quad \text{for all } f \in L_p(\Omega) \text{ when } 1 \le p < \infty \tag{5.1}$$

and

$$|\lambda_{\xi,m} f| \le K_{10} \|f\|_{\infty,\Omega} \quad \text{for all } f \in L_\infty(\Omega) \text{ when } p = \infty \tag{5.2}$$

for some constant K_{10}. Finally, suppose that the corresponding quasi-interpolation operator

$$Q_m f = \sum_{\xi \in \Gamma}^{N} (\lambda_{\xi,m} f) \phi_\xi \tag{5.3}$$

reproduces polynomials in the sense that

$$Q_m P = P \qquad \text{for all} \qquad P \in \mathcal{P}_m. \tag{5.4}$$

Then there exists a constant C depending only on the constants $K_1, \ldots, K_7$ appearing in Lemmas 2.1, 3.1, 3.2, 4.2, 4.5, and 4.6, and the constants ℓ, K_8, K_9, K_{10} above such that if $f \in W_p^{m+1}(\Omega)$, then

$$\left\| D_x^\alpha D_y^\beta (f - Q_m f) \right\|_{p,\Omega} \le C |\triangle|^{m+1-\alpha-\beta} |f|_{m+1,p,\Omega} \tag{5.5}$$

for all $0 \le \alpha + \beta \le m$ and all $1 \le p \le \infty$.

Proof: We present the proof for $1 \le p < \infty$; the proof for $p = \infty$ is similar and simpler. For a fixed triangle T in $\triangle$, let $U := \bigcup \{\sigma(\phi_\xi) : T \subset \sigma(\phi_\xi)\}$. If we write $\mathcal{T}$ for the set of triangles making up U, then in our earlier notation $U = U_\mathcal{T}$. By H1), $U_\mathcal{T} \subset \text{star}^{2\ell+1}(v)$ for some vertex v of T. By Lemma 4.6 there exists a polynomial g of degree m so that

$$\left\| D_x^\alpha D_y^\beta (f - g) \right\|_{p,U_\mathcal{T}} \le K_7 |U_\mathcal{T}|^{m+1-\alpha-\beta} |f|_{m+1,p,U_\mathcal{T}}. \tag{5.6}$$

Using (5.4), we have

$$\left\| D_x^\alpha D_y^\beta (f - Q_m f) \right\|_{p,T} \le \left\| D_x^\alpha D_y^\beta (f - g) \right\|_{p,T} + \left\| D_x^\alpha D_y^\beta Q_m (f - g) \right\|_{p,T}.$$

Since $T \subset U_\mathcal{T}$, we can apply (5.6) to estimate the first term. We now examine the second term in more detail.

For each $\xi \in \Sigma_T$, let T_ξ be the triangle in (5.1). Now by H2), (3.7), (3.8), (5.1), and (5.6) for $\alpha = \beta = 0$, and Lemmas 4.2, we have

$$\begin{aligned}
&\int_T |\lambda_{\xi,m}(f-g)|^p |D_x^\alpha D_y^\beta \phi_i|^p \, dx dy \\
&\qquad \le \left[\frac{K_5 K_{10}}{\rho_T^{\alpha+\beta}} \right]^p \frac{A_T}{A_{T_\xi}} \; \|f - g\|_{p,T_\xi}^p \; \|\phi_\xi\|_{\infty,T}^p \\
&\qquad \le K_3^2 \left[\frac{K_5 K_7 K_8 K_{10}}{\rho_T^{\alpha+\beta}} |U_\mathcal{T}|^{m+1} |f|_{m+1,p,U_\mathcal{T}} \right]^p \\
&\qquad \le \left(K_{11} |\triangle|^{m+1-\alpha-\beta} |f|_{m+1,p,U_\mathcal{T}} \right)^p,
\end{aligned}$$

where $K_{11} := [2\ell]^{(m+1+\alpha+\beta)} K_3^{(\alpha+\beta+2/p)} K_5 K_7 K_8 K_{10}$. In view of H3), we get

$$\begin{aligned} \|D_x^\alpha D_y^\beta Q_m(f-g)\|_{p,T}^p &= \int_T \Big| \sum_{\xi\in\Sigma_T} \lambda_{\xi,m}(f-g) D_x^\alpha D_y^\beta \phi_\xi \Big|^p \, dx dy \\ &\le [K_{12} |\triangle|^{m+1-\alpha-\beta} |f|_{m+1,p,U_T}]^p, \end{aligned} \tag{5.7}$$

where $K_{12} := K_9^{1-1/p} K_{11}$.

To complete the proof, we now add (5.6) and (5.7) together and sum over all triangles $T \in \triangle$. Since U_T contains other triangles besides T, some triangles appear more than once in the sum on the right. However, a given triangle T_R appears on the right only if it is associated with a triangle T_L on the left which lies in the set $\text{star}^{2\ell+1}(v)$, for some vertex v of T_R. But then Lemma 3.1 implies that there is a constant K_{13} depending only on ℓ and $\theta_\triangle$ such that T_R enters at most K_{13} times on the right. We conclude that

$$\|D_x^\alpha D_y^\beta f - Q_m f\|_{p,\Omega}^p \le K_{13}(K_7^p + K_{11}^p)[|\triangle|^{m+1-\alpha-\beta} |f|_{m+1,p,\Omega}]^p,$$

and taking the p-th root, we get (5.5). □

Clearly, we could have normalized the splines ϕ_ξ appearing in Theorem 5.1 so that the constant $K_8 = 1$. However, we have not done that here since in using this result later, it is more convenient to normalize our splines in a different way.

§6. Domain Points and Smoothness Conditions

It is well-known that the space of splines $\mathcal{S}_d^0(\triangle)$ is in one-to-one correspondence with the set of *domain points*

$$\mathcal{D}_\triangle = \{\xi_{ijk}^T : \ T \text{ is a triangle in } \triangle\}, \tag{6.1}$$

where the ξ_{ijk}^T are defined in (4.2). For each point $\xi \in \mathcal{D}_\triangle$, let γ_ξ be the linear functional such that for any spline $s \in \mathcal{S}_d^0(\triangle)$,

$$\gamma_\xi s := \text{the Bézier coefficient of } s_T \text{ associated with the domain point } \xi, \tag{6.2}$$

where s_T is the polynomial which agrees with s on T. Suppose $\mathcal{S}$ is a linear subspace of $\mathcal{S}_d^0(\triangle)$. We recall [3] that a subset Γ of $\mathcal{D}_\triangle$ is called a *determining set for* $\mathcal{S}$ provided that for any $s \in \mathcal{S}$, the coefficients of s are uniquely determined by the set $\{c_\xi\}_{\xi\in\Gamma}$. Γ is called a *minimal determining set* for $\mathcal{S}$ if there is no determining set with fewer elements. There is a convenient way to recognize when a given determining set is minimal. Suppose that for each $\xi \in \Gamma$, it is possible to construct a spline $\phi_\xi \in \mathcal{S}$ such that

$$\gamma_\eta \phi_\xi = \delta_{\eta,\xi}, \qquad \text{all } \eta \in \Gamma. \tag{6.3}$$

Then as shown in [3], the splines ϕ_ξ are linearly independent and form a basis for $\mathcal{S}$.

When Γ is a minimal determining set, the splines ϕ_ξ satisfying (6.3) can be constructed as follows. Given $\xi \in \Gamma$, to construct ϕ_ξ, we first set the coefficients of ϕ_ξ corresponding to domain points $\eta \in \Gamma$ so that (6.3) holds. Then we solve for the remaining coefficients of ϕ_ξ taking care to satisfy all of the smoothness conditions required to make ϕ_ξ lie in $\mathcal{S}$. In Sect. 9 below we shall construct a basis of locally supported splines for a certain super-spline subspace $\mathcal{S}$ of $\mathcal{S}_d^r(\triangle)$.

We devote the remainder of this section to a discussion of how to use smoothness conditions between adjacent polynomial pieces of a spline to solve for coefficients. Suppose $T = \langle v_1, v_2, v_3 \rangle$ and $\widetilde{T} = \langle v_4, v_2, v_3 \rangle$ are two adjacent triangles which share a common edge $e = \langle v_2, v_3 \rangle$. Let $\{B_{ijk}^d\}$ and $\{\tilde{B}_{ijk}^d\}$ be the Bernstein-Bézier basis polynomials associated with T and $\widetilde{T}$, respectively. Then it is well-known (cf. [4] and [9]) that the two polynomials

$$p(v) := \sum_{i+j+k=d} c_{ijk} B_{ijk}^d(v) \tag{6.4}$$

and

$$\tilde{p}(v) := \sum_{i+j+k=d} \tilde{c}_{ijk} \tilde{B}_{ijk}^d(v), \tag{6.5}$$

join together with smoothness C^r across the edge e if and only if

$$\tilde{c}_{mjk} = \sum_{\nu+\mu+\kappa=m} c_{\nu,j+\mu,k+\kappa} B_{\nu\mu\kappa}^m(v_4), \quad \text{all } \ j+k = d-m \ \text{ and } \ m = 0, \ldots, r. \tag{6.6}$$

Assuming that the coefficients appearing on the right-hand side of (6.6) are known, we can use the equation to solve for $\tilde{c}_{mjk}$. The following lemma shows that this is a stable process.

Lemma 6.1. *Suppose s is a spline in $\mathcal{S}_d^r(\triangle)$, and that p and $\tilde{p}$ are its restrictions to a pair of adjoining triangles T and $\widetilde{T}$ as described above. Suppose the coefficients $\{c_{ijk}\}_{i \le r}$ of p are known, and that $C := \max_{i \le r} |c_{ijk}|$. Then the coefficients $\{\tilde{c}_{mjk}\}_{m \le r}$ of $\tilde{p}$ can be computed from (6.6), and are bounded by KC, where K is a constant depending only on the smallest angle $\theta_\triangle$ in the triangulation.*

Proof: Suppose

$$v_4 = \alpha v_1 + \beta v_2 + \gamma v_3. \tag{6.7}$$

We claim that the α, β, γ are bounded by a constant depending only on $\theta_\triangle$. Indeed, each of them is a ratio of the areas of two triangles which share a common edge. The area of the triangle T with edges e and $\tilde{e}$ separated by an angle θ is given by $A_T = \frac{1}{2}|e||\tilde{e}|\sin\theta$. Now by (3.9), the edges of T and of $\widetilde{T}$ are of comparable size with a constant depending only on $\theta_\triangle$, and the result follows. $\square$

The smoothness conditions can also be used in a different way to compute coefficients. Given a vertex v, we define the *ring of radius* m *around* v to be the set $R_m(v) := \{\eta : \operatorname{dist}(\eta, v) = m\}$. The *disk of radius* m *around* v is $\mathcal{D}_m(v) := \{\eta : \operatorname{dist}(\eta, v) \le m\}$. We also define the *arc* $a_{m,e}(v)$ associated with an edge $e := \langle v, u\rangle$ as the set of domain points in the ring $R_m(v)$ whose distance to e is at most r. Here we recall that if $T = \langle v_1, v_2, v_3\rangle$, then the *distance of the domain point* ξ^T_{ijk} *from the vertex* v_1 is defined to be $\operatorname{dist}(\xi^T_{ijk}, v) := d - i$, with similar definitions for the other two vertices, while the *distance of* ξ^T_{ijk} *from the edge* $\langle v_2, v_3\rangle$ is i, with similar definitions for the other two edges.

Lemma 6.2. *Suppose T and $\widetilde{T}$ are a pair of neighboring triangles as in Lemma 6.1, and that we know the coefficients of a spline $s \in \mathcal{S}^r_d(\triangle)$ for all domain points in the disk $\mathcal{D}_{m-1}(v)$ with $m \ge r$. Let $c_i := c^T_{i,d-m,m-i}$ be the coefficients of $p := s|_T$ in the arc $a_{m,e}(v_2)$ and let $\tilde{c}_i := c^{\widetilde{T}}_{i,d-m,m-i}$ be those of $\tilde{p} := s|_{\widetilde{T}}$. Suppose that the coefficients c_i and $\tilde{c}_i$ are known for $i \in K := \{0, \ldots, r-2q, r-q+1, \ldots, r\}$ for some q with $r+1 \ge 2q$. Let $C := \max_{i\in K}\{|c_i|, |\tilde{c}_i|\}$. Then the coefficients c_i and $\tilde{c}_i$ are uniquely determined for $i \in L := \{r-2q+1, \ldots, r-q\}$, and are bounded by KC, where K is a constant depending only on d, the smallest angle $\theta_\triangle$ in the triangulation, and the size of α^{-1} and γ^{-1}, where α, β, γ are as in (6.7).*

Proof: Versions of the first assertion can be found in [5,8,11]. To bound the size of the computed coefficients, we recall from Lemma 3.3 of [11] that the vector

$$x := (c_{r-q}, \ldots, c_{r-2q+1}, \tilde{c}_{r-2q+1}, \ldots, \tilde{c}_{r-q})$$

is uniquely determined by a system of equations of the form $Mx = y$, where M is a nonsingular matrix with

$$\det M = \kappa \alpha^{i_1}\gamma^{i_2} \begin{vmatrix} \frac{1}{q!} & \frac{1}{(q-1)!} & \cdots & \frac{1}{1!} \\ \vdots & & \ddots & \vdots \\ \frac{1}{(2q-1)!} & \frac{1}{(2q-2)!} & \cdots & \frac{1}{q!} \end{vmatrix},$$

for some constants i_1, i_2 and κ depending only on r, q, d. Now the arguments in the proof of Lemma 6.1 provide a bound on the components of y, while $\det M$ is bounded away from zero by a constant depending on the size of α^{-1} and γ^{-1}. □

Lemma 6.2 cannot be used when the edge e is *degenerate*, i.e., when $\gamma = 0$ in (6.7). In fact, since we want to control the size of computed coefficients, we cannot use the lemma whenever γ is small. This will have an effect on the way in which we construct a minimal determining set for our super-spline space.

§7. Near-Degenerate Edges and Near-Singular Vertices

We need generalizations of the well-known concepts of a degenerate edge and a singular vertex.

Definition 7.1. *Suppose $T = \langle v_1, v_2, v_3\rangle$ and $\widetilde{T} = \langle v_4, v_2, v_3\rangle$ are two triangles which share an edge $e = \langle v_2, v_3\rangle$. Suppose that α, β, γ are the barycentric coordinates of v_4 relative to T as defined in (6.7). Then we say that the edge e is δ-near-degenerate at v_2 provided $\gamma < \delta$. We write $\mathcal{E}^{\delta}_{ND}(v_2)$ for the collection of all such edges.*

In the case where $e \in \mathcal{E}^0_{ND}(v_2)$, the edges $\langle v_1, v_2\rangle$ and $\langle v_4, v_2\rangle$ are collinear, and the edge $e = \langle v_2, v_3\rangle$ is a classical *degenerate edge*. We are interested in near-degenerate edges for small δ. In this case, the cardinality of $\mathcal{E}^{\delta}_{ND}(u)$ can only be one, two, or four. Moreover, no edge can be near-degenerate at both ends.

Definition 7.2. *If v is a vertex with $\#\mathcal{E}^{\delta}_{ND}(v) = 4$, then we call v a δ-near-singular vertex. We write $\mathcal{V}^{\delta}_{NS}$ for the set of all such vertices.*

If $v \in \mathcal{V}^0_{NS}$, then the vertex v is a classical *singular* vertex formed by the intersection of two lines. For small δ, it is impossible for two neighboring vertices to both belong to $\mathcal{V}^{\delta}_{NS}$ since as we observed above, no edge can be near-degenerate at both ends. We also note that if $v \notin \mathcal{V}^{\delta}_{NS}$, then there must be at least one edge attached to v which does not belong to $\mathcal{E}^{\delta}_{ND}(v)$.

The following lemma will be used in the Section 9 to deal with near-singular vertices. Given a triangle T, let

$$\mu := r + \bar{r}, \qquad \bar{r} := \lfloor (r+1)/2 \rfloor, \tag{7.1}$$

and define

$$\mathcal{K}^T := \bigcup_{k=0}^{\bar{r}-1} \{\xi^T_{i,d-i-k,k}\}_{i=r+1}^{\mu-k}, \qquad \mathcal{L}^T := \bigcup_{k=0}^{\bar{r}-1} \{\xi^T_{i,d-i-k,k}\}_{i=\mu-k+1}^{\mu+\bar{r}-2k},$$

$$\widetilde{\mathcal{K}}^T := \bigcup_{j=0}^{\bar{r}-1} \{\xi^T_{i,j,d-i-j}\}_{i=r+1}^{\mu-j}, \qquad \widetilde{\mathcal{L}}^T := \bigcup_{j=0}^{\bar{r}-1} \{\xi^T_{i,j,d-i-j}\}_{i=\mu-j+1}^{\mu+\bar{r}-2j}$$

These sets are illustrated in Fig. 1.

Lemma 7.3. *Suppose $v \in \mathcal{V}^{\delta}_{NS}$ is attached to the four neighbors $v_1, \ldots, v_4$ (in counterclockwise order). Let $\triangle_v$ be the corresponding triangulation consisting of the four triangles $T_i := \langle v, v_i, v_{i+1}\rangle$, $i = 1, \ldots, 4$, where v_5 is identified with v_1. Let*

$$\Gamma_v := \{\xi \in \mathcal{D}^T_{d-r-1}(v) : \; \xi \notin \mathcal{L}^T \cup \widetilde{\mathcal{L}}^T \cup \mathcal{K}^T \cup \widetilde{\mathcal{K}}^T), \tag{7.2}$$

where T is any one of the triangles $T_1, \ldots, T_4$, and let $s \in \mathcal{S}^{d-r-1}_d(\triangle_v)$. Then if δ is sufficiently small, the coefficients of s associated with domain points in the disk $\mathcal{D}_{d-r-1}(v)$ are uniquely determined by the coefficients associated with domain points in the set

$$\Lambda_v := \Gamma_v \cup \bigcup_{\ell=1}^{4} \mathcal{K}^{T_\ell}. \tag{7.3}$$

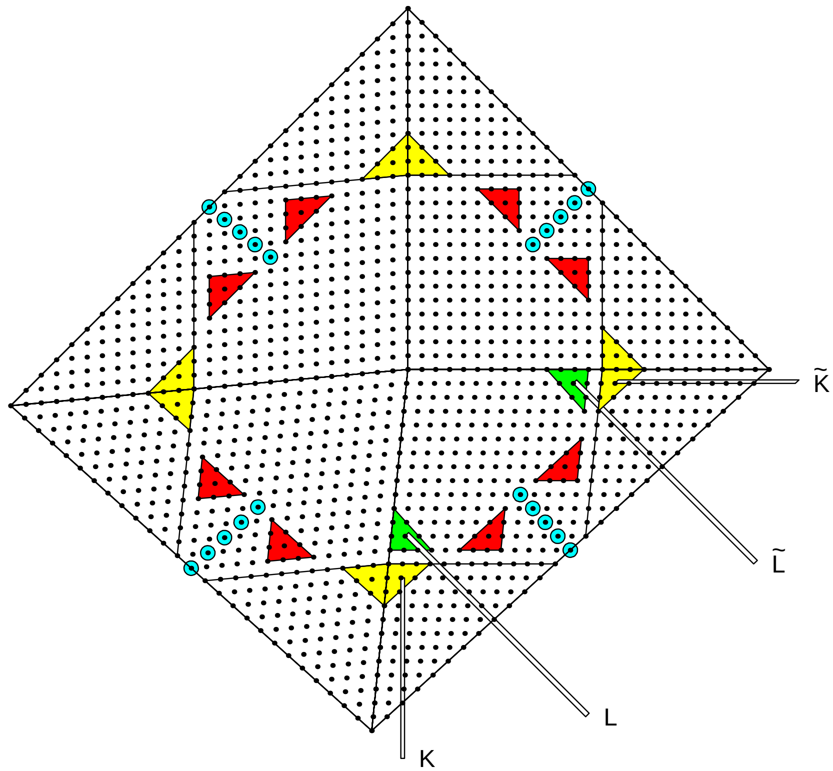

Fig. 1. Domain points in Lemma 7.3 with $r = 8$, $\bar{r} = 4$, $\mu = 12$, and $d = 26$.

Moreover, there exists a positive constant δ_0 depending only on d and the smallest angle $\theta_\triangle$ in $\triangle$ such that if $C := \max_{\xi\in\Lambda_v} |c_\xi|$, then $|c_\xi| \le KC$ for all $\xi \in \mathcal{D}_{d-r-1}(v)$, where K is a constant depending only on d and $\theta_\triangle$.

Proof: Without loss of generality, we may assume that $T = T_1$. Let

$$\begin{aligned} v_3 =& \alpha_1 v + \alpha_2 v_1 + \alpha_3 v_2 \\ v_4 =& \beta_1 v + \beta_2 v_1 + \beta_3 v_2. \end{aligned}$$

Suppose that all of the coefficients of s corresponding to domain points in Λ_v have been fixed. Since s is in C^{d-r-1} around the vertex v, it suffices to show that the unspecified coefficients in $T \cap \mathcal{D}_{d-r-1}(v)$ (namely those with subscripts lying in $\mathcal{L}$ and in $\widetilde{\mathcal{L}}$) are uniquely determined by the smoothness conditions. To this end we write down all smoothness conditions of the form (6.6) across the edges $e_1 := \langle v, v_1\rangle$ and $e_2 := \langle v, v_2\rangle$ which involve these coefficients. Suppose we put them into a vector in the order

$$c_{r+2,\tilde{d},\bar{r}-1}, c_{r+3,\tilde{d},\bar{r}-2}, c_{r+4,\tilde{d}-1,\bar{r}-2}, \ldots, c_{\mu,\tilde{d},0}, \ldots, c_{\mu+\bar{r},\tilde{d}-\bar{r},0}, \tag{7.4}$$

followed by

$$c_{r+2,\bar{r}-1,\tilde{d}}, c_{r+3,\bar{r}-2,\tilde{d}}, c_{r+4,\bar{r}-2,\tilde{d}-1}, \ldots, c_{\mu,0,\tilde{d}}, \ldots, c_{\mu+\bar{r},0,\tilde{d}-\bar{r}}. \tag{7.5}$$

where $\tilde{d} = d - \mu - 1$. (Here we have suppressed the superscript T on the coefficients to simplify the notation). The vector c has length $2m$ with $m := 1 + 2 + \cdots + \bar{r} = \binom{\bar{r}+1}{2}$. Note that the coefficients in both (7.4) and (7.5) fall naturally into subsets of size $1, 2, \ldots, \bar{r}$.

We also need to exercise some care in the order in which we write down the smoothness conditions. We start with those associated with edge e_2. As the first equation, we write the $C^{d-\mu}$ condition which involves only the coefficient $c_{r+2,\tilde{d},\bar{r}-1}$ from $\mathcal{L}$. Next we write two conditions, namely the $C^{d-\mu}$ and $C^{d-\mu+1}$ conditions which involve only the three coefficients from $\mathcal{L}$ with third subscript $k \geq \bar{r} - 2$. Finally, we write the $\bar{r}$ conditions for $C^{d-\mu}$ up to C^{d-r-1} which involve all the coefficients in $\mathcal{L}$. So far this is a total of m conditions. We now repeat the process for the conditions across the edge e_1, and end up with a system of the form

$$\begin{pmatrix} A & B \\ \tilde{B} & \tilde{A} \end{pmatrix} c = R, \tag{7.6}$$

where all four blocks in the matrix are of size $m \times m$.

We now examine these blocks in detail. The matrix A is a lower triangular block matrix of the form

$$A = \begin{pmatrix} A_1 & & & \\ \times & A_2 & & \\ \times & \times & \ddots & \\ \times & \times & \ldots & A_{\bar{r}} \end{pmatrix},$$

where

$$A_i = \alpha_1^{i^2} \alpha_2^{\kappa_i - i^2} C_i$$

is an $i \times i$ matrix with $\kappa_i := \sum_{j=0}^{i-1}(d - \mu + j)$. Here $C_i := M_i(\frac{1}{(m+n+1)!})_{m,n=0}^{i-1}$, where M_i is a nonzero product of factorials. The matrix $\tilde{A}$ has a similar structure with

$$\tilde{A}_i = \beta_1^{i^2} \beta_3^{\kappa_i - i^2} C_i.$$

Now observe that every entry of B involves some positive power of α_3, while every entry of $\tilde{B}$ involves some positive power of β_2. The remaining α_i and β_i are bounded away from 0 by a constant depending on the smallest angle $\theta_\triangle$ in $\triangle$. Let $D(\delta)$ be the determinant of the matrix in (7.6). Then $D(0) = \det(A)\det(\tilde{A})$ is bounded below by a positive constant D_0 which depends only on d and $\theta_\triangle$. But then by continuity, there exists a δ_0 depending only on d and $\theta_\triangle$ such that $D(\delta) \geq D_0/2$ for all $\delta \leq \delta_0$. $\square$

§8. Propogation

In the following section we are going to use the approach described in the previous section to construct a set of locally supported splines $\{\phi_\xi\}_{\xi\in\Gamma}$ which satisfy the duality condition (6.3) and properties H1)–H3) of Theorem 5.1. This requires a careful choice of Γ. As observed in [3,10,11], to this end it is useful to separate the domain points in $\mathcal{D}$ into certain subsets. For the remainder of the paper, let

$$\begin{aligned}
\mathcal{D}_\mu^T(v_\ell) &:= \{\xi \in \mathcal{D}_T : \operatorname{dist}(\xi, v_\ell) \le \mu\} \\
\mathcal{A}^T(v_\ell) &:= \{\xi \in \mathcal{D}_T : \operatorname{dist}(\xi, v_\ell) > \mu,\ \operatorname{dist}(\xi, e_\ell) \le r, \\
&\qquad\qquad \operatorname{dist}(\xi, e_{\ell+2}) \le r\} \\
\mathcal{C}^T &:= \{\xi \in \mathcal{D}^T : \operatorname{dist}(\xi, v_j) < d - r,\quad j = 1,2,3\},
\end{aligned} \tag{8.1}$$

where we define $e_\ell := \langle v_\ell, v_{\ell+1}\rangle$ (identifying v_4 with v_1). We also define

$$\begin{aligned}
\mathcal{F}^T(e_\ell) &:= \{\xi \in \mathcal{D}_T : \operatorname{dist}(\xi, e_\ell) \le r\} \\
\mathcal{E}^T(e_\ell) &:= \{\xi \in \mathcal{F}^T(e_\ell) : |\operatorname{dist}(\xi, v_\ell) - \operatorname{dist}(\xi, v_{\ell+1})| \le d - 3r - 2\} \\
\mathcal{G}_L^T(e_\ell) &:= \{\xi \in \mathcal{F}^T(e_\ell) :\ \operatorname{dist}(\xi, v_\ell) < \operatorname{dist}(\xi, v_{\ell+1}) \text{ and} \\
&\qquad\qquad \xi \notin D_\mu^T(v_\ell) \cup A^T(v_\ell) \cup \mathcal{E}^T(e_\ell)\} \\
\mathcal{G}_R^T(e_\ell) &:= \{\xi \in \mathcal{F}^T(e_\ell) :\ \operatorname{dist}(\xi, v_\ell) > \operatorname{dist}(\xi, v_{\ell+1}) \text{ and} \\
&\qquad\qquad \xi \notin D_\mu^T(v_{\ell+1}) \cup A^T(v_{\ell+1}) \cup \mathcal{E}^T(e_\ell).\}
\end{aligned}$$

The following lemma is implicit in several earlier papers [3, 10,11].

Lemma 8.1. *Suppose $T := \langle v_1, v_2, v_3\rangle$ and $\widetilde{T} := \langle v_4, v_2, v_3\rangle$ are two adjoining triangles sharing the edge $e := \langle v_2, v_3\rangle$, and that $e \notin \mathcal{E}_{ND}(v_2) \cup \mathcal{E}_{ND}(v_3)$. Suppose s is a spline in $\mathcal{S}_d^r(\triangle)$ whose coefficients are known for all domain points in $\mathcal{D}_\mu^T(v_2)$, $\mathcal{D}_\mu^T(v_3)$, and $\mathcal{E}^T(e)$. Suppose the coefficients are also known for all points in any two of the sets $\mathcal{A}^T(v_2)$, $\mathcal{A}^{\widetilde{T}}(v_2)$ or $\mathcal{G}_L^T(e)$, and for all points in any two of the sets $\mathcal{A}^T(v_3)$, $\mathcal{A}^{\widetilde{T}}(v_3)$, or $\mathcal{G}_R^{\widetilde{T}}(e)$. Then all unspecified coefficients of s in $\{\xi \in \mathcal{D}_{2r}(v_2) \cup \mathcal{D}_{2r}(v_3) : d(\xi, e) \le r\}$ are uniquely determined by the smoothness conditions.*

Proof: We alternately compute the coefficients in the arcs $a_{m,e}(v_2)$ and $a_{m,e}(v_3)$ for each $m = \mu+1, \ldots, 2r$, using Lemma 6.1 or Lemma 6.2, depending on which coefficients are given. □

Note that in Lemma 8.1, if e is degenerate at v_2, we cannot choose both $\mathcal{A}^T(v_2)$ and $\mathcal{A}^{\widetilde{T}}(v_2)$. In order to control the size of coefficients (cf. Lemma 6.2) we should also avoid this choice whenever e is near-degenerate at v_2. The analogous observation holds at v_3. A careful examination of Lemma 8.1 shows that if s has nonzero coefficients for some points in $\mathcal{D}_{2r}(v_2)$, then the computed coefficients can be nonzero for some points in $\mathcal{D}_{2r}(v_3)$. We refer to this as *propogation.* We are particularly concerned about getting nonzero coefficients in one of the sets $\mathcal{A}^T(v_3)$ or $\mathcal{A}^{\widetilde{T}}(v_3)$, since these can then propogate further. The following lemma shows how such propogation can be stopped.

Lemma 8.2. *Let T and $\widetilde{T}$ be as in Lemma 8.1 where $v_3 \notin \mathcal{V}_{NS}$. Suppose $s \in \mathcal{S}_d^r(\triangle)$ is a spline whose coefficients are zero for all domain points in some set Γ_0 which includes $\mathcal{D}_\mu^T(v_2)$, $\mathcal{D}_\mu^T(v_3)$, $\mathcal{E}^T(e)$, $\mathcal{A}^{\widetilde{T}}(v_3)$, and $\mathcal{G}_R^{\widetilde{T}}(v_3)$. In addition, suppose one of the following holds:*

1) *Γ_0 contains $\mathcal{G}_L^T(v_2)$,*
2) *Γ_0 contains $\mathcal{A}^T(v_2)$ and $\mathcal{A}^{\widetilde{T}}(v_2)$.*

Then the coefficients of s associated with points in $\mathcal{A}^T(v_3)$ must be zero.

Proof: In case 1), a careful examination of the smoothness conditions shows that in applying Lemma 6.1 to compute coefficients for points in $\mathcal{A}^T(v_3)$, we always get zero. In case 2), using Lemma 6.1 for the arcs around v_2 leads to zero coefficients for points in $\mathcal{G}_L^T(v_2)$, and the claim follows as before. □

§9. A Space of Super-splines with a Stable Local Basis

Let δ_0 be the constant defined in Lemma 7.3, and set $\rho := (\rho_1, \ldots, \rho_n)$ with

$$\rho_i = \begin{cases} d - r - 1, & v \in \mathcal{V}_{NS}^{\delta_0} \\ \mu, & \text{otherwise}, \end{cases} \tag{9.1}$$

where μ is defined in (7.1). We shall prove Theorem 1.1 by applying Theorem 5.1 to the *super-spline space*

$$\mathcal{SS} := \mathcal{S}_d^{r,\rho}(\triangle) = \{s \in S_d^r(\triangle) : \ s \in C^{\rho_i}(v_i), \ i = 1, \ldots, n\},$$

where $s \in C^{\rho_i}(v_i)$ means that the derivatives up to order ρ_i of the polynomial pieces $s_T := s|_T$ on triangles T sharing the vertex v_i all have the same values at v_i.

In the sequel we hold δ_0 fixed, and so for ease of notation we drop it from the notation. In particular, given any triangulation $\triangle$ whose smallest angle exceeds $\theta_\triangle$, we write $\mathcal{V}_{NS} := \mathcal{V}_{NS}^{\delta_0}(\triangle)$ and $\mathcal{E}_{ND} := \mathcal{E}_{ND}^{\delta_0}(\triangle)$ for the sets of near-singular vertices and near-degenerate edges in $\triangle$, respectively. Let

$$\mathcal{V}_i := \{v : \ \# \, \mathcal{E}_{ND}(v) = i\}, \qquad i = 0, 1, 2.$$

Our aim now is to construct a stable basis for $\mathcal{SS}$. Following the discussion in Sect. 6, we need to describe an appropriate minimal determining set Γ for $\mathcal{SS}$ in such a way that the corresponding set of basis functions $\{\phi_\xi\}_{\xi \in \Gamma}$ possess properties H1) – H3) of Theorem 5.1. To get these properties requires considerable care in the choice of Γ.

Theorem 9.1. *Choose the set Γ as follows:*

1) *For each vertex $v \notin \mathcal{V}_{NS}$, pick a triangle T with vertex at v and choose all points in the set $\mathcal{D}_\mu^T(v)$.*

2) *For each vertex $v \in \mathcal{V}_{NS}$, pick a triangle T with first vertex at v and choose all points in the set*

$$\Gamma_v := \{\xi \in \mathcal{D}^T_{d-r-1}(v) : \ \xi \notin \mathcal{L}^T \cup \widetilde{\mathcal{L}}^T \cup \mathcal{K}^T \cup \widetilde{\mathcal{K}}^T\}. \tag{9.2}$$

3) *For each edge $e := \langle v, u\rangle$ with $v, u \notin \mathcal{V}_{NS}$, include the set $\mathcal{E}^T(e)$, where T is a triangle containing the edge e. If e is a boundary edge, there is only one such triangle, while if it is an interior edge, we can choose either of the two triangles containing e. If e is a boundary edge, also include the two sets $\mathcal{G}^T_L(e)$ and $\mathcal{G}^T_R(e)$.*

4) *Suppose $v \notin \mathcal{V}_{NS}$ is connected to $v_1, \ldots, v_n$ in clockwise order, and suppose $1 \le i_1 < \cdots < i_k < n$ are such that $e_{i_j} \in \mathcal{E}_{ND}(v_{i_j}) \cup \mathcal{E}_{ND}(v)$, where $e_i := \langle v, v_i\rangle$ for $i = 1, \ldots, n$. Let $J_v := \{i_1, \ldots, i_k\}$. Define $T_i := \langle v, v_i, v_{i+1}\rangle$ for $i = 1, \ldots, n-1$, and let $T_0 := T_n := \langle v, v_n, v_1\rangle$ if v is an interior vertex.*

   a) *Include the sets $\mathcal{G}_L^{T_{i_j-1}}(e_{i_j})$ for all $1 \le j \le k$ such that $v_{i_j} \notin \mathcal{V}_{NS}$.*

   b) *Include the sets $\mathcal{A}^{T_i}(v)$ for all $1 \le i \le n-1$ such that $i \notin J_v$.*

   c) *Include $\mathcal{A}^{T_n}(v)$ if v is an interior vertex.*

5) *For all triangles $T = \langle v, u, w\rangle$ with $u, v, w \notin \mathcal{V}_{NS}$, include the set $\mathcal{C}^T$.*

Then Γ is a minimal determining set for $\mathcal{SS}$, and there exists a corresponding basis for $\mathcal{SS}$ consisting of splines $\{\phi_\xi\}_{\xi\in\Gamma}$ satisfying properties H1) – H3) of Theorem 5.1.

Proof: We claim that Γ is well-defined. In particular, a simple geometric argument shows that for any interior vertex $v \notin \mathcal{V}_{NS}$, there is always at least one edge attached to v which is not near degenerate at either end. In the numbering of the edges in item 4) above, we can choose this edge to be $\langle v, v_n\rangle$. We now show that Γ is a determining set, i.e., if we prescribe the coefficients of a spline $s \in \mathcal{SS}$ corresponding to all the points in Γ, then all other coefficients of s can be uniquely computed. This can be done as follows:

Step 1. We first work on the disks of the form $\mathcal{D}_\mu(v)$ for $v \notin \mathcal{V}_{NS}$. Note that s is in $C^\mu(v)$ and Γ includes all points in one subtriangle intersected with $\mathcal{D}_\mu(v)$. Then all coefficients in the disk $\mathcal{D}_\mu(v)$ can be uniquely computed using Lemma 6.1.

Step 2. For each $v \in \mathcal{V}_{NS}$, we use Lemma 7.3 on the disk $\mathcal{D}_{d-r-1}(v)$.

Step 3. For each $v \notin \mathcal{V}_{NS}$, we use Lemma 8.1 on the disk $\mathcal{D}_{2r}(v)$. We proceed by first doing all rings of size $\mu+1$, then all of size $\mu+2$, etc., until we have completed the rings of size $2r$. In computing coefficients in a ring $R_m(v)$, we process one arc $a_{m,e}(v)$ after another, always proceeding in a *clockwise* direction. To show that this process works, we have to show how to start it, and that once started we can continue all the way around the vertex. Let $v_1, \ldots, v_n$ be the neighboring vertices as in hypothesis 4) of the theorem. These have been numbered so that $e_n \notin \mathcal{E}_{ND}(v) \cup \mathcal{E}_{ND}(v_n)$. This assures that Γ includes $\mathcal{A}^{T_n}(v)$. Now we can apply Lemma 8.1 to the arc a_{m,e_n} since

a) if $v_1 \in \mathcal{V}_{NS}$, we know the coefficients of s for points in $\mathcal{G}_L^{T_n}(e_n)$ in as much as they are included in $\mathcal{D}_{d-r-1}(v_1)$,

b) if $e_1 \in \mathcal{E}_{ND}(v_1)$ but $v_1 \notin \mathcal{V}_{NS}$, Γ includes $\mathcal{G}_L^{T_n}(e_n)$,

c) if $e_1 \in \mathcal{E}_{ND}(v)$, Γ includes $\mathcal{G}_L^{T_n}(e_n)$,

d) otherwise $e_1 \notin \mathcal{E}_{ND}(v) \cup \mathcal{E}_{ND}(v_1)$, and Γ includes $\mathcal{A}^{T_n}(v)$ and $\mathcal{A}^{T_1}(v)$.

Once we have computed a_{m,e_n}, we then have the coefficients for points in $\mathcal{A}^{T_1}(v)$, and the process can be repeated on the arc a_{m,e_1}, and then on around the vertex.

Step 4. Suppose T and $\widetilde{T}$ are the two triangles sharing an interior edge $e = \langle v, u \rangle$ with $v, u \notin \mathcal{V}_{NS}$, and that $\mathcal{E}^T(e)$ is included in Γ but $\mathcal{E}^{\widetilde{T}}(e)$ is not. Then the coefficients in $\mathcal{E}^{\widetilde{T}}(e) \setminus [\mathcal{D}_{2r}(v) \cup \mathcal{D}_{2r}(u)]$ can be computed from the smoothness conditions of Lemma 6.1.

For each $\xi \in \Gamma$, we now construct a locally supported ϕ_ξ which satisfies the duality condition (6.3). First we set the coefficient corresponding to ξ to 1, and the coefficients corresponding to all other $\eta \in \Gamma$ to 0. We then solve for the remaining coefficients of ϕ_ξ as described above. We note that the computed coefficients remain bounded by a constant depending only on d and $\theta_\triangle$. In particular, Lemma 6.2 is only used to compute coefficients in a ring $R_m(v)$ when $v \notin \mathcal{V}_{NS}$, so that the numbers α^{-1} and γ^{-1} entering into the bound on the size of the coefficients in Lemma 6.2 are themselves bounded by a constant depending on d and $\theta_\triangle$. This assures that the ϕ_ξ satisfy hypothesis H2) of Theorem 5.1.

Since Γ is a determining set and ϕ_ξ satisfy (6.3), by the discussion in Sect. 6 we conclude that Γ is a minimal determining set with dim $\mathcal{SS} = \#\Gamma$, and $\{\phi_\xi\}_{\xi \in \Gamma}$ is a basis for $\mathcal{SS}$. Given $\xi \in \Gamma$, we now discuss the support properties of ϕ_ξ. Let $\Gamma_0(\xi) = \Gamma \setminus \{\xi\}$. Then all of the coefficients of ϕ_ξ associated with points in $\Gamma_0(\xi)$ are *set* to zero. We consider several cases depending on where ξ lies.

Case 1: Suppose $\xi \in C^T$ for some triangle T. Since the coefficients corresponding to points in C^T do not enter any smoothness conditions, we conclude that the only nonzero coefficient of ϕ_ξ is the one corresponding to ξ, and thus the support of ϕ_ξ is T.

Case 2: Suppose $\xi \in \mathcal{E}^T(e)$ where $e := \langle v, u \rangle$ is a boundary edge of a triangle T, and that $\xi \notin \mathcal{D}_{2r}(v) \cup \mathcal{D}_{2r}(u)$. Then the coefficient corresponding to ξ does not enter any smoothness conditions, and thus remains the only nonzero coefficient of ϕ_ξ. It follows that the support of ϕ_ξ is T.

Case 3: Suppose $T = \langle v_1, v_2, v_3 \rangle$ and $\widetilde{T} = \langle v_4, v_2, v_3 \rangle$ are two triangles sharing an interior edge $e = \langle v_2, v_3 \rangle$ with $v_2, v_3 \notin \mathcal{V}_{NS}$. Let $\xi \in \mathcal{E}^T(e)$, and suppose $\xi \notin \mathcal{D}_{2r}^T(v_2) \cup \mathcal{D}_{2r}^T(v_3)$. Then the coefficients of ϕ_ξ corresponding to points in $\mathcal{D}_{2r}^T(v_2) \cup \mathcal{D}_{2r}^T(v_3)$ will be zero, but using the smoothness conditions, we can get nonzero coefficients for points in the set $\mathcal{E}^{\widetilde{T}}(e)$. Since all other coefficients are zero, we conclude that the support of ϕ_ξ is $T \cup \widetilde{T}$.

Case 4: Suppose $\xi \in \mathcal{D}_{2r}(u)$ where $u \notin \mathcal{V}_{NS}$. We assume u is an interior vertex (the case where it is a boundary vertex is similar). Let $u_1, \ldots, u_n$ and $w_1, \ldots, w_m$ be the vertices in clockwise order which lie on the boundaries of $\text{star}(u)$ and of $\text{star}^2(u)$, respectively. Note that $\Gamma_0(\xi)$ includes the disks $\mathcal{D}_\mu(v)$ for all $v \neq u$. It also includes the disks $\mathcal{D}_{d-r-1}(v)$ for all $v \in \mathcal{V}_{NS}$. There are two subcases:

a) If $u_i \notin \mathcal{V}_{NS}$, then the nonzero coefficients can propogate to the disk $\mathcal{D}_{2r}(u_i)$. However, we claim that they cannot further propogate around the vertices w_j. Indeed, since we process the arcs around w_j in *clockwise* order, to show that propogation along the edge $e_{ij} := \langle u_i, w_j \rangle$ is blocked, it suffices to show that the computed coefficients associated with points in $\mathcal{A}^{T_{ij}}(w_j)$ are zero, where T_{ij} is the triangle with vertices u_i, w_j, v in counter-clockwise order for some v. This is automatic if $w_j \in \mathcal{V}_{NS}$ since $\mathcal{A}^{T_{ij}}(w_j) \subset \mathcal{D}_{d-r-1}(w_j) \subset \Gamma_0(\xi)$. Now if $w_j \notin \mathcal{V}_{NS}$ and $e_{ij} \in \mathcal{E}_{ND}(u_i) \cup \mathcal{E}_{ND}(w_j)$, Lemma 8.2 insures that the coefficients associated with $\mathcal{A}^{T_{ij}}(w_j)$ are zero. Finally, if e_{ij} is not near-degenerate at either end, then propogation is again blocked since by the choice of Γ (see item 4), $\Gamma_0(\xi)$ includes the set $\mathcal{A}^{T_{ij}}(w_j)$.

b) If $u_i \in \mathcal{V}_{NS}$, then applying Lemma 7.3, nonzero coefficients in $\mathcal{D}_{2r}(u)$ can propogate to the disk $\mathcal{D}_{2r}(w_j)$ around the vertex w_j which lies on the opposite side from the near singular vertex u_i, where w_j is a vertex connecting to u_i. Note that $w_j \notin \mathcal{V}_{NS}$. Now treating u_i as u and arguing as in subcase a), we see that it cannot propogate any further.

We conclude that the support of ϕ_ξ lies in

$$\text{star}(u) \cup \bigcup_{u_i \notin \mathcal{V}_{NS}} \text{star}(u_i) \bigcup_{u_i \in \mathcal{V}_{NS}} \text{star}^2(u_i) \quad \subset \quad \text{star}^3(u).$$

Case 5: Suppose $\xi \in \Gamma_u$ where $u \in \mathcal{V}_{NS}$. All coefficients associated with points in the disks of the form $\mathcal{D}_\mu^T(v)$ with $v \notin \mathcal{V}_{NS}$ are zero. Let $v_1, \ldots, v_4$ be the vertices attached to v. Since it is impossible for two near-singular vertices to be neighbors, $v_i \notin \mathcal{V}_{NS}$ for $i = 1, \ldots, 4$. Now nonzero coefficients associated with points in Γ_u may propogate to points in the disks of radius $2r$ around the vertices $v_1, \ldots, v_4$. However, arguing as in Case 5a), we see that they cannot propogate any further, and thus the support of ϕ_ξ is contained in $\text{star}^2(u)$.

We have shown that the splines $\{\phi_\xi\}_{\xi \in \Gamma}$ satisfy properties H1)–H2) of Theorem 5.1. It remains to verify that the ϕ_ξ satisfy H3) of the theorem. Fix $T := \langle v_1, v_2, v_3 \rangle$, and let Σ_T be the set of ξ such that the support $\sigma(\phi_\xi)$ includes T. By the support properties of the ϕ_ξ, each $\xi \in \Sigma_T$ must lie in a triangle which is contained in $\cup_{i=1}^3 \text{star}^3(v_i)$. Now by Lemma 3.1 the number of such triangles is bounded by a constant C depending only on $\theta_\triangle$. Since each triangle contains at most $\binom{d+2}{2}$ domain points, it follows that the cardinality of Σ_T is bounded $C\binom{d+2}{2}$. $\square$

We conclude this section by showing that a natural renorming of the splines $\{\phi_\xi\}_{\xi \in \Gamma}$ in Theorem 5.1 provides a p-stable basis for $\mathcal{SS}$.

Theorem 9.2. *Fix $1 \le p \le \infty$. Let $\Phi := \{\psi_\xi = (A_{T_\xi})^{-1/p}\phi_\xi\}_{\xi\in\Gamma}$, where for each ξ, T_ξ is the triangle containing ξ. Then Φ forms a p-stable basis for $\mathcal{S}$ in the sense that there exist constants K_{12} and K_{13} dependent only on $\theta_\triangle$ such that*

$$K_{12}\|c\|_p \;\le\; \|\sum_{\xi\in\Gamma} c_\xi\psi_\xi\|_p \;\le\; K_{13}\|c\|_p \tag{9.3}$$

for all choices of the coefficient vector $c = (c_\xi)_{\xi\in\Gamma}$.

Proof: We consider the case $1 \le p < \infty$ as the case $p = \infty$ is similar (and simpler). First we establish the upper bound in (9.3). Suppose $s = \sum_{\xi\in\Gamma} c_\xi\psi_\xi$. Fix a triangle T, and let Σ_T be the set appearing in H3). By the uniform boundedness of the ϕ_ξ,

$$\int_T |s|^p = \int_T |\sum_{\xi\in\Sigma_T} c_\xi(A_{T_\xi})^{-1/p}\phi_\xi|^p \le K_8^p K_9^{p-1} \max_{\xi\in\Sigma_T} \frac{A_T}{A_{T_\xi}} \sum_{\xi\in\Sigma_T} |c_\xi|^p$$

where K_8 and K_9 are the constants in H2 and H3 of Theorem 5.1. For each $\xi \in \Sigma_T$, T and T_ξ both lie in $\sigma(\phi_\xi)$. Thus, Lemma 3.2 with $\ell = 3$ implies $\max_{\xi\in\Sigma_T} A_T/A_{T_\xi} \le K_3^2$.

We now sum over all triangles T. A given c_ξ can appear more than once on the right-hand side. In fact, the number of times it appears is equal to the number of triangles in the support of ϕ_ξ. Since $\sigma(\phi_\xi)$ is contained in $\mathrm{star}^3(v_\xi)$ for some vertex v_ξ, the number of triangles it contains is bounded by the constant K_2 with $\ell = 3$ in Lemma 3.1. Thus,

$$\|s\|_p^p = \sum_{T\in\triangle} \int_T |s|^p \le K_2 K_3^2 K_8^p K_9^{p-1} \;\|c\|_p^p,$$

and the proof of the upper bound in (9.3) is complete.

We now establish the lower bound in (9.3). Given a triangle T, we first estimate the size of the coefficients c_ξ for $\xi \in T$. For these ξ, we have $T_\xi = T$, and in view of the normalization, the Bernstein-Bézier coefficient of the polynomial s_T which agress with s on T is $c_\xi(A_T)^{-1/p}$. Now applying Lemma 4.1, we get

$$\sum_{\xi\in T\cap\Gamma} |c_\xi|^p = A_T \sum_{\xi\in T\cap\Gamma} |c_\xi(A_T)^{-1/p}|^p \le K_4^p\|s_T\|_{p,T}^p.$$

Summing over all T, we get

$$\|c\|_p^p \le \sum_{T\in\triangle}\sum_{\xi\in T\cap\Gamma} |c_\xi|^p \le K_4^p\|s\|_{p,\Omega}^p,$$

and the proof is complete. □

§10. Proof of Theorem 1.1

We are now in a position to apply Theorem 5.1 to establish our main theorem. Let $\{\phi_\xi\}_{\xi\in\Gamma}$ be the basis functions for $\mathcal{SS}$ constructed in the previous section. We now define corresponding linear functionals and an associated quasi-interpolation operator.

Choose $\xi \in \Gamma$, and suppose T_ξ is a triangle in which ξ lies. Then for any function $f \in L_1(\Omega)$, we define

$$\lambda_{\xi,m} f := \gamma_\xi(F_{m,B_{T_\xi}} f),$$

where $F_{m,B_{T_\xi}} f$ is the averaged Taylor polynomial associated with f and the disk B_{T_ξ}, and γ_ξ is the functional which when applied to a polynomial written in B-form, picks off the Bézier coefficient corresponding to the domain point ξ (cf. (6.2)). Note that $\lambda_{\xi,m}$ is a linear functional, and the value of $\lambda_{\xi,m} f$ depends only on values of f on the triangle T_ξ.

We have already seen in the previous section that the basis functions ϕ_ξ satisfy the hypotheseses H1) – H3) of Theorem 5.1. Using Lemmas 4.1 and 4.5, we have

$$|\lambda_{\xi,m} f| = |\gamma_\xi(F_{m,B_{T_\xi}} f)| \le \frac{K_4}{A_{T_\xi}^{1/p}} \|F_{m,B_{T_\xi}} f\|_{p,T_\xi} \le \frac{K_4 K_6}{A_{T_\xi}^{1/p}} \|f\|_{p,T_\xi}.$$

This shows that hypothesis H4) of Theorem 5.1 is satisfied.

To check H5), we have to show that Q_m reproduces polynomials of degree m. Given $f \in \mathcal{P}_m$, let $\sum_{\xi\in\Gamma} a_\xi \phi_\xi$ be its unique expansion in terms of ϕ_ξ. By Lemma 4.4, $F_{m,B_{T_\xi}} f = f$ for each $\xi \in \Gamma$. Thus, $\lambda_{\xi,m} f = \gamma_\xi F_{m,B_{T_\xi}} f = \gamma_\xi f = a_\xi$ for all $\xi \in \Gamma$, which implies that $Q_m f = f$. When $m = d$, $F_{d,B_{T_\xi}} f = f|_{T_\xi}$ for any spline $f \in \mathcal{SS}$. Then the same argument shows that Q_d reproduces all of $\mathcal{SS}$.

We have now verified that Q satisfies all of the hypotheses of Theorem 5.1, and our main result Theorem 1.1 follows immediately.

§11. Remarks

Remark 1. The basis splines constructed in Theorem 9.1 have maximal support on sets of the form $\mathrm{star}^3(v)$. The approximation results for the uniform norm given in [7] are based on a different super-spline space. The supports of their basis elements can be much larger, depending on the smallest angle in the triangulation.

Remark 2. General super-spline spaces with variable degrees of additional smoothness at the vertices were introduced in [14]. For $d \ge 3r+2$, local bases for them were constructed in [11]. However, the focus there was on dimension, and so the bases were constructed without concern for their stability in the presence of near-singular vertices or near-degenerate edges.

Remark 3. It is easy to see that when $d \geq 4r+1$, the supports of the basis splines constructed in Theorem 9.1 are at most star(v), recovering well-known finite-element results.

Remark 4. The estimate (1.1) can be generalized further by measuring the error on the left-hand side in a q norm, where $1 \leq p \leq q \leq \infty$. In this case the exponent of $|\triangle|$ on the right-hand side is replaced by $m+1-\alpha-\beta+1/p-1/q$. (See [13] for the univariate case).

Remark 5. When $d < 3r+2$ the appproximation order by splines has been established only for special triangulations, see [12].

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