Eric Schechter

Equivalents of Mingle and Positive Paradox

Abstract. Relevant logic is a proper subset of classical logic. It does not include among its theorems any of

positive paradox	$A \to (B \to A)$
mingle	$A \to (A \to A)$
linear order	$(A \to B) \vee (B \to A)$
unrelated extremes	$(A \wedge \overline{A}) \to (B \vee \overline{B})$

This article shows that those four formulas have different effects when added to relevant logic, and then lists many formulas that have the same effect as positive paradox or mingle.

Keywords: Relevant logic, mingle, positive paradox, Sugihara, matrix.

1. Introduction

This paper investigates some extensions of relevant logic; results are summarized at the beginning of Section 3. All results in this article refer to propositional logic; we will not consider predicate logic. Classical reasoning will be used for the metalogic throughout this paper. Also, our structural rules will be classical — i.e., in a derivation of $\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \vdash \mu$, the hypothesis λ_i's may be used in any order, and each λ_i may be used once, or more than once, or not at all.

Since relevant logic includes modus ponens, from any theorem of implication $\vdash P \to Q$ we easily obtain an inference rule $P \vdash Q$; we shall refer to that as the **inferential corollary** of $\vdash P \to Q$. But the corollary may be *strictly* weaker, as relevant logic does not obey the classical Deduction Theorem. For instance, among the formulas below, 26 is provable in relevant logic but 75 is not, though 26 is the inferential corollary of 75.

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2. Relevant logic

By *relevant logic* we shall mean the propositional logic given by the following two assumed inference rules:

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1. $\{A,\ A \to B\} \vdash\ B$ (modus ponens)
2. $\{A,\ B\} \vdash\ A \wedge B$ (adjunction)

and twelve axiom schemes:

3. $\vdash A \to [(A \to B) \to B]$ (assertion)
4. $\vdash [A \to (A \to B)] \to (A \to B)$ (contraction)
5. $\vdash (A \to B) \to [(B \to C) \to (A \to C)]$ (suffixing)
6. $\vdash (A \wedge B) \to A$ ($\wedge$-elimination)
7. $\vdash (A \wedge B) \to B$ ($\wedge$-elimination)
8. $\vdash A \to (A \vee B)$ ($\vee$-introduction)
9. $\vdash B \to (A \vee B)$ ($\vee$-introduction)
10. $\vdash [(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$ ($\wedge$-introduction)
11. $\vdash [(B \to A) \wedge (C \to A)] \to [(B \vee C) \to A]$ ($\vee$-elimination)
12. $\vdash [A \wedge (B \vee C)] \to [(A \wedge B) \vee C]$ (distributive)
13. $\vdash (A \to \overline{B}\,) \to (B \to \overline{A}\,)$ (contrapositive)
14. $\vdash \overline{\overline{A}} \to A$ (double negation)

A thorough introduction to relevant logic — including other, equivalent axiomatizations — can be found in [1] or [3].

For later reference, we also state (without proof) some of the basic theorems and inference rules that can be proved in relevant logic. The following list is not intended to be exhaustive in any way; it merely is intended to supply a modicum of intuition and to serve the needs of later parts of this article. We take $P \leftrightarrow Q$ as an abbreviation for $(P \to Q) \wedge (Q \to P)$.

15. $\vdash A \to A$ (identity)
16. $\{A \to B,\ B \to C\} \vdash\ A \to C$ (transitive)
17. $A \to (B \to C) \vdash\ B \to (A \to C)$ (permutation, inferential)
18. $A \to B \vdash\ (B \to C) \to (A \to C)$ (suffixing, inferential)
19. $\vdash (A \to B) \to [(C \to A) \to (C \to B)]$ (prefixing)
20. $A \to B \vdash\ (C \to A) \to (C \to B)$ (prefixing, inferential)
21. $[(A \to B) \to B] \to B \vdash\ A \to B$ (variant of contraction)
22. $\{A \to B,\ A \to C\} \vdash\ A \to (B \wedge C)$ ($\wedge$-introduction, inferential)

23. $\{B \to A,\, C \to A\} \vdash (B \vee C) \to A$ ($\vee$-elimination, inferential)
24. $\vdash (A \wedge B) \leftrightarrow (B \wedge A)$ and $\vdash (A \vee B) \leftrightarrow (B \vee A)$ (commutative)
25. $\vdash A \to (A \wedge A)$ and $\vdash (A \vee A) \to A$ (idempotent)
26. $A \to B \vdash (A \wedge C) \to (B \wedge C)$ ($\wedge$-suffixing)
27. $A \to B \vdash (A \vee C) \to (B \vee C)$ ($\vee$-suffixing)
28. $\vdash [A \wedge (A \to B)] \to B$ (conjunctive detachment)
29. $\vdash A \to \overline{\overline{A}}$ (converse of double negation)
30. $\vdash \overline{A \vee B} \leftrightarrow (\overline{A} \wedge \overline{B})$, $\vdash \overline{A \wedge B} \leftrightarrow (\overline{A} \vee \overline{B})$ (De Morgan's Laws)
31. $\vdash \left(\overline{A} \to A\right) \to A$ (reductio)
32. $\{A \to B,\, A \to \overline{B}\} \vdash \overline{A}$ (proof by contradiction, inferential)
33. $\{\overline{A} \to B,\, A \to B\} \vdash B$ (proof by cases, inferential)
34. $\vdash A \vee \overline{A}$ (excluded middle)
35. $\{B \to C,\, \overline{A} \to C\} \vdash (A \to B) \to C$ (mixing inference)
36. $\vdash (A \to B) \to (\overline{A} \vee B)$ (disjunctive consequence)
37. $\vdash [A \to (B \to C)] \leftrightarrow \left[\overline{A \to \overline{B}} \to C\right]$ (basic cotenability)

Using relevant logic, we can also prove this substitution principle:

> *Let A and A' be two formulas that satisfy $\vdash A \leftrightarrow A'$. Let X be a formula in which A appears at least once as a subformula, and let X' be the formula obtained from X by replacing one of the occurrences of subformula A with A'. Then $\vdash X \leftrightarrow X'$.*

For instance, A and $\overline{\overline{A}}$ can be used interchangeably.

The proofs given in the remainder of this paper are only sketches; for brevity we shall omit the most obvious steps. In particular, we shall freely use double negation and its converse (14, 29), identity (15), transitivity (16), commutativity (24), excluded middle (34), and substitution, sometimes without explicitly mentioning them.

3. A scale of irrelevancies

We now consider these five additions to relevant logic:

$$\begin{array}{rl}
\text{positive paradox} & A \to (B \to A) \\
\text{mingle} & A \to (A \to A) \\
\text{linear order} & (A \to B) \vee (B \to A) \\
\text{unrelated extremes} & (A \wedge \overline{A}) \to (B \vee \overline{B}) \\
\text{no addition at all} & \quad\text{—}
\end{array}$$

It is well known that adding positive paradox yields classical logic. Adding mingle yields a logic known as RM (for "relevant plus mingle"). Making no addition at all yields just relevant logic, included in the list for comparison.

In this section, we will show that each of those additions is stronger than the one below it. For instance, if we add mingle to relevant logic as an axiom scheme, then linear order becomes provable; but adding linear order does not make mingle provable. (This scale of five levels is not intended to be exhaustive in any respect; it simply contains the five levels that were of greatest interest to the author.) In Sections 4 and 5 we will investigate an assortment of equivalents of mingle and positive paradox.

Positive paradox yields mingle. Obvious.

Mingle does not yield positive paradox. The logic RM can be characterized by the following semantics, sometimes known as "Sugihara's matrix." For semantic values use the integers, with $0, 1, 2, 3, \ldots$ true and $-1, -2, -3, \ldots$ false. Evaluate formulas by these rules:

$$A \vee B = \max\{A, B\}, \qquad A \wedge B = \min\{A, B\}, \qquad \overline{A} = -A,$$

$$A \to B \ = \left\{ \begin{array}{ll} \max\{-A, B\} & \text{if } A \le B, \\ \min\{-A, B\} & \text{if } A > B. \end{array} \right.$$

It is fairly simple to verify that this interpretation makes tautological (i.e., always true) all the axioms of relevant logic, plus mingle, and also makes both of the assumed inference rules of relevant logic truth-preserving. Hence every theorem of RM is tautological in this interpretation. But positive paradox is not tautological. (It can also be shown that every tautology of this interpretation is a theorem of RM, but that is considerably harder; a proof can be found in [1].)

Mingle yields linear order. Here we borrow from Section 4 the fact that mingle and formula 53 are equivalent; thus we may assume $\vdash \overline{A \to B} \to (B \to A)$. On the other hand, we have $\overline{A \to B} \vee (A \to B)$ as an instance of 34. Combine those using 27 to obtain linear order.

Linear order does not yield mingle. That is shown by the following interpretation. Let -2 be false and let $-1, +1, +2$ be true values. Define

$$A \vee B = \max\{A, B\}, \qquad A \wedge B = \min\{A, B\}, \qquad \overline{A} = -A,$$

and define implication by this table:

$A \to B$	$B = -2$	$B = -1$	$B = +1$	$B = +2$
$A = -2$	$+2$	$+2$	$+2$	$+2$
$A = -1$	-2	-1	$+1$	$+2$
$A = +1$	-2	-2	-1	$+2$
$A = +2$	-2	-2	-2	$+2$

It is not hard to verify that this matrix is sound for the axioms and inference rules of relevant logic as well as linear order, but not mingle.

We remark that this interpretation falsifies $A \vdash A \to (A \to A)$, the inferential corollary of 46; but it makes valid $(A \to B) \to A \vdash A$, the inferential corollary of 68.

Linear order yields unrelated extremes. Abbreviate $\alpha = A \wedge \overline{A}$ and $\beta = B \wedge \overline{B}$. Then we also have $\overline{\alpha} = A \vee \overline{A}$ and $\overline{\beta} = B \vee \overline{B}$, by De Morgan's Laws. Both $\overline{\alpha}$ and $\overline{\beta}$ are theorems of relevant logic, and what we want to prove is $\alpha \to \overline{\beta}$.

step	formula(s)	reason
[1]	$\alpha \to A, \ \ \beta \to B$	$\wedge$-elimination (6)
[2]	$A \to \overline{\alpha}, \ \ B \to \overline{\beta}$	$\vee$-introduction (8)
[3]	$\alpha \to \overline{\alpha}, \ \ \beta \to \overline{\beta}$	[1], [2], transitive (16)
[4]	$(\overline{\beta} \to \alpha) \to (\overline{\alpha} \to \beta)$	contrapositive (13)
[5]	$(\overline{\alpha} \to \beta) \to (\overline{\alpha} \to \overline{\beta})$	[3], prefixing (20)
[6]	$\overline{\alpha} \to [(\overline{\alpha} \to \beta) \to \overline{\beta}\,]$	[5], permutation (17)
[7]	$\alpha \to [(\overline{\alpha} \to \beta) \to \overline{\beta}\,]$	[3],[6], transitive (16)
[8]	$\alpha \to [(\overline{\beta} \to \alpha) \to \overline{\beta}\,]$	[7],[4], suffixing, prefixing
[9]	$(\overline{\beta} \to \alpha) \to (\alpha \to \overline{\beta})$	[8], permutation (17)
[10]	$(\alpha \to \overline{\beta}) \to (\alpha \to \overline{\beta})$	identity (15)
[11]	$(\alpha \to \overline{\beta}) \vee (\overline{\beta} \to \alpha)$	linear order
[12]	$\alpha \to \overline{\beta}$	[9], [10], [11], $\vee$-elimination (23)

Unrelated extremes does not yield linear order. The following interpretation is known as "KR" in some of the literature. Let $\Omega = \{1, 2\}$; we use the subsets of Ω as semantic values. Take false values $\emptyset$, $\{1\}$ and true values $\{2\}$, $\{1,2\}$. Define

$$A \vee B = A \cup B, \qquad A \wedge B = A \cap B, \qquad \overline{A} = \Omega \setminus A.$$

Implication will be interpreted as follows:

$A \to B$	$B = \emptyset$	$B = \{1\}$	$B = \{2\}$	$B = \{1,2\}$
$A = \emptyset$	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$	$\{1,2\}$
$A = \{1\}$	$\emptyset$	$\{2\}$	$\emptyset$	$\{1,2\}$
$A = \{2\}$	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$
$A = \{1,2\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\{1,2\}$

It is not hard to verify that this matrix is sound for the axioms and inference rules of relevant logic as well as unrelated extremes, but not linear order.

We remark that this matrix also makes tautological the formulas $A \to (B \vee \overline{B})$ and $(A \wedge \overline{A}) \to B$, both stronger than unrelated extremes; thus neither of those yields linear order. Also, this matrix makes valid the inference rules $A \vdash A \to (A \to A)$ and $\{A, \overline{A} \vee B\} \vdash B$, so those do not yield linear order; those are the inferential corollaries of 46 and 71 respectively.

Relevant logic does not yield unrelated extremes. Belnap [2] showed soundness of relevant logic in a certain 8-valued semantics with this interesting property: If $P \to Q$ is a tautology of Belnap's semantics, then the formulas P and Q must share at least one propositional variable symbol; P and Q cannot be unrelated. Thus (for instance) the formula scheme $(A \wedge \overline{A}) \to (B \vee \overline{B})$ cannot be a theorem scheme in relevant logic.

4. Equivalents of mingle

We will show that mingle (formula 38) is equivalent to the other formulas and inference rules listed below. The list is not intended to be exhaustive in any formal sense. Members of this list were selected for their simplicity and their appeal to the author's intuition.

38. $\vdash A \to (A \to A)$
39. $\vdash (A \to B) \to [A \to (B \to B)]$
40. $A \to B \ \vdash\ A \to (B \to B)$
41. $\vdash (A \to C) \to \big\{(B \to C) \to \big[A \to (B \to C)\big]\big\}$
42. $A \to C \ \vdash\ (B \to C) \to \big[A \to (B \to C)\big]$

43. $\{A \to C,\, B \to C\} \vdash A \to (B \to C)$ (*Remark: compare with* 35)
44. $\vdash (A \to B) \to \big[A \to (A \to B)\big]$ (*Remark: compare with* 4)
45. $A \to B \vdash A \to (A \to B)$
46. $\vdash A \to (A \to (A \to A))$
47. $\vdash \overline{A} \to (\overline{A} \to \overline{A})$
48. $\vdash \overline{A} \to (A \to A)$
49. $\vdash A \to (\overline{A} \to A)$ (*Remark: compare with* 31)
50. $\vdash (\overline{A} \wedge B) \to (A \to B)$ (*Remark: compare with* 72)
51. $\{\overline{A}, B\} \vdash A \to B$
52. $\vdash \overline{A \to A} \to (B \to B)$
53. $\vdash \overline{A \to B} \to (B \to A)$
54. $\overline{A \to B} \vdash B \to A$
55. $A \vdash \overline{A} \to A$
56. $\vdash \overline{A \to A} \to (A \to A)$
57. $\vdash \overline{A} \to [A \to (A \to A)]$.

We will prove the equivalences in this order:

$$
\begin{array}{ccccccccccccc}
42 & \Rightarrow & 43 & \Rightarrow & 40 & & & & 49 & \Rightarrow & 50 & \Rightarrow & 51 & \Rightarrow & 52 \\
\Uparrow & & & & \Downarrow & & & & \Uparrow & & & & & & \Downarrow \\
41 & \Leftarrow & 39 & \Leftarrow & 38 & \Leftrightarrow & 47 & \Leftrightarrow & 48 & & & & & & 53 \\
\Downarrow & & & & \Uparrow & & & & \Uparrow & & & & & & \Downarrow \\
44 & \Rightarrow & 45 & \Rightarrow & 46 & & & & 57 & \Leftarrow & 56 & \Leftarrow & 55 & \Leftarrow & 54
\end{array}
$$

Here " $P \Rightarrow Q$ " does not actually mean "P implies Q." Rather, it means that if we add P to relevant logic, as an additional axiom or an additional assumed inference rule, then Q becomes provable. Following are the proofs:

$38 \Rightarrow 39$: Start from $B \to (B \to B)$ and apply prefixing (20).

$39 \Rightarrow 41$: Start from $(B \to C) \to [B \to (C \to C)]$. Use permutation a couple of times to get $C \to \big[(B \to C) \to (B \to C)\big]$. Prefixing then yields $(A \to C) \to \big\{A \to \big[(B \to C) \to (B \to C)\big]\big\}$. Now permutation yields 41.

$41 \Rightarrow 42 \Rightarrow 43$: Inferential corollaries.

$43 \Rightarrow 40$: Substitute $C = B$.

$40 \Rightarrow 38$: Substitute $B = A$.

$41 \Rightarrow 44$: $(A \to B) \to [(A \to B) \to (A \to (A \to B))]$ is an instance of 41. Combine that with contraction (4), to obtain 44.

$44 \Rightarrow 45$: Inferential corollary.

$45 \Rightarrow 46$: Applying 45 once with $B = A$ yields $\vdash A \to (A \to A)$. Then applying 45 with $B = (A \to A)$ yields 46.

$46 \Rightarrow 38$: Apply contraction (4).

$38 \Leftrightarrow 47$: Substitute $\overline{A}$ for A.

$47 \Leftrightarrow 48$: Use the contrapositive law (13).

$48 \Rightarrow 49$: Use permutation (17).

$49 \Rightarrow 50$:

[1]	$(\overline{A} \wedge B) \to \overline{A}$	$\wedge$-elimination (6)
[2]	$(\overline{A} \wedge B) \to B$	$\wedge$-elimination (7)
[3]	$(\overline{A} \wedge B) \to \left[\overline{\overline{A} \wedge B} \to (\overline{A} \wedge B)\right]$	49
[4]	$\left[\overline{\overline{A} \wedge B} \to (\overline{A} \wedge B)\right] \to \left[\overline{\overline{A} \wedge B} \to \overline{A}\right]$	[1], prefixing (20)
[5]	$\left[\overline{\overline{A} \wedge B} \to \overline{A}\right] \to [A \to (\overline{A} \wedge B)]$	contrapositive
[6]	$[A \to (\overline{A} \wedge B)] \to (A \to B)$	[2], prefixing (20)
[7]	$(\overline{A} \wedge B) \to (A \to B)$	[3][4][5][6] transitive (16)

$50 \Rightarrow 51$: Inferential corollary.

$51 \Rightarrow 52$: $\left\{\overline{\overline{A \to A}}, B \to B\right\} \vdash \overline{A \to A} \to (B \to B)$ is an instance of 51.

$52 \Rightarrow 53$: By substitution, switch roles of A and B; thus 52 yields $\vdash \overline{B \to B} \to (A \to A)$. By permutation, $\vdash A \to \big(\overline{B \to B} \to A\big)$. By contraposition, $\vdash A \to [\overline{A} \to (B \to B)]$. Several more permutations yield $\vdash B \to [\overline{A} \to (A \to B)]$. Then contrapositive again: $\vdash B \to (\overline{A \to B} \to A)$. Finally, one more permutation yields 53.

$53 \Rightarrow 54$: Inferential corollary.

$54 \Rightarrow 55$:

[1]	$(A \to \overline{A}) \to \overline{A}$	reductio (31)
[2]	$A \to \overline{A \to \overline{A}}$	[1], contrapositive
[3]	A	hypothesis of 55
[4]	$\overline{A \to \overline{A}}$	[3], [2], modus ponens
[5]	$\overline{A} \to A$	[4], 54

55 ⇒ 56: $A \to A \vdash \overline{A \to A} \to (A \to A)$ is an instance of 55.

56 ⇒ 57: By permutation we have $A \to \big(\overline{A \to A} \to A\big)$. Then $A \to [\overline{A} \to (A \to A)]$ by contraposition. Another permutation yields 57.

57 ⇒ 48: We have $[A \to (A \to A)] \to (A \to A)$ by contraction. Combine that with 57, using transitivity.

5. Equivalents of positive paradox

We will show that positive paradox (formula 58) is equivalent to the other formulas listed below:

58. $\vdash A \to (B \to A)$
59. $\vdash B \to (A \to A)$
60. $A \vdash B \to A$
61. $\overline{A} \vdash A \to B$
62. $\vdash \overline{A} \to (A \to B)$
63. $\vdash A \to [(B \to B) \wedge A]$
64. $A \vdash B \to (B \wedge A)$ (*Remark: compare with* 2)
65. $\vdash A \to [B \to (B \wedge A)]$
66. $A \to C \vdash (B \to C) \to [(A \vee B) \to C]$ (*Remark: compare* 10, 22)
67. $\vdash (A \to C) \to \{(B \to C) \to [(A \vee B) \to C]\}$
68. $\vdash ((A \to B) \to A) \to A$ (Peirce's Law)
69. $\vdash (A \to B) \to (A \to A)$
70. $\vdash (A \to B) \to (B \to B)$
71. $\overline{A} \vee B \vdash A \to B$ (*Remark: compare with* 36)
72. $\vdash (\overline{A} \vee B) \to (A \to B)$
73. $\vdash (A \to B) \to [A \to (A \wedge B)]$

74. $\vdash (A \to B) \to [(A \vee B) \to B]$

75. $\vdash (A \to B) \to [(A \wedge C) \to (B \wedge C)]$ (*Remark: compare with* 26)

76. $\vdash (A \to B) \to [(A \vee C) \to (B \vee C)]$ (*Remark: compare with* 27)

Proofs of equivalence. Instead of loops of reasoning, as in Section 4, we only need to show that the formulas above form a tree: We must show that each of the formulas listed yields positive paradox. That reduction can be explained as follows:

All of the results 58–76 are easily verified to be tautologous in the classical two-valued (true/false) interpretation. Since that interpretation is well known to be complete for classical logic, all the results 58–76 are provable in classical logic. But it is well known that adding positive paradox (formula 58) to relevant logic yields classical logic; therefore adding positive paradox yields all of 58–76.

Thus, it remains to show that each of 58–76 yields positive paradox (formula 58). Considering the items 58–76 as a list, it suffices to show that each item listed after 58 yields some earlier item in the list.

$59 \Rightarrow 58$: Use permutation (17).

$60 \Rightarrow 59$: $(A \to A) \vdash B \to (A \to A)$ is an instance of 60.

$61 \Rightarrow 60$: Substitute $\overline{A}$ for A and $\overline{B}$ for B, and use contrapositive (13).

$62 \Rightarrow 61$: Inferential corollary.

$63 \Rightarrow 59$: We have $B \to [(A \to A) \wedge B]$ as an instance of 63. We also have $[(A \to A) \wedge B] \to (A \to A)$ as an instance of $\wedge$-elimination (6). Combine those using transitivity (16).

$64 \Rightarrow 63$: Start with $B \to B$ as an instance of identity (15). Apply 64 to obtain $A \to [A \wedge (B \to B)]$. Then use commutativity (24).

$65 \Rightarrow 64$: Inferential corollary.

$66 \Rightarrow 58$:

[1]	$B \to [(B \to A) \to A]$	assertion (3)
[2]	$\{[(B \to A) \to A] \to A\} \to (B \to A)$	[1], suffixing (18)
[3]	$A \to A$	identity (15)
[4]	$[(B \to A) \to A] \to \{[A \vee (B \to A)] \to A\}$	[3], 66
[5]	$[A \vee (B \to A)] \to \{[(B \to A) \to A] \to A\}$	[4], permutation (17)
[6]	$A \to [A \vee (B \to A)]$	$\vee$-introduction (8)
[7]	$A \to (B \to A)$	[6][5][2] transitive (16)

$67 \Rightarrow 66$: Inferential corollary.

$68 \Rightarrow 58$: (This proof is taken from Meyer [4].)

[1]	$\{[A \to (B \to A)] \to A\} \to A$	instance of 68
[2]	$[A \to (B \to A)] \to \left[\overline{A \to \overline{B}} \to A\right]$	cotenability (37)
[3]	$\left(\left\{[A \to (B \to A)] \to A\right\} \to A\right) \to \left(\left\{\left[\overline{A \to \overline{B}} \to A\right] \to A\right\} \to A\right)$	[2]; suffixing (18)
[4]	$\left\{\left[\overline{A \to \overline{B}} \to A\right] \to A\right\} \to A$	[1], [3], MP
[5]	$\overline{A \to \overline{B}} \to A$	[4], (21)
[6]	$A \to (B \to A)$	[5], cotenability (37)

$69 \Rightarrow 59$:

[1]	$\left(A \to \overline{B \to \overline{A}}\right) \to (A \to A)$	69
[2]	$\left[B \to \left(A \to \overline{B \to \overline{A}}\right)\right] \to [B \to (A \to A)]$	[1], prefixing (20)
[3]	$\left[\overline{B \to \overline{A}} \to \overline{B \to \overline{A}}\right] \to [B \to (A \to A)]$	[2], cotenability (37)
[4]	$\overline{B \to \overline{A}} \to \overline{B \to \overline{A}}$	identity (15)
[5]	$B \to (A \to A)$	[4], [3], modus ponens

$70 \Rightarrow 69$: Substituting $\overline{B}$ for A and $\overline{A}$ for B transforms 70 into $\vdash (\overline{B} \to \overline{A}) \to (\overline{A} \to \overline{A})$. Now two uses of contrapositive laws yield 69.

$71 \Rightarrow 60$: We may rewrite 71 as $\overline{B} \vee A \vdash B \to A$, and then make use of $A \vdash \overline{B} \vee A$, which is a corollary of 9.

$72 \Rightarrow 71$: Inferential corollary.

$73 \Rightarrow 69$:

[1]	$(A \to B) \to [A \to (A \wedge B)]$	73
[2]	$(A \wedge B) \to A$	$\wedge$-elimination (6)
[3]	$[A \to (A \wedge B)] \to (A \to A)$	[2], prefixing (20)
[4]	$(A \to B) \to (A \to A)$	[1], [3], transitive (16)

$74 \Rightarrow 73$: By replacing A with $\overline{B}$ and B with $\overline{A}$, we may rewrite 74 as $(\overline{B} \to \overline{A}) \to [(\overline{B} \vee \overline{A}) \to \overline{A}]$. By De Morgan's Laws and commutativity, we may rewrite $\overline{B} \vee \overline{A}$ as $\overline{A \wedge B}$; thus 74 is equivalent to $(\overline{B} \to \overline{A}) \to (\overline{A \wedge B} \to \overline{A})$.

Finally, use the contrapositive laws.

75 ⇒ 73:

[1]	$(A \to B) \to [(A \wedge A) \to (B \wedge A)]$	75
[2]	$A \to (A \wedge A)$	idempotent (25)
[3]	$[(A \wedge A) \to (B \wedge A)] \to [A \to (B \wedge A)]$	[2], suffixing (18)
[4]	$(B \wedge A) \to (A \wedge B)$	commutative (24)
[5]	$[A \to (B \wedge A)] \to [A \to (A \wedge B)]$	[4], prefixing (20)
[6]	$(A \to B) \to [A \to (A \wedge B)]$	[1], [3], [5], trans. (16)

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[1]	$(A \to B) \to [(A \vee B) \to (B \vee B)]$	76
[2]	$(B \vee B) \to B$	idempotent (25)
[3]	$[(A \vee B) \to (B \vee B)] \to [(A \vee B) \to B]$	[2], prefixing (20)
[4]	$(A \to B) \to [(A \vee B) \to B]$	[1], [3], transitive (16)

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Eric Schechter
Department of Mathematics
Vanderbilt University
1326 Stevenson Center
Nashville, TN 37240, USA
eric.schechter@vanderbilt.edu