REVIEWS

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The Heart of Mathematics: An Invitation to Effective Thinking. By Edward B. Burger and Michael Starbird. Key College Publishing, 2000, xvii + 646 pp., $79.95.

Reviewed by **Marion D. Cohen**

First and foremost: This book is wonderful. Written as a text for a required course for students who will not take higher mathematics courses, it can also entertain and instruct mathematics majors and mathematicians in general. Here's a quick run through the table of contents: numbers (pigeonhole principle, Fibonacci, primes, modular arithmetic, irrationals, finite and infinite decimals), everything you wanted to know about infinity but didn't think to ask (Cantor's cardinals made accessible to all—but, no, they don't go into inaccessible cardinals!), Geometric Gems (Pythagoras, the art gallery problem, the golden mean, tilings, Platonic solids, non-Euclidean and higher-dimensional geometries), non-metric geometry ("equivalence by distortion", Möbius and Klein, $V - E + F$, knots and links, fixed points, and hot loops), chaos and fractals, and finally, Risky Business (probability, including the likelihood of coincidences and statistical pitfalls).

At first I didn't quite pay attention to, or believe the claim of, the title. Living up to it is, after all, a tall order. However, this book does seem to me to capture "the heart of mathematics", taking "heart" in the senses of both "essence" and "spirit". And by "mathematics" the authors mean, to my great liking, genuine pure mathematics. Not that they don't mention applications, computers, and physics; but the spirit is always mathematical.

This is very possibly the best "mathematics for the non-mathematician" book that I have seen—and that includes popular (non-textbook) books that one would find in a general bookstore. This one doesn't lapse into the pitfalls often found in such books. It preaches the beauty and fascination of mathematics in the introduction and then follows through. It does not cut mathematical corners: Its explanations are complete and accurate, its theorems are stated precisely, and its proofs are intuitive and often colloquially presented without being sloppy.

The book could also be of benefit to mathematicians. Teachers often say that they learn something new every time they teach a course, but they usually mean something subtle. I have to confess that I learned some non-subtle things from reading this book. For example, I hadn't heard of Newcomb's Paradox or Conway tiles, and I hadn't realized that no one knows whether the builders of the Parthenon were consciously aware of the golden mean.

One of the strongest aspects of the book is its writing style. Of all the user-friendly texts I've seen, this is the friendliest. The authors make a real attempt to take into account the anxieties, doubts, and inexperience of their readers. (They may not go far enough in some respects, a point I return to later.) They connect mathematical ideas with human experience, and they offer many gentle insights about the loveliness and seriousness of mathematics. Here are some examples. "Reading mathematics is much different from reading about many other subjects. Here's how to read mathematics: We read a sentence or two, stop reading ... realize we're completely confused ..." (p. xi). Concerning dimensions higher than three (p. 306): "We will succeed in building insights without experience .... If you can conquer infinity and the fourth dimension, what *can't* you do?" (Yes, mathematics is empowering.) And, in the section on the fixed point theorem (p. 393): "We close this section with the secure feeling that, even in an ever-changing world, some things remain the same."

The jokes are great. (Some, the authors say, are *not* so great—but sometimes it's the corniest joke that gets the most appreciation or contributes most to the understanding.) To wit: chapter and section titles such as "Prime Cuts of Numbers", "The Platonic Solids Turn Amorous" (referring to duality), and "The Band That Wouldn't Stop Playing" (meaning Möbius), and theorem titles such as "All Line Segments are Created Equal" (that is, they have the same cardinality). Concerning the sum of the angles in a triangle and what happens when it's a right triangle (p. 211): it's "one of the rare occasions when two wrongs ... make a right." And some good clean fun, "The Story of Fermat's Last Theorem" (p. 74) begins, "It was a dark and stormy night ..."

The authors must be wonderful teachers; at any rate, their book is a wonderful teacher, saturated with comments that might be subtly instructive for professors as well as students. On p. 65, concerning modular arithmetic: "We were introduced to long division ... in the third or fourth grade—we weren't impressed. ... The basic reality of long division is that either it comes out even or there is a remainder." On p. 502, in hopes of giving readers a feeling for why integral dimensions won't always do for fractals: "It cannot be two-dimensional since it's just lines, yet it cannot be one-dimensional since its infinite fuzziness seems to take up a lot of area." And finally, an example of an indirect proof brought to life (p. 361): "Could there be a regular solid that we have not thought of? If there were, then that *mysterahedron* would satisfy the formula $V - E + F = 2$ .... That fact will lead to its demise and show us that the mysterious *mysterahedron* is actually a *nonexistahedron* ..."

Other delectable and practical pedagogical devices include a mathematically enthusiastic "Welcome!" section at the beginning, an analogous "Farewell" at the end, and in between, to get readers of all levels comfortable with fractals, a seven-page gallery of pictures (with some optional fine print) simply to be enjoyed. Puzzles make their appearance, as do unsolved problems, paradoxes, artwork, and literary quotes. I like the idea of stating certain theorems more than once. I also find it refreshing when the authors say (p. 3): "Who says that deep ideas and important consequences come only from hard work?"

The authors avoid a phenomenon found in many other texts, that of asking a provocative question and then not answering it. However, I wasn't thrilled to find that most of the exercises don't have answers or partial answers in the back of the book.

Here, finally, is a compliment about a supplement (I take responsibility for *that* bad joke): The book comes with a Manipulative Kit consisting of materials for building the Platonic solids, 3D glasses, several puzzles, "cool dice", and a CD-ROM with approximately one activity per book section. Some of the materials I sampled seemed a bit meager or corny, but they're fun and support the positive spirit of the book.

Nit-picking: Because I enjoyed this book so much, I indulged in the luxury of reading every word, imagining myself teaching from it, and thinking of ways in which I might do things differently. Of course, there's no one correct user-friendly style, nor is there one best way to explain any particular concept, so the following reservations and suggestions should be taken with a grain of salt.

In Section 2.7 on decimals, the authors begin by motivating decimals as a uniform method of labeling all points on the line. However, there are actually many such methods, such as continued fractions. If this were my book, I'd give a more accurate description of the state of affairs; it would take only a sentence or two.

The authors wisely begin the chapter on transfinite numbers with a short section on one-to-one correspondences. Not so wisely, they subtitle that section "What does infinity mean?" Infinity is mentioned at length in that section, but it is *not* defined, and a student could find the title confusing. ("I read through the whole section and I still don't know what they're saying infinity means—did I miss something?") In any case, the authors do establish that a proper subset of an infinite set can have the same cardinality as the whole. After this, it is perplexing to find on p. 152: "... but it's physically impossible to have infinitely many ping-pong balls—if we did, they would take up all the space in our universe." (Centering a ping-pong ball about every point whose coordinates are multiples of a million light-years would give an infinite set of balls and still leave plenty of room for human beings and planets to knock around!)

Much of the subsequent material in this chapter is based on the idea of finding sets of higher and higher cardinality. But why would students believe that such sets exist, and why would they be motivated to look for them? I would take a slightly more honest approach and say more, perhaps inviting students to try to find a way of choosing points $x_1, x_2, \ldots$ in the unit interval so that they fill up the interval completely.

On p. 297, after discussing non-Euclidean plane geometry, the authors say, "We have just caught a glimpse of three types of geometries: planar, spherical, and hyperbolic. Which is our universe?" A student might well respond, "Huh? Our universe isn't a two-dimensional surface ..."

The authors occasionally omit or minimize nontrivial calculations. This seems an unnecessary evil, a way to lose students; a phrase such as "After a somewhat long but straightforward calculation ..." could provide reassurance. Examples of this phenomenon occur on p. 66 in the discussion of factoring, and on p. 357, where the counting of dots and edges in the "scribble scrabble" is not easy.

An error on p. 359: "$V - E + F$" isn't a *formula*.

On p. 460 the authors contradict themselves when they say, "Fractal does not have a rigorous mathematical meaning." On a previous page, a fractal has been defined as something whose parts are similar to the whole.

My biggest concern about the quality of the presentation lies in the chapters on chaos. Why are there two of them, and why are they not consecutive? And the definition of chaos that is offered—essentially, a situation where small discrepancies in the input yield big ones in the output—seems too vague. What about the function $f(x) = 1{,}000{,}000x$? That's not considered chaotic, is it?

Bone-picking: The authors seem naive, even a bit starry-eyed, about students' abilities, or at any rate the mathematical abilities of non-mathematics majors. For example, in the chapter on fractals they assume that their readers are quite comfortable with logarithms. But freshmen often come to college with misconceptions and negative attitudes concerning logarithms, or no knowledge at all. It seems unrealistic to employ logarithms without first reviewing the definitions and the laws that are needed.

The authors also probably overestimate students' *interest*. True, in the "Welcome!" section they say, "If you are not intrigued by the romance of the subject, that's fine too, because at least you will have a firmer understanding of what it is you are judging." But, as far as I could discover, that is the only place among the 600+ pages where any allowance is made for not being so "intrigued". Perhaps students in their neck of the woods are different from those in mine. In my experience, there are a few who find themselves swept away by "cool" mathematics presented in an enticing and friendly manner, but most students are taking a mathematics course because they have to, not because they want to. Many are overworked with jobs, family responsibilities, and other courses, and they are not prepared to be friendly users of even the most user-friendly course. In fact, some students are so upset or angry at being forced to take a mathematics course that they resist enjoying it or admitting that they enjoy it, and the more the the teacher or the other students convey their enjoyment, the more resistant they become. So, for best results, teachers and textbooks need to be very gentle and low-key with these students.

"We do not have modest goals for this book," the authors admit in the "Welcome!" section. "We want you to look at your life, your habits of thought, and your perception of the world in a new way. And we hope you enjoy the view." I hope so too, but I've seen too many students who are suspicious of anything other than what they're used to. "If a course is easy and enjoyable and pertinent to my life," they might rationalize, "then we must not be learning anything." I believe that teachers and textbook writers should indeed strive to present material in user-friendly ways that are pertinent to students' lives, but this is sometimes easier said than done, and it needs to be handled carefully.

Keeping all this in mind, if this were my book, I would do things a bit differently in the following ways.

1) The authors occasionally offer platitudes about mathematical life and work. I would qualify them so as to be more honest or avoid them entirely. For example: "Looking at an issue from a new point of view often enables us to understand it more effectively." Not often enough, in my experience with mathematical research. Perhaps even more wishful is the following thinking: "Failing is good—it is an effective way to build new insights and discover the truth." I only wish I could count on that!
2) To statements such as "What do you think of when you contemplate the notion of *number*? Perhaps you think about the power of counting ... perhaps your mind drifts to mod clock arithmetic ... perhaps you now *see* and appreciate subtle distinctions ... " I would append statements such as "Or perhaps *nothing* in this book has turned you on—at least not yet." In other words, I would not give the impression that I want or expect students to learn to love mathematics; such expectations can make them feel anxious and interfere with the learning process. To me it seems sufficient for them to see that it is possible for *some* people to love mathematics and that they should begin to *respect* mathematicians.
3) Upon reading the "Welcome!" section, a student could wonder, "Yeah, but what type of questions are going to be on the *final*?" To counteract this, I would pose and answer that question right there. I would assure readers that there will be some straightforward exercises and not only "Invitations to Further Thought", which students can interpret as "Oh, no! Essay questions!"
4) I would omit, alter, or make optional the "In Your Own Words" exercises at the end of every chapter. One, for example, runs, "Write a short essay describing the most interesting discovery you made in exploring this section's material. If any material seemed puzzling or even unbelievable, address that as well." To me that seems a bit unrealistic and starry-eyed. (I write poetry about the emotions involved in doing mathematics, and I don't always feel that I could do these exercises.) And again, students freeze at "essay questions". It might be better to ask simply which discoveries were most interesting or surprising, without

asking students to justify their selections. Or perhaps the exercise just quoted could also include "Describe the *most confusing* or *least favorite* items in this section." Students can learn from that, too, and the honesty and humor might relax them.

Indeed, allowing for and pointing out individual differences can teach students more about mathematics and mathematicians. Students can be let in on the secret that loving mathematics doesn't necessarily mean loving *all* mathematics, just as, for me, loving this book doesn't mean loving *all* of it. It would be a shame, though, not to re-emphasize that I love all but epsilon of it.

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The Integral: An Easy Approach after Kurzweil and Henstock. By Lee Peng Yee and Rudolf Výborný. Cambridge University Press, 2000, xii + 311 pp., $39.95 softcover.

Reviewed by **J. Alan Alewine and Eric Schechter**

A Simple Definition. Riemann's integral of 1867 can be summarized as

$$\int f(t)\,dt = \lim \sum f(\tau_i)(t_i - t_{i-1}).$$

This summary conceals some of the complexity—for example, the limit is of a net, not a sequence—but it displays what we wish to emphasize: The integral is formed by combining the values $f(\tau_i)$ in a very direct fashion.

The values of f are used less directly in Lebesgue's integral (1902), which can be described as $\lim_{n\to\infty} \int_a^b g_n(t)\,dt$. The approximating functions g_n must be chosen carefully, using deep, abstract notions of measure theory. Simpler definitions are possible—for example, functional analysts might consider the metric completion of $C[0, 1]$ using the L^1 norm—but such a definition does not give us easy access to the Lebesgue integral's simple and powerful properties such as the Monotone Convergence Theorem. We generally think in terms of those simple properties, rather than the various complicated definitions, when we actually use the Lebesgue integral.

The KH integral (also known as the gauge integral, the generalized Riemann integral, etc.) was discovered or invented independently by Kurzweil and Henstock in the 1950's; it has attracted growing interest in recent years. It offers the best of both worlds: a powerful integral with a simple definition. In fact, its definition is nearly identical to that of the Riemann integral, as we now show. For any *tagged partition*

$$P: \qquad a = t_0 < t_1 < t_2 < \cdots < t_n = b; \qquad \tau_i \in [t_{i-1}, t_i]$$

of the interval $[a, b]$, let us abbreviate $f(P) = \sum_{i=1}^n f(\tau_i)(t_i - t_{i-1})$. For any given function δ on $[a, b]$, we say that P is δ*-fine* if $t_i - t_{i-1} < \delta(\tau_i)$ for all i. Finally, a number v is the *Riemann integral* (respectively, the *KH integral*) of a function $f : [a, b] \to \mathbb{R}$ if for each constant $\varepsilon > 0$ there exists a constant $\delta > 0$ (respectively, a function $\delta : [a, b] \to (0, +\infty)$) such that whenever P is a δ-fine tagged partition of $[a, b]$, then $|v - f(P)| < \varepsilon$.

We emphasize that the function $\delta(\tau)$, though positive at each τ, need not be bounded below by a positive constant. In applications we generally take $\delta(\tau)$ to be extra small at locations τ where f behaves erratically, so that those locations have less effect on the summation $\sum f(\tau_i)(t_i - t_{i-1})$. Think of approximating the region under a curve as a union of thin rectangles, as in a calculus course: The Riemann integral simply requires that all the rectangles be narrower than a certain constant width, but the KH integral uses more sophisticated spacing.

For example, let $1_{\mathbb{Q}}$ denote the characteristic function of the rationals. This function is discontinuous everywhere; it is a standard example of a bounded function that is not Riemann integrable on [0, 1]. Nevertheless, the KH integral $\int_0^1 1_{\mathbb{Q}}(t)dt$ exists and equals zero. Indeed, let $p_1, p_2, p_3, \ldots$ be any enumeration of the rationals, and let

$$\delta(\tau) \quad = \quad \begin{cases} 2^{-j-1}\varepsilon & \text{if} \quad \tau = p_j, \\ 1 & \text{if} \quad \tau \notin \mathbb{Q}; \end{cases}$$

the proof follows easily. It is less elementary to produce a bounded function that is not KH integrable on [0, 1]; that requires the Axiom of Choice.

For a second example, consider the function $f(t)$ equal to $t^{-1}\sin(t^{-2})$ on (0, 1], and vanishing at $t = 0$. This function is neither Riemann nor Lebesgue integrable, but it is KH integrable; that can be shown using

$$\delta(\tau) \quad = \quad \begin{cases} \sqrt{\varepsilon} & \text{if } \tau = 0, \\ \min\left\{\tau/2, \varepsilon\tau^4/24\right\} & \text{if } 0 < \tau \le 1. \end{cases}$$

Finding and working with such functions $\delta(\tau)$ may require skill and effort, but it does not require a deep, abstract theory.

A More General Integral. Generalizing the constant δ to a function $\delta(\tau)$ obviously yields a wider class of integrands, but it is surprising just how much wider. It turns out that

$$\left\{\begin{array}{l}\text{Riemann}\\ \text{integrable}\\ \text{functions}\end{array}\right\} \subsetneq \left\{\begin{array}{l}\text{Lebesgue}\\ \text{integrable}\\ \text{functions}\end{array}\right\} \subsetneq \left\{\begin{array}{l}\text{KH}\\ \text{integrable}\\ \text{functions}\end{array}\right\}.$$

The classes of KH integrable functions and Lebesgue integrable functions are closely related. Indeed, it can be shown that

$$\begin{array}{l}\text{a function } f \text{ is Lebesgue integrable if and}\\ \text{only if both } f \text{ and } |f| \text{ are KH integrable.}\end{array} \tag{A}$$

If f is KH integrable on $[a, b]$, then f is also KH integrable on every subinterval, but not necessarily on every measurable subset. (We say that f is KH integrable on $S \subseteq [a, b]$ if $f 1_S$ is KH integrable on $[a, b]$, where 1_S is the characteristic function of S.) In fact,

$$\begin{array}{l}f \text{ is Lebesgue integrable if and only if } f \text{ is KH}\\ \text{integrable on every measurable subset of } [a, b].\end{array} \tag{B}$$

For motivation, we suggest an analogy: KH integrable functions are like convergent series, whereas Lebesgue integrable functions are like *absolutely* convergent series (motivation for (A)); any subseries of an absolutely convergent series is also convergent (motivation for (B)).

Kurzweil was led to Riemann-like integrals by his investigations of differential equations $u'(t) = f(t, u(t))$. To see the connection between the two topics, note that some initial value problems can be restated as integral equations:

$$u(t) = u_0 + \int_0^t f(s, u(s))\, ds.$$

For some kinds of applications the integrals $\int_S f(s, u(s))\, ds$, over arbitrary measurable sets S, are not relevant; we need to consider only integrals $\int_a^b f(s, u(s))\, ds$ over intervals. Thus we may use KH integrals and KH techniques; see [**6**] or [**14**].

The KH integral brings wider applicability to differential equations, to Fourier analysis (mentioned later in this review), and to some other branches of analysis, because the KH integral can integrate more functions. However, $t^{-1}\sin(t^{-2})$ is typical of the new functions: They are erratic, more often cited for pathological counterexamples than for useful applications. Thus, the chief benefit of the KH theory may not be its wider applicability, but rather its concrete and elementary formulations of ideas that we already know in the Lebesgue theory. For example, a set $S \subseteq [a, b]$ is Lebesgue measurable if and only if its characteristic function 1_S is KH-integrable, in which case its Lebesgue measure is $\int_a^b 1_S(t)\, dt$.

For brevity, we have defined the KH integral only on a compact interval, but the basic ideas extend easily to bounded or unbounded regions in finite-dimensional Euclidean space, and to measures other than Lebesgue measure. For example, it is shown in Theorem 4.1.1 of [**12**] or Theorem 24.35 of [**13**] that the measures μ on the Borel subsets of an interval $[a, b]$ can be expressed as KH-Stieltjes integrals:

$$\mu_\varphi(S) = \int_a^b 1_S(t) d\varphi(t) = \lim_P \sum_{i=1}^n 1_S(\tau_i)\, [\varphi(t_i) - \varphi(t_{i-1})]$$

for a suitable function φ of bounded variation.

The preceding characterization of measures makes good use of the special properties of intervals in $\mathbb{R}$, admittedly a rather special setting. The KH integral can be extended to a more abstract and general setting of "division spaces" (see, for example, [**9**]), and the resulting theory is applied to the Wiener and Feynman integrals in [**11**], but this theory is more complicated. Still, it is a continuation of the ideas of the Riemann integral; we do not have to start over with a whole new approach involving σ-algebras. This may make the KH approach attractive to scientists and engineers.

Teaching the KH Integral. Where, if at all, does the KH integral belong in our standard analysis curriculum? Bartle [**1**] suggests that it could replace the Lebesgue integral, while Gordon [**8**] says that it should not. Perhaps the difference in their opinions reflects different audiences. For example, the KH approach permits us to avoid, or at least postpone, the notion of σ-algebras. Such abstract notions are insightful and valuable for a mathematically advanced audience, but may be less accessible for undergraduates or for scientists and engineers.

At many American universities today, the standard analysis curriculum is in three stages:

1. freshman calculus, introducing the Riemann integral but omitting most proofs;
2. an advanced undergraduate course, typically titled "Introduction to Real Analysis", which includes (among other things) those omitted proofs; and
3. a graduate course on the Lebesgue integral.

We consider each of these stages.

(1) *Freshman calculus*. The KH integral simplifies and strengthens some classical results. For example, one half of the Fundamental Theorem of Calculus says

> If $G : [a, b] \to \mathbb{R}$ is differentiable [and G' is continuous], then G' is integrable and $\int_a^b G'(t)\,dt = G(b) - G(a)$.

The continuity assumption, or some other assumption like it, is needed for Riemann integrability; that assumption can be omitted entirely if we use the KH integral.

Another improvement on calculus is Hake's Theorem (Theorem 2.8.3 in the Lee-Výborný book):

> The KH integral $\int_a^b f$ exists if and only if $\lim_{r \downarrow a} \int_r^b f$ exists, in which case they are equal.

In effect, this says that we do not need to define an "improper" KH integral, analogous to the improper Riemann integral of calculus; any improper KH integral is also a proper KH integral. (Hake's Theorem is not valid for Lebesgue integrals: $\lim_{r \downarrow 0} \int_r^1 t^{-1} \sin(t^{-2})\,dt = 0.312\ldots$, but $\int_0^1 t^{-1} \sin(t^{-2})\,dt$ does not exist as a Lebesgue integral.)

For the sake of improvements such as these, new calculus teachers might be tempted to introduce the KH integral, but experienced teachers may be less optimistic about their students' abilities. Most of our calculus students can learn computations, but lack the mathematical maturity for proofs. For them, the Fundamental Theorem of Calculus is simply the equation $\int_a^b G'(t)\,dt = G(b) - G(a)$, and discussions about the integrability of $1_{\mathbb{Q}}$ are just gibberish. The value of the calculus course to these students does not lie chiefly in its proofs.

(2) *The advanced undergraduate course*. A general goal of this course is to enable students to understand analysis proofs. That is accomplished more specifically by practicing techniques of ε-δ, convergent sequences, limsups, and the like. As it happens, those techniques are also the main tools in the KH theory. Thus, adding the KH integral to the advanced undergraduate course would require only small alterations in that course.

At least two textbooks are already available that support such a course: DePree and Swartz [**5**] and the third edition of Bartle and Sherbert [**4**]. Each of these books covers the material of a conventional undergraduate analysis course, but then adds a chapter on the KH integral. The additional chapter appears late enough in the book so that it is not crucial; thus teachers who are hesitant about the KH integral can adopt it at their own pace.

(3) *The graduate course on Lebesgue integration.* This course traditionally commits a large amount of time to plowing through the terminology and lemmas of set theory, σ-algebras, measurable functions, inner and outer measure, etc., eventually arriving at $L^1[0, 1]$ and related theory. Introducing the KH theory into this course, either in addition to or instead of the Lebesgue theory, would change the course substantially, for the two theories are based on very different methods—for example, ε-δ inequalities versus σ-algebras.

Nevertheless, the *results* of the two theories are closely related, as we noted in (A) and (B). Consequently, results from either theory can be used as tools in the development of the other theory. Gordon's book [**7**] develops the Lebesgue theory first, and then uses some of its results in developing the KH theory. The first few chapters of this book would fit the traditional graduate course with few alterations; the book lies somewhere between textbook and research monograph.

The Book under Review. The book of Lee and Výborný goes in the other direction, developing the KH theory first and then using it to develop the Lebesgue theory. It is written to be used on several different levels, as explained in its preface: Chapters 1–3 and 6–7 might replace the standard courses that we have called "Stage 2" and "Stage 3". Even Chapter 1, on the Riemann integral, can be read on different levels, according as one includes or omits the material marked "optional". Chapters 4–5 are more advanced reading intended for specialists in integration theory.

The book is rich in examples and applications. We were fascinated by Example 1.4.5, a construction over a page long producing an everywhere differentiable function whose derivative is bounded but not Riemann integrable. The applications include things such as Corollary 7.5.3, which states that the Fourier series of a KH integrable function is Abel summable almost everywhere to that function. Also included are some of the latest discoveries. For instance, in 1993 Výborný formulated the notion of "negligible variation", which Bartle used in 1997 to characterize indefinite KH-integrals; that characterization is Theorem 3.9.1.

The book covers not only the KH integral but also an assortment of "other" integrals—notably, the Denjoy integral (1912), the Perron integral (1914), and the SL integral (formulated by Lee and Výborný in 1993 based on the strong Lusin condition). Actually, these "other" integrals all turn out to be equivalent to the KH integral—that is, they yield the same classes of integrable functions and the same numerical values for the integrals. Though the KH definition is already quite enough for beginners, the alternate definitions yield additional insights that may be helpful in certain kinds of research. For example, the value of $\int_a^b f(t)\,dt$ is not affected if we alter f on a set of measure zero; that fact follows only indirectly from the KH definition, but very directly from the Denjoy or SL definition. Thus either of those definitions, and its associated ideas and techniques, might be useful in investigations that involve discarding null sets.

One more topic that deserves mention is that of convergence theorems—that is, sufficient conditions for $\int_a^b f_n \to \int_a^b f$. The Dominated Convergence Theorem for Lebesgue integrals is a special case of the Vitali Convergence Theorem, which can be generalized to the KH setting as follows (Theorem 3.7.5):

> *Equiintegrability Theorem*. Suppose (f_n) is a sequence of KH integrable functions on $[a, b]$, convergent pointwise to some function f. Suppose that (f_n) is *KH equiintegrable*, in the sense that
>
> > for each $\varepsilon > 0$ there exists a function $\delta : [a, b] \to (0, +\infty)$, such that whenever P is a δ-fine tagged partition, then $\sup_n \left| f_n(P) - \int_a^b f_n \right| < \varepsilon$.
>
> Then f is KH integrable and $\int_a^b f_n \to \int_a^b f$.

That theorem is probably general enough for a first course on integration, and for anyone except a specialist in integration theory, but some deeper results with weaker but more complicated hypotheses are included in the advanced chapters.

The theory of the KH integral has not yet settled down to a classical formulation, but already it is worthy of a place in our standard curriculum. The book of Lee and Výborný serves well as an introduction and reference for anyone interested in this topic. Other good sources are Gordon [**7**], which covers much of the same material from a somewhat different perspective, and the forthcoming book of Bartle [**3**].

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