an overview of my book and my course

by Eric Schechter
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Version of March 3, 2006. If you've downloaded the PDF file, press the [space key] to advance to the next display.

Introduction

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analogy	practice (how to do it)	theory (why it works)
cars	driving lessons	auto mechanics
pastry	cookbooks	organic chemistry
proofs	other math courses	a course in logic

Logics considered in this talk

classical

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Logics considered in this talk

comparative

classical

crystal

fuzzy

constructive

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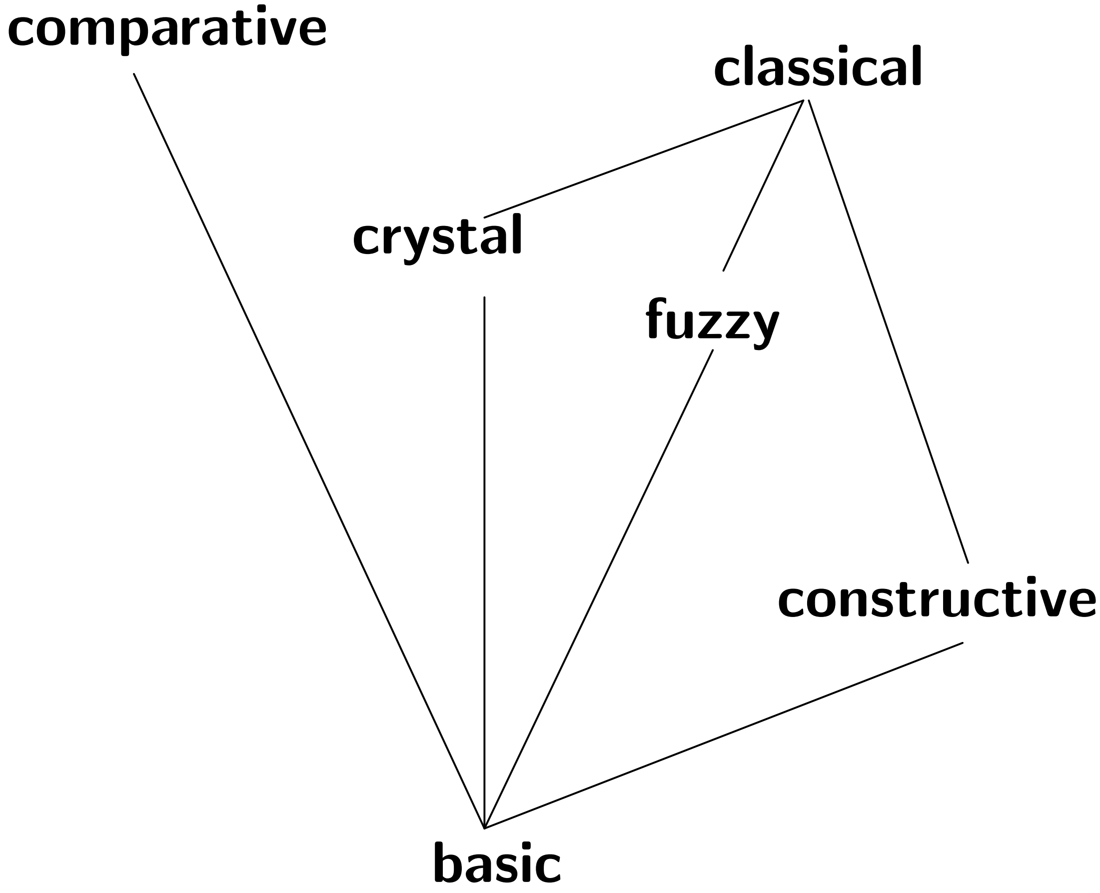

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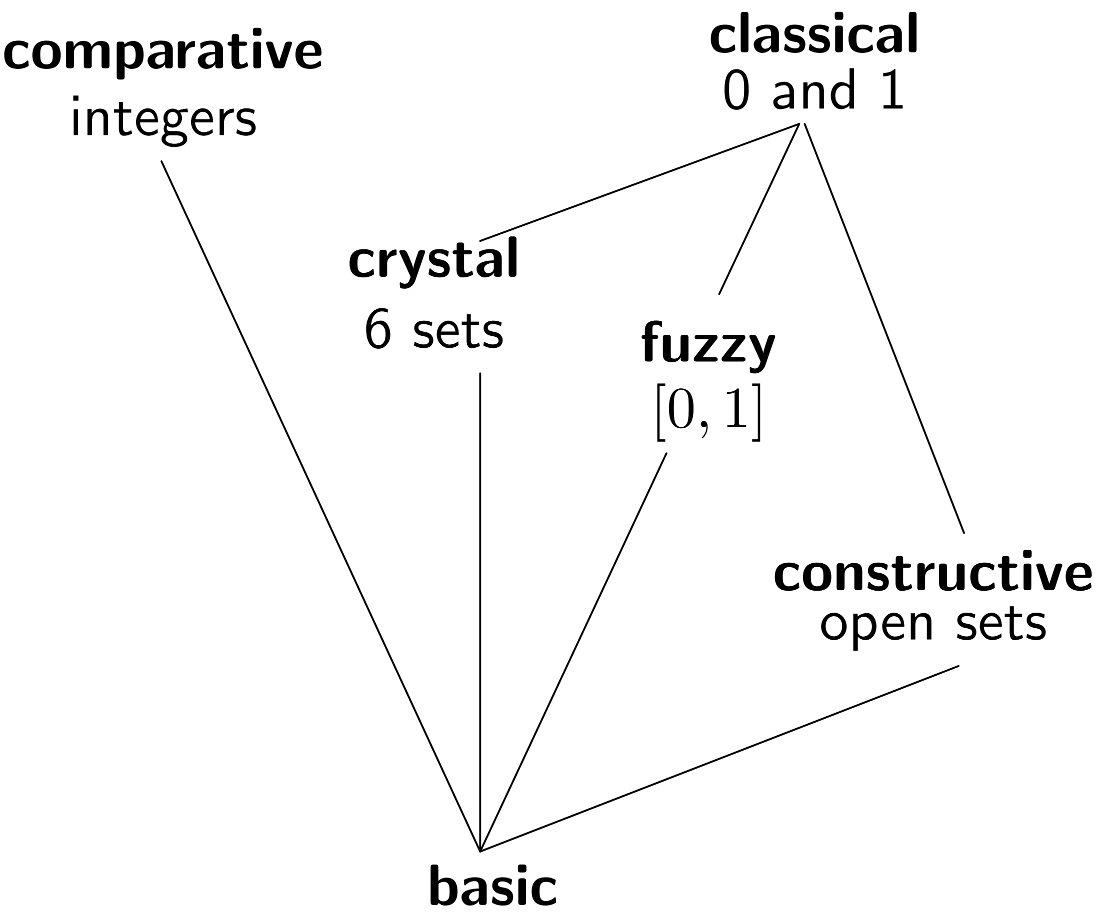

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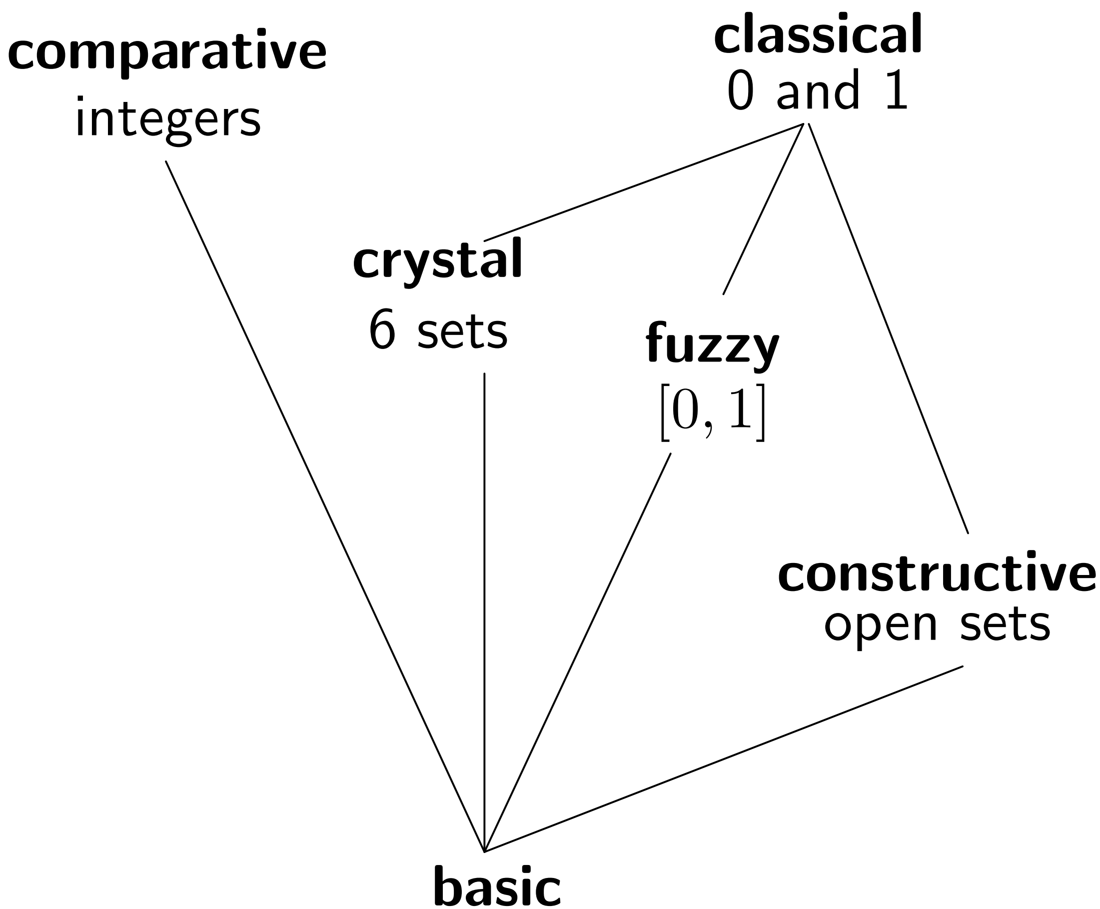

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Different logics have different sets of truths, computed using different maths.

I'll begin with evaluations (semantics), and end with axiomatizations (syntactics).

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— true for a classical logician, but nonsense for anyone else. Our thoughts are closer to *relevant* logic.

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- Reasoning requires *questioning*, not just memorizing. We must teach *doubt*. That's easier if we have multiple possibilities. For instance, to see the significance of $(\neg\neg P) \rightarrow P$, it helps to ask "what happens in logics where $(\neg\neg P) \rightarrow P$ isn't always true?"

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- In the classical-only course, true/false tables are too easy, reducing proofs to mere ritual. An omitted step will hardly be noticed if the student already knows that the conclusion is true. (Analogously, in Euclidean-only geometry, pictures demonstrate isolated facts.)

Classical logic

Two-valued logic

inputs									
p									
F									
T									

Two-valued logic

inputs	not								
p	$\neg p$								
F	T								
T	F								

Two-valued logic

inputs		not	or			exclu. middle				
p	q	$\neg p$	$p \vee q$			$q \vee \neg q$				
F	F	T	F			T				
F	T	T	T			T				
T	F	F	T			T				
T	T	F	T			T				

Two-valued logic

inputs p	q	not $\neg p$	or $p \vee q$	and $p \wedge q$		exclu. middle $q \vee \neg q$	contra-diction $p \wedge \neg p$			
F	F	T	F	F		T	F			
F	T	T	T	F		T	F			
T	F	F	T	F		T	F			
T	T	F	T	T		T	F			

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inputs		not	or	and	imply					
p	q	$\neg p$	$p \vee q$	$p \wedge q$	$p \rightarrow q$					
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F	T	T	T	F	T					
T	F	F	T	F	F					
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F	**F**	T	F	F	**T**					
F	**T**	T	T	F	**T**					
T	F	F	T	F	F					
T	T	F	T	T	T					

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p	q	$\neg p$	$p \vee q$	$p \wedge q$	$p \rightarrow q$				
F	**F**	T	F	F	**T**				
F	**T**	T	T	F	**T**				
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A falsehood implies anything — i.e., if p is false then $p \rightarrow q$ is true.

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F	**T**	T	T	F	**T**					
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If pigs have wings *then* it is now raining in Pittsburgh.

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inputs		not	or	and	imply		contra-diction	explosion	
p	q	$\neg p$	$p \vee q$	$p \wedge q$	$p \rightarrow q$		$p \wedge \neg p$	$(p \wedge \neg p) \rightarrow q$	
F	F	T	F	F	T		F	T	
F	T	T	T	F	T		F	T	
T	F	F	T	F	F		F	T	
T	T	F	T	T	T		F	T	

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$(p \wedge \neg p) \rightarrow q$ (Releventists call this "explosion")

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p	q	$\neg p$	$p \vee q$	$p \wedge q$	$p \rightarrow q$				
F	F	T	F	F	T				
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If the Yankees win the pennant next year then $1 + 1 = 2$.

Two-valued logic

inputs		not	or	and	imply	exclu. middle			superfluous hypothesis	positive paradox
p	q	$\neg p$	$p \vee q$	$p \wedge q$	$p \rightarrow q$	$q \vee \neg q$			$p \rightarrow (q \vee \neg q)$	$q \rightarrow (p \rightarrow q)$
F	F	T	F	F	T	T			T	T
F	T	T	T	F	T	T			T	T
T	F	F	T	F	F	T			T	T
T	T	F	T	T	T	T			T	T

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$p \rightarrow (q \vee \neg q)$ ("superfluous hypothesis")

$q \rightarrow (p \rightarrow q)$ (Releventists call this "positive paradox")

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p	q	$\neg p$	$p \vee q$	$p \wedge q$	$p \rightarrow q$					
F	F	T	F	F	T					
F	T	T	T	F	T					
T	F	F	T	F	F					
T	T	F	T	T	T					

relabeling. . .

Using math to study logic

inputs		not	or	and	implies
p	q	$\neg p$	$p \vee q$	$p \wedge q$	$p \rightarrow q$
0	0	1	0	0	1
0	1	1	1	0	1
1	0	0	1	0	0
1	1	0	1	1	1

$0 =$ false

$1 =$ true

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p	q	$\neg p$	$p \vee q$	$p \wedge q$	$p \rightarrow q$
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0	1	1	1	0	1
1	0	0	1	0	0
1	1	0	1	1	1
		$1 - p$			

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0	1	1	1	0	1
1	0	0	1	0	0
1	1	0	1	1	1
		$1 - p$	$\max\{p, q\}$		

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1	0	0	1	0	0
1	1	0	1	1	1
		$1 - p$	$\max\{p, q\}$	$\min\{p, q\}$	

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0	1	1	1	0	1
1	0	0	1	0	0
1	1	0	1	1	1
		$1-p$	$\max\{p,q\}$	$\min\{p,q\}$	$\min\{1,\ 1-p+q\}$

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Multivalued logics

Łukasiewicz's 3-valued logic

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0	1	1	1	0	1
1/2	0	1/2	1/2	0	1/2
1/2	1/2	1/2	1/2	1/2	1
1/2	1	1/2	1	1/2	1
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1/2	1/2	1/2	1/2	1/2	1
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or more simply

$$\begin{aligned} \neg p &= 1-p, \\ p \vee q &= \max\{p, q\}, \\ p \wedge q &= \min\{p, q\}, \\ p \rightarrow q &= \min\{1,\ 1-p+q\}. \end{aligned}$$

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Note that $\neg \frac{1}{2} = \frac{1}{2}$.

Fuzzy logic: infinitely many values

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The 100th student is $4\frac{1}{2}$ feet tall, so p_{100} is absolutely false, and $[\![p_{100}]\!] = 0$.

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The 100th student is $4\frac{1}{2}$ feet tall, so p_{100} is absolutely false, and $[\![p_{100}]\!] = 0$.

Interpolating, it seems reasonable to assign $[\![p_i]\!] = 1 - \frac{i}{100}$.

(continued next slide)

Tall people continued

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$$(A \to (A \to B)) \to (A \to B).$$

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$$(A \to (A \to B)) \to (A \to B).$$

But contraction fails in fuzzy logic, e.g. when $[\![A]\!] = 1/2$ and $[\![B]\!] = 0$. More about that later.

Relevant logics

Aristotle's comparisons

If there are two things both more desirable than something, the one which is more desirable to a greater degree is more desirable than the one more desirable to a less degree.

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For comparisons, we need a different logic. . . .

Comparative logic

$$\text{false values} = \{\ldots, -3, -2, -1\}, \qquad \text{true values} = \{0, 1, 2, 3, \ldots\},$$

$$\neg\, p = -p, \qquad p \vee q = \max\{p, q\}, \qquad p \wedge q = \min\{p, q\}, \qquad p \to q = q - p.$$

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- On the other hand, this logic affirms some irrelevancies. For instance, "unrelated extremes" $(p \wedge \neg p) \to (q \vee \neg q)$ is always true in this logic.

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Note: A few slides from now I'll use the fact that, in this logic,

$$\neg\, 0 \;=\; 0 \wedge 0 \;=\; 0 \vee 0 \;=\; 0 \to 0 \;=\; 0.$$

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One logic with particularly strong relevance properties is *crystal logic* . . .

Crystal logic: sets for values

6 semantic values

$\{-2, -1, +1, +2\}$

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$\{-1, +2\}$

$\{+1, +2\}$

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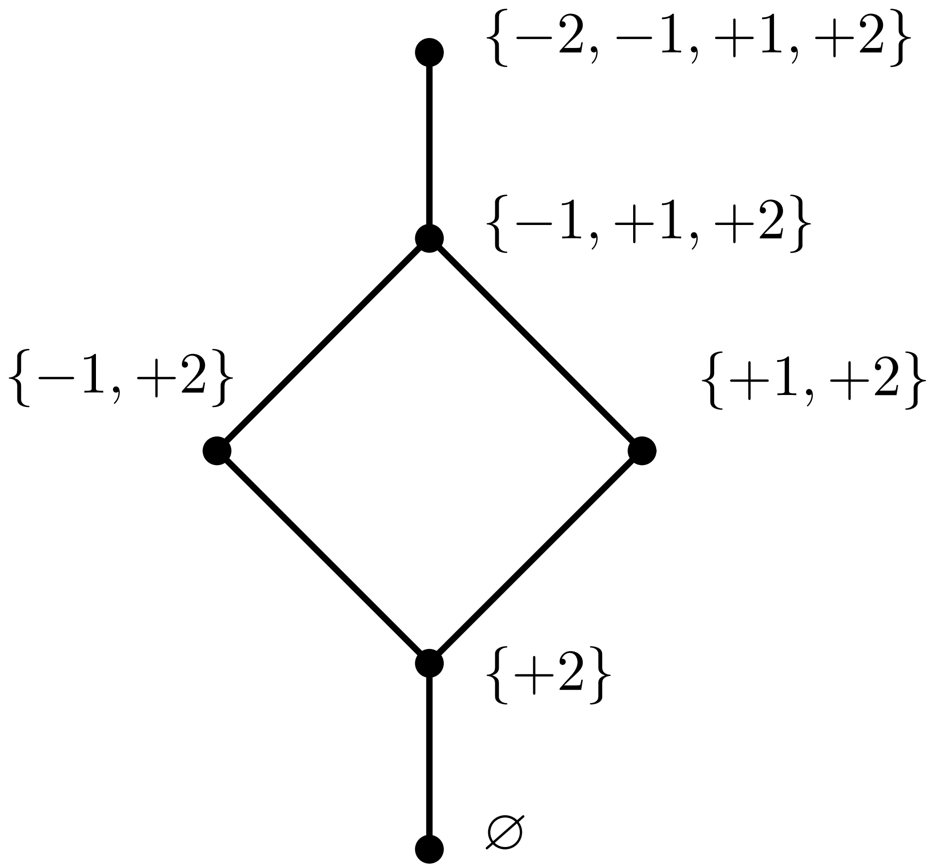
Crystal logic: sets for values



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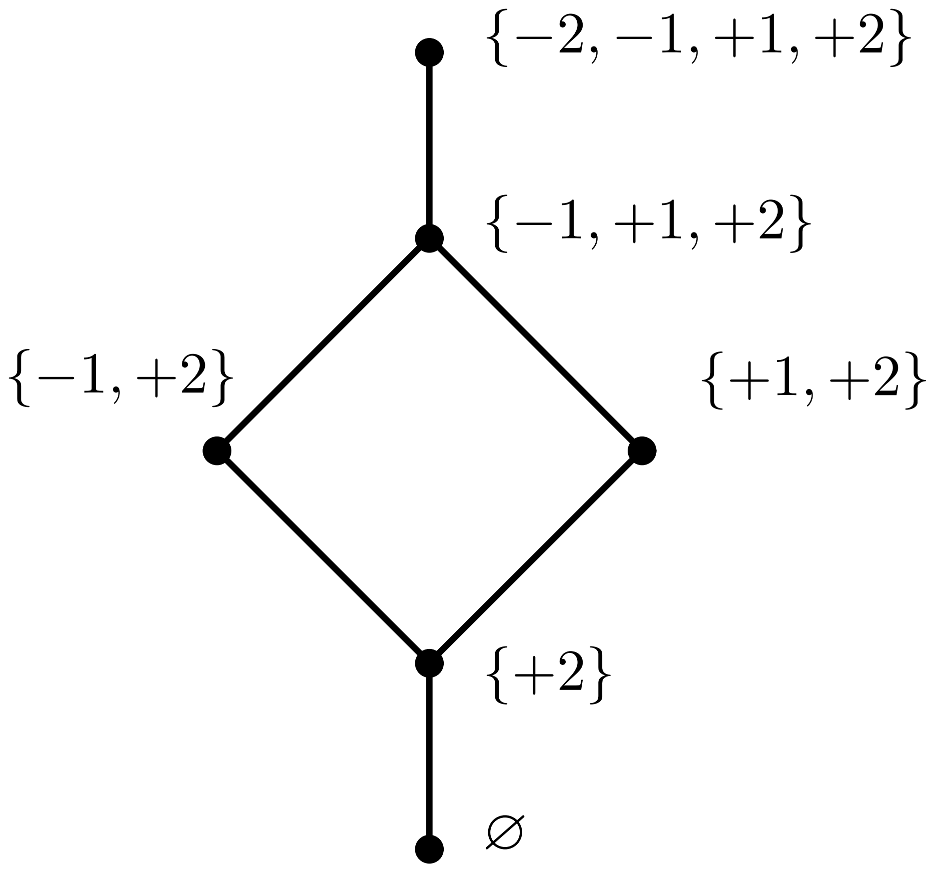

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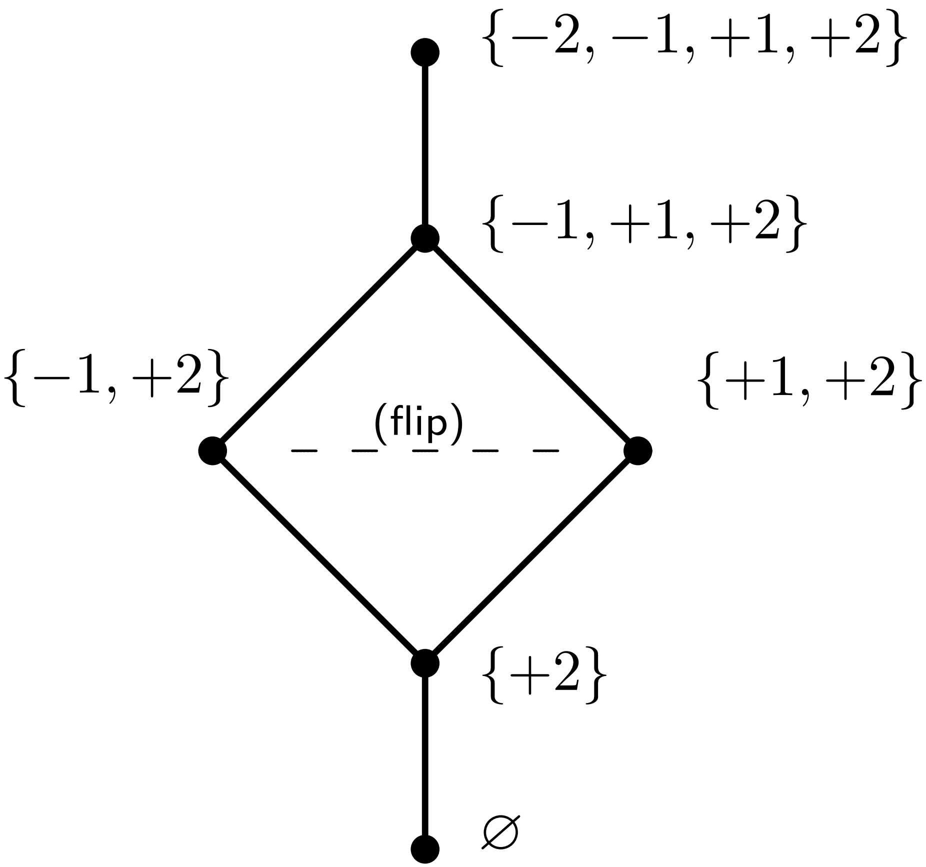

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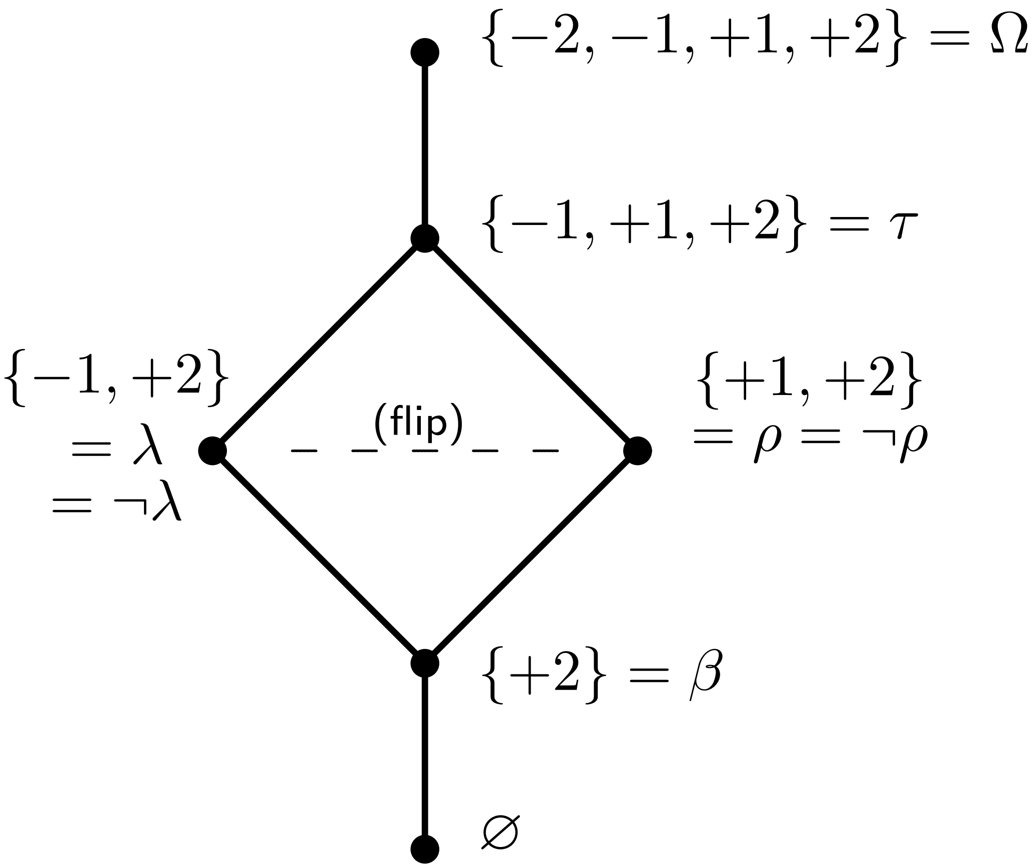

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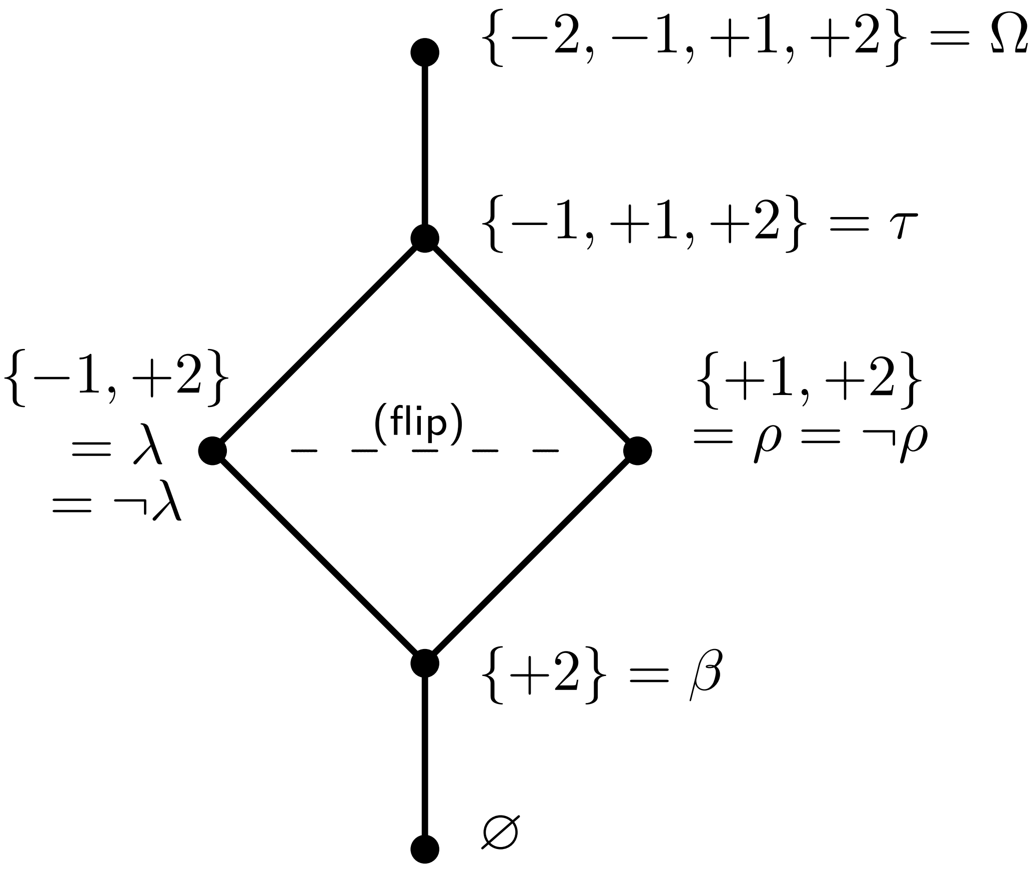

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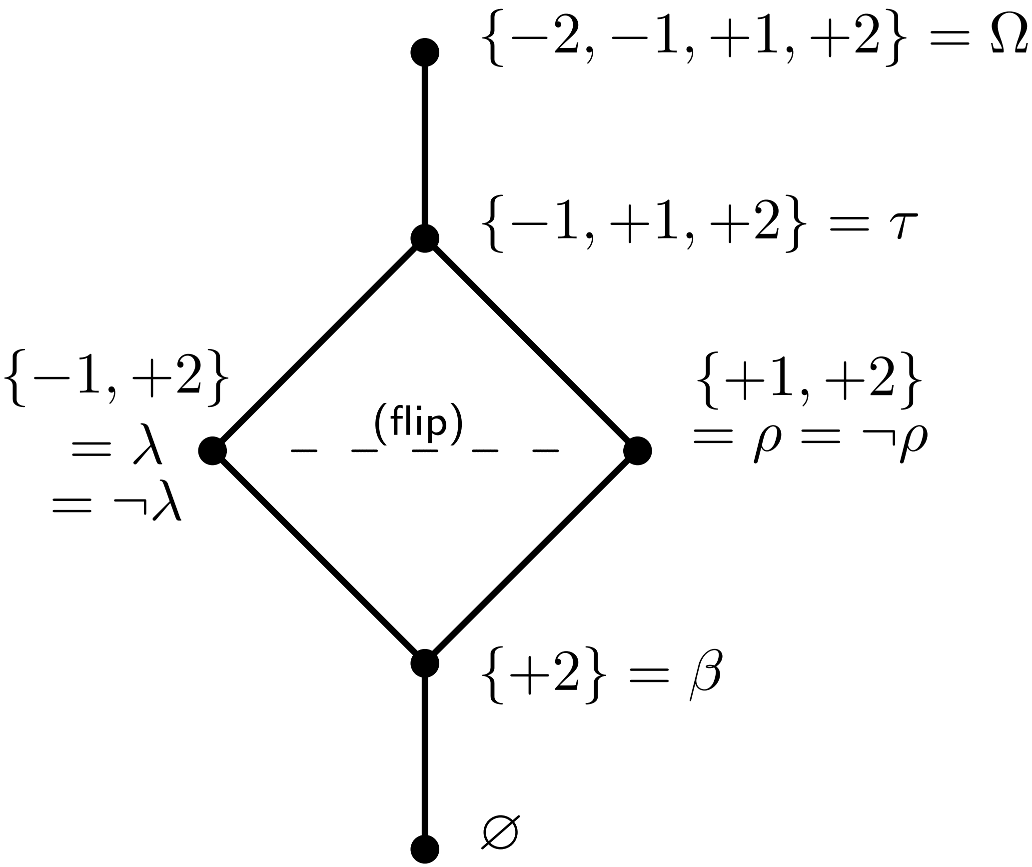

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Note that $\lambda \vee \lambda = \lambda \wedge \lambda = \lambda \to \lambda = \neg\lambda = \lambda$
and $\rho \vee \rho = \rho \wedge \rho = \rho \to \rho = \neg\rho = \rho.$

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$S \to T$		$T =$					
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	λ	$\varnothing$	$\varnothing$	λ	$\varnothing$	λ	Ω
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or equivalently

$$S \to T = \begin{cases} \Omega & \text{if } S = \varnothing \text{ or } T = \Omega, \\ \varnothing & \text{if } S \text{ is not a subset of } T, \\ T & \text{if } S = \beta, \\ \neg S & \text{if } T = \tau, \\ S & \text{if } (S,T) \text{ is } (\lambda,\lambda) \text{ or } (\rho,\rho). \end{cases}$$

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- includes my "basic logic" (discussed later), so it's not bizarre; and
- prevents irrelevant implications — for instance, $(p \wedge \neg p) \to (q \vee \neg q)$ is not always true. More generally . . .

Relevance Principles

Suppose that formulas A and B are irrelevant to each other, i.e., they have no variables in common. Can $A \to B$ still be tautological (i.e., always true)?

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(2) In **comparative logic**, $A \to B$ is a tautology if and only if both B and $\neg A$ are tautologies, as in "unrelated extremes"

$$\underbrace{(p \wedge \neg p)}_{A} \to \underbrace{(q \vee \neg q)}_{B}.$$

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$$\underbrace{p}_{A} \to \underbrace{(q \vee \neg q)}_{B} \qquad \text{or} \qquad \underbrace{(p \wedge \neg p)}_{A} \to \underbrace{q}_{B}.$$

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I'll prove part of (2). (Its other parts and (1) and (3) are proved similarly.)

A relevance proof

In comparative logic (i.e., with subtraction for implication), if A and B share no variables and B is not a tautology, then $A \rightarrow B$ is not a tautology.

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- ☐ Then $[\![A]\!] = 0$, since $0 \vee 0 \;=\; 0 \wedge 0 \;=\; 0 \rightarrow 0 \;=\; \neg 0 \;=\; 0$.

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- Then $[\![A]\!] = 0$, since $0 \vee 0 \; = \; 0 \wedge 0 \; = \; 0 \to 0 \; = \; \neg 0 \; = \; 0.$
- Then $[\![A \to B]\!] \; = \; [\![B]\!] - [\![A]\!] \; < \; 0.$ So $A \to B$ isn't always true.

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Even to someone who speaks this language, and is familiar with conditions (i) and (ii), it is not <u>obvious</u> that there is any relation between those conditions. In fact, that relation is the whole point of the theorem.

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Lemma 1. $\sqrt{2}$ is irrational.

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So a and b <u>exist</u>. But we still don't know what they <u>are</u>!

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Two philosophies of mathematics

Classical philosophy

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  But it's just his *proof* that is nonconstructive. His *theorem* can be made constructive via other proofs — for instance, use Gelfond-Schneider theorem to prove that j is irrational, or more simply just take $a = \sqrt{2}$ and $b = \log_2 9$.

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On the other hand, some mathematical results (such as the Axiom of Choice) are *inherently* nonconstructive, and rejected altogether by constructivists.

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But we don't know which, and perhaps we never will. Consequently, some constructivists might say that it is *neither* "true" or "false" — like the statement

> Luke Skywalker's favorite color is red.

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$P \vee \neg P$ is false (for instance) when $[\![P]\!] = (0, 1) \cup (1, 2)$.

AXIOM SYSTEMS

Example of proving a theorem from some axioms

Assumptions:

$$\{A,\ A \to B\} \vdash B$$ "detachment"

$$\vdash C \to (D \to C)$$ "positive paradox"

$$\vdash [E \to (F \to G)] \to [(E \to F) \to (E \to G)]$$ "self-distribution"

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> Only if we assume that the symbol "$\to$" has some meaning close to the usual meaning of "implies." But we don't want to assume that. In axiomatic logic, we start with no meaning at all for symbols such as "$\to$"; they're just symbols. They obtain only the meanings given to them by our axioms.

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> Many choices of axioms are possible; we'll discuss those soon. But some choices work better than others. For instance, we find that
>
> {detachment, positive paradox, self-dist.} $\Rightarrow$ identity, but
> {detachment, positive paradox, identity} $\not\Rightarrow$ self-dist.

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Theorem: $\vdash X \to X$ (“identity”).

Proof of theorem:

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(4)	$[X \to (X \to X)] \to (X \to X)$	detach. with $A =$ (2), $A \to B =$ (3)
(5)	$X \to X$	detach. with $A =$ (1), $A \to B =$ (4)

Axioms for classical logic, divided into two parts

$$\begin{array}{ll} \{A,\ A\to B\}\vdash B, & \{A,\ B\}\vdash A\wedge B \\ (A\wedge B)\to A, & A\to(A\vee B) \\ (A\wedge B)\to B, & B\to(A\vee B) \\ A\to A, & (A\to\neg B)\to(B\to\neg A) \\ \multicolumn{2}{l}{[A\to(B\to C)]\to[B\to(A\to C)]} \\ \multicolumn{2}{l}{(A\to B)\to[(C\to A)\to(C\to B)]} \\ \multicolumn{2}{l}{[(A\to B)\wedge(A\to C)]\to[A\to(B\wedge C)]} \\ \multicolumn{2}{l}{[(B\to A)\wedge(C\to A)]\to[(B\vee C)\to A]} \\ \multicolumn{2}{l}{[A\wedge(B\vee C)]\to[(A\wedge B)\vee C]} \end{array}$$

“Basic” logic. This is the uncontroversial, “vanilla” part. *Most* logics satisfy these axioms. They are numerous, but each is fairly simple by itself.

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A \to (B \to A) & \text{positive paradox} \\
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Non-basic axioms. Add just *some* of these spices to get nonclassical logics.

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A book on just classical logic uses a shorter list of stronger axioms.

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But such pairings are hard to find, and harder to prove.

A few examples of completeness pairings

name	**values:**	**axioms:** basic, plus ...
classical	$\{0,1\}$	positive paradox, double negation, contraction
Łukasiewicz	$\{0,\frac{1}{2},1\}$	positive paradox, double negation, $((A \to B) \to B) \to (A \vee B)$, and $(A \to (A \to \neg A)) \to (A \to \neg A)$,
fuzzy	$[0,1]$	positive paradox, double negation, and $((A \to B) \to B) \to (A \vee B)$
comparative	integers	$((A \to B) \to B) \to A$, $(A \to A) \leftrightarrow \neg(A \to A)$
crystal	6 sets	contraction, double negation, $A \vee (A \to B)$, and $((\neg A) \wedge B) \to (((\neg A) \to A) \vee (A \to B))$
constructive	open sets	positive paradox, contraction, and explosion

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fuzzy	$[0,1]$	positive paradox, double negation, and $((A\to B)\to B)\to(A\vee B)$	h
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$\surd$ = proved in my book; h = too hard to prove in my book. ■