Character rigidity for lattices in higher-rank groups

Jesse Peterson

October 27, 2014

Abstract

We show that if Γ is an irreducible lattice in a higher rank center-free semi-simple Lie group with no compact factors and having property (T) of Kazhdan, then Γ is operator algebraic superrigid, i.e., any unitary representation of Γ which generates a II_1 factor extends to a homomorphism of the group von Neumann algebra $L\Gamma$. This generalizes results of Margulis, and Stuck and Zimmer, and answers in the affirmative a conjecture of Connes. We also show a general operator algebraic superrigidity result for irreducible lattices in products of property (T) groups, generalizing results of Bader and Shalom, and Creutz.

1 Introduction

Let G be a semi-simple Lie group with trivial center, no compact factors, and real rank at least 2, and let $\Gamma < G$ be an irreducible lattice in G. Margulis' superrigidity theorem [Mar75] states that if H is a simple algebraic group over a local field and $\pi : \Gamma \to H$ is a homomorphism whose image is Zariski dense in H, then either $\pi(\Gamma)$ is precompact or else π extends to a continuous homomorphism $G \to H$.

There is a rich analogy between the interaction of a lattice in a group $\Gamma < G$ and the interaction between a countable group in its von Neumann algebra $\Gamma < \mathcal{U}(L\Gamma)$, and based on the superrigidity result of Margulis and the cocycle version due to Zimmer [Zim80], Connes suggested that, at least in the property (T) setting, the analogy could be pursued further so that there should be a similar superrigidity phenomenon for such groups embedded in their group von Neumann algebras (See [Jon00] or the introduction to [CP13] where this conjecture is discussed in detail).

Concerning the Lie groups themselves, a classical result of Segal and von Neumann shows that they have no non-trivial continuous representations which generate a finite von Neumann algebra [SvN50]. Concerning lattices, the first examples where "operator algebraic superrigidity" was verified were obtained by Bekka [Bek07] who showed that this holds for the groups $SL_n(\mathbb{Z})$ for $n \geq 3$. Further examples were found in [PT13] where the same results were obtained for the groups $SL_2(A)$, where $A = \mathcal{O}$ is a ring of integers (or, more generally, $A = \mathcal{O}S^{-1}$ a localization) with infinitely many units. The first examples of higher-rank groups having *arbitrary* irreducible lattices operator algebraic superrigid were found in [CP13], where this was shown to be the case for irreducible lattices in certain product groups where one of the product factors is an algebraic group over a non-archimedian local field.

The purpose of this work is to give a complete solution to Connes conjecture:

Theorem A. *Let G be a property (T) semi-simple Lie group with trivial center, no compact factors, and real rank at least 2, and let $\Gamma < G$ be an irreducible lattice in G, if M is a finite factor, and $\pi : \Gamma \to \mathcal{U}(M)$ is a representation such that $\pi(\Gamma)'' = M$, then either M is finite dimensional or else π extends to an isomorphism $L\Gamma \xrightarrow{\sim} M$.*

The methods used to prove this theorem also adapt to the abstract setting of irreducible lattices in products of groups. Combining these methods with results from [CP13] we obtain a similar operator algebraic superrigidity statement in this setting. This removes the totally disconnectedness assumption from the results in [CP13].

Theorem B. *Suppose that G_1 and G_2 are non-compact second countable locally compact groups such that G_1 has property (T), and $\Gamma < G_1 \times G_2$ is an irreducible lattice. Suppose that any trace preserving ergodic action of G_i on a finite von Neumann algebra must be properly outer when restricted to Γ. Then if M is a finite factor and $\pi : \Gamma \to \mathcal{U}(M)$ is a homomorphism such that $\pi(\Gamma)'' = M$, then either M is finite dimensional, or else π extends to an isomorphism $L\Gamma \xrightarrow{\sim} M$.*

A character on a discrete group Γ is a conjugation invariant function τ which is of positive type, and is normalized so that $\tau(e) = 1$ (or equivalently, a tracial state on the full group C^*-algebra $C^*\Gamma$). If M is a finite von Neumann algebra with trace τ, and $\pi : \Gamma \to \mathcal{U}(M)$ is a representation, then $\gamma \mapsto \tau(\pi(\gamma))$ gives a character. Moreover, the GNS-construction shows that every character arises in this way. The space of characters forms a Choquet simplex, and the extreme points correspond to representations which generate a finite factor [Tho64]. Operator algebraic superrigidity is therefore equivalent to the following character rigidity property:

Theorem C. *Let Γ be as in Theorems A or B. If $\tau : \Gamma \to \mathbb{C}$ is an extreme point in the space of characters, then either τ is almost periodic, or else $\tau = \delta_e$.*

One consequence of Operator algebraic/character rigidity is that it implies a rigidity phenomenon for non-free probability measure-preserving actions (see [DM12, Theorem 2.11], or [PT13, Theorem 3.2]).

Corollary D. *Let Γ be as in Theorems A or B. Then any probability measure-preserving ergodic action of Γ on a standard Lebesgue space is essentially free.*

The previous corollary generalizes results of Stuck and Zimmer [SZ94] in the setting of semi-simple Lie groups, and Creutz and the author [CP12] (also [Cre13]) in the setting of product groups. By considering actions of quotient groups, this previous corollary in turn implies that Γ is almost simple, i.e., the only normal subgroups of Γ are either finite index or trivial. This then generalizes the normal subgroup theorema of Margulis [Mar78], in the setting of simple Lie groups, and Bader and Shalom [BS06], in the setting of product groups.

That this generalizes Margulis' normal subgroup theorem is especially noteworthy as our strategy will follow closely that of Margulis. Specifically, suppose that $\pi : \Gamma \to \mathcal{U}(M)$ is a representation which generates a II_1 factor M, and such that π does not extend to an isomorphism $L\Gamma \xrightarrow{\sim} M$, our strategy to show that M is finite dimensional is to show that it is both amenable [Con76], and has property (T) [CJ85]. As is the case for quotient groups, property (T) for the von Neumann algebra M follows from property (T) of Γ (this accounts for the property (T) assumption in Theorems A and B), thus the main difficulty is to show amenability. For the normal subgroup theorem Margulis obtains amenability by exploiting amenability of a minimal parabolic subgroup $P < G$, and then proving a general "factor theorem" showing that Γ-invariant σ-subalgebras of the Borel σ-algebra of G/P are automatically G-invariant.

A key step in the proof of Margulis' factor theorem is his use of contracting automorphisms in order to show that the cross sections of certain measurable subsets must almost always be contained in the Γ-invariant σ-subalgebra [Mar91, Lemma IV.2.7]. We use this result in order to show that the essential ranges of certain Γ-equivariant operators in a "non-commutative boundary" will commute with the elements of Γ which are not orthogonal to the identity in M. An ergodicity type argument then entails that the non-commutative boundary must be trivial, hence showing that M is amenable. A detailed outline of this strategy is presented in [CP13, Section 2].

Connes' conjecture described above implicitly concerns the case when the lattices have property (T). This is quite natural, given his rigidity results regarding the fundamental group of von Neumann algebras associated with property (T) groups [Con80]. The question of operator algebraic/character rigidity, however, is also interesting in the case of irreducible lattices in higher-rank semi-simple groups without property (T). For example, the lattices considered in [PT13] are all of this form and satisfy this rigidity. The results in this paper give the "amenability half" of a proof in this direction (see Theorems 4.5 and 5.3 below where property (T) is not assumed). Thus, to prove character rigidity for such lattices it suffices to prove a "property (T) half" analogous to the situation for normal subgroups considered in [Mar79], and [Sha00].

2 Preliminaries

This work is primarily concerned with the interactions between lattices in semi-simple algebraic groups, and the finite von Neumann algebras they generate. A reference for the former is [Mar91], while a reference for the latter is [Dix81].

A von Neumann algebra is a self-adjoint unital subalgebra M of bounded operators on a Hilbert space $\mathcal{H}$, which is closed in the strong operator topology. The strong and weak topologies give the same closed convex sets and hence M is also closed in the weak operator topology. Moreover, by von Neumann's double commutant theorem, this is also equivalent to requiring that M is equal to its double commutant $M'' \subset \mathcal{B}(\mathcal{H})$. The von Neumann algebra M is a factor if it has trivial center $\mathcal{Z}(M) = \mathbb{C}$.

A von Neumann algebra acting on a separable Hilbert space is finite if and only if there exists a normal faithful tracial state τ on M, i.e., τ is a weak (equivalently strong) operator topology continuous linear functional such that $\tau(1) = 1$, $\tau(y^*y) \geq 0$ for all $y \in M$, and equals 0 if and only if $y = 0$, and $\tau(yz) = \tau(zy)$ for all $y, z \in M$. For finite factors, the trace is unique.

Given a finite von Neumann algebra M with a normal faithful tracial state τ, the GNS-construction with respect to τ gives rise to the standard representation of M on $\mathcal{B}(L^2(M,\tau))$. Thus, we may view M both as an algebra of bounded operators on $\mathcal{B}(L^2(M,\tau))$ and as a dense subspace of the Hilbert space $L^2(M,\tau)$. When we wish to emphasize the latter perspective we will write $\hat{x}$ for an element $x \in M$ when it is viewed as an element in $L^2(M,\tau)$. We will also write $P_{\hat{x}}$ to denote the rank 1 projection onto $\mathbb{C}\hat{x} \subset L^2(M,\tau)$. When M is a finite factor, the trace is unique, and so, in this case, we use the notation L^2M, for $L^2(M,\tau)$.

The tracial property of τ entails that the conjugation map $M \ni \hat{x} \mapsto \hat{x^*}$ extends to a conjugate linear isometry $J : L^2(M,\tau) \to L^2(M,\tau)$. We than have $JyJ\hat{x} = \widehat{xy^*}$, and from this we see that $JMJ \subset M'$. In fact, we have equality $JMJ = M'$, and thus, conjugation by J gives an anti-isomorphism between M and M'.

If Γ is a countable group, the group von Neumann algebra is the von Neumann algebra $L\Gamma \subset \mathcal{B}(\ell^2\Gamma)$ generated by the left-regular representation of Γ. This is a finite von Neumann algebra as $L\Gamma \ni x \mapsto \langle x\delta_e, \delta_e\rangle$ is easily seen to give a normal faithful tracial state. This is a factor if and only if the group is ICC, i.e., every non-trivial conjugacy class is infinite.

If A is an abelian von Neumann algebra acting on a separable Hilbert space, then there exists a standard Borel space X, together with a Borel probability measure μ such that $A \cong L^\infty(X,\mu)$, where $L^\infty(X,\mu)$ is viewed as a von Neumann subalgebra of $\mathcal{B}(L^2(X,\mu))$ consisting of multiplication operators $M_f\xi = f\xi$, for $f \in L^\infty(X,\mu)$, and $\xi \in L^2(X,\mu)$. Any quasi-invariant automorphism $\alpha \in \mathrm{Aut}^*(X,\mu)$, gives rise to an automorphism $\alpha^* \in \mathrm{Aut}(A)$ given by $\alpha^*(f)(x) = f(\alpha^{-1}(x))$. Moreover, every automorphism of A arrises in this way.

Given a von Neumann algebra M acting on a separable Hilbert space, the strong and weak operator topologies are not, in general, Polish. However, these topologies do give a

Polish structure on any closed ball in M, and both topologies give the same Borel structure to M, which is standard. If (X, μ) is a standard probability space, we may then consider $L^\infty(X, \mu; M)$, the space of all essentially bounded Borel functions from X to M, where functions are identified if they agree almost everywhere. This is a von Neumann subalgebra of $\mathcal{B}(L^2(X, \mu; \mathcal{H}))$, where the multiplication is given by $(f\xi)(x) = f(x)\xi(x)$ for $f \in L^\infty(X, \mu; M)$, $\xi \in L^2(X, \mu; \mathcal{H})$, and $x \in X$. We have a natural embedding $M \subset L^\infty(X, \mu; M)$ given by constant functions, and $L^\infty(X, \mu) \subset L^\infty(X, \mu; M)$ given by scalar valued functions.

An operator $y_0 \in M$ is said to be in the essential range of $f \in L^\infty(X, \mu; M)$ if for any strong operator topology neighborhood O of y_0, there exists a Borel set $E \subset X$ with $\mu(E) > 0$, such that $f(x) \in O$, for all $x \in E$. If $N \subset M$ is a von Neumann subalgebra then an operator $f \in L^\infty(X, \mu; M)$ is contained in $L^\infty(X, \mu; N)$ if and only if the essential range of f is contained in N.

Every function in $L^\infty(X, \mu; M)$ is a strong operator topology limit of uniformly bounded simple functions, and hence under the identification $L^2(X, \mu; \mathcal{H}) \cong L^2(X, \mu)\overline{\otimes}\mathcal{H}$ we see that $L^\infty(X, \mu; M)$ is generated as a von Neumann algebras by operators of the form $f \otimes y$ for $f \in L^\infty(X, \mu)$ and $y \in M$. Thus, we have a canonical identification $L^\infty(X, \mu; M) \cong L^\infty(X, \mu)\overline{\otimes}M$. If $M = L^\infty(Y, \nu)$ for some other standard probability space (Y, ν), then we have the further identification $L^\infty(X, \mu; L^\infty(Y, \nu)) \cong L^\infty(X, \mu)\overline{\otimes}L^\infty(Y, \nu) \cong L^\infty(X \times Y, \mu \times \nu)$. In general, for von Neumann algebras M and N we have a natural embedding $M \to M\overline{\otimes}N$ given by $M \ni x \mapsto x \otimes 1 \in M\overline{\otimes}N$. Because of this, in the sequel we will often regard M as a subalgebra of $M\overline{\otimes}N$ and hence we will only occasionally write $x \otimes 1$ for emphasis.

A von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$ is injective (or amenable) if there exists a linear contractive idempotent $E : \mathcal{B}(\mathcal{H}) \to M$ (which is not necessarily normal). Injectivity is independent of the Hilbert space on which M is represented. For separable II_1 factors, a celebrated result of Connes [Con76] states that, up to isomorphism, the hyperfinite II_1 factor R is the unique injective separable II_1 factor.

If G is a locally compact second countable group and (X, μ) is a standard Borel space, then we shall consider actions of G on X such that the action map $G \times X \ni (g, x) \mapsto gx \in X$ is Borel, and such that the action is quasi-invariant, i.e., for $g \in G$, and $E \subset X$ we have $\mu(gE) = 0$ if and only if $\mu(E) = 0$. We denote by $\sigma : G \to \mathrm{Aut}(L^\infty(X, \mu))$ the corresponding group homomorphism described above. This homomorphism is continuous when $\mathrm{Aut}(L^\infty(X, \mu))$ is endowed with the topology of pointwise weak (or, equivalently, strong) operator topology convergence.

The Koopman representation associated to the action $G \curvearrowright (X, \mu)$ will be denoted by $\sigma^0 : G \to \mathcal{U}(L^2(X, \mu))$ and is given by $\sigma^0_g(\xi)(x) = \xi(g^{-1}x)(\frac{dg\mu}{d\mu})^{1/2}(x)$ for $g \in G$, $\xi \in L^2(X, \mu)$, and $x \in X$. This representation is continuous from G to $\mathcal{U}(L^2(X, \mu))$ where the latter is endowed with the weak (or, equivalently, strong) operator topology. When we view

$L^\infty(X,\mu)$ as a subalgebra of $\mathcal{B}(L^2(X,\mu))$ then conjugation by the Koopman representation implements the action on $L^\infty(X,\mu)$, i.e., $\sigma_g(f) = \sigma_g^0 f \sigma_{g^{-1}}^0$ for all $g \in G$, and $f \in L^\infty(X,\mu)$.

If Γ is a countable group, then a character τ on Γ is a conjugation invariant function which is of positive type and which we normalize so that $\tau(e) = 1$. If M is a finite von Neumann algebra with a normal faithful tracial state τ, and $\pi : \Gamma \to \mathcal{U}(M)$ is a representation, then the function $\gamma \mapsto \tau(\pi(\gamma))$ is easily seen to be a character.

Conversely, given a character τ on Γ, the associated GNS-representation gives a Hilbert space $\mathcal{H}$, together with a representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$, and a unit cyclic vector $\xi_0 \in \mathcal{H}$, such that $\tau(\gamma) = \langle \pi(\gamma)\xi_0, \xi_0 \rangle$ for all $\gamma \in \Gamma$. If we let M be the von Neumann algebra generated by $\pi(\Gamma)$, then the state $M \ni x \mapsto \langle x\xi_0, \xi_0 \rangle$ (which we also denote by τ) is easily seen to be a normal faithful tracial state on M, and hence M is finite.

If M is not a factor then there exists $p \in \mathcal{P}(M)$, a central projection, $p \neq 0, 1$. We then have that $\tau_p(\gamma) = \frac{\tau(p\pi(\gamma))}{\tau(p)}$, and $\tau_{1-p}(\gamma) = \frac{\tau((1-p)\pi(\gamma)}{\tau(1-p)}$ are again characters on Γ, which are distinct from τ, and such that $\tau = \tau(p)\tau_p + \tau(1-p)\tau_{1-p}$. Conversely, if $\tau = \alpha\tau_1 + (1-\alpha)\tau_2$ is a non-trivial convex combination and $(\pi_i, \mathcal{H}_i, \xi_i)$ are the associate GNS-constructions for $i = 1, 2$, then the diagonal representation $\gamma \mapsto \pi_1(\gamma) \oplus \pi_2(\gamma)$ extends to an embedding of M into $\mathcal{B}(\mathcal{H}_1) \oplus \mathcal{B}(\mathcal{H}_2)$. The trace $M \ni x \mapsto \langle x\xi_1, \xi_1 \rangle$ then gives a trace on M which is distinct from τ, and hence M is not a factor. Thus, we see that extreme points in the space of characters correspond to unitary representations of Γ which generate a finite factor [Tho64].

If we consider the character on Γ which is given by the Dirac function at the identity, then the corresponding GNS-representation is the left-regular representation, and the corresponding von Neumann algebra is the group von Neumann algebra $L\Gamma$. If $\pi : \Gamma \to \mathcal{U}(M)$ is a representation into a tracial von Neumann algebra M, then this extends to an embedding $\tilde{\pi} : L\Gamma \to M$ if and only if we have $\tau \circ \pi = \delta_e$.

If $\Sigma \lhd \Gamma$ is a normal subgroup and $\tau_0 : \Sigma \to \mathbb{T}$ is a character, then this character may be induced to a character on Γ by setting $\tau(\gamma) = 0$ if $\gamma \not\in \Sigma$, and $\tau(\gamma) = \tau_0(\gamma)$ otherwise. If τ is an extreme character then so is τ_0. If τ_0 is an extreme character, then a sufficient condition for τ is be extreme is if Γ/Σ is ICC, although this is not necessary in general. If Γ has a non-trivial center $\mathcal{Z}(\Gamma)$, then any representation which generates a finite factor $\pi : \Gamma \to \mathcal{U}(M)$ must send $\mathcal{Z}(\Gamma)$ to the center $\mathcal{Z}(M) = \mathbb{C}$. Thus, any extreme point in the space of character for Γ must also be extreme when restricted to $\mathcal{Z}(\Gamma)$ and hence $\chi = \tau_{|\mathcal{Z}(\Gamma)}$ is a homomorphism.

3 Induced actions and ergodicity

Let G be a locally compact group, and $\Gamma < G$ a lattice. Suppose M is a finite von Neumann algebra with a normal faithful trace τ, and $\theta : \Gamma \to \mathrm{Aut}(M, \tau)$ is a trace preserving action of Γ on M. We let $\alpha : G \times G/\Gamma \to \Gamma$ be a cocycle corresponding to the identity map $\Gamma \to \Gamma$, e.g., $\alpha(g,x) = \psi(gx)^{-1} g \psi(x)$, where $\psi : G/\Gamma \to G$ is a Borel section. We may then induce the action of Γ on M to an action of G on $L^\infty(G/\Gamma)\overline{\otimes}M$ as $\tilde{\theta}_g(f)(x) = \theta_{\alpha(g,g^{-1}x)}(f(g^{-1}x))$. Since θ is trace preserving, we have that $\tilde{\theta}$ preserves the trace on $L^\infty(G/\Gamma)\overline{\otimes}M$ given by $\int \otimes \tau$.

We consider the action $L : G \to \mathrm{Aut}(L^\infty G)$ (resp. $R : G \to \mathrm{Aut}(L^\infty G)$ induced by left (resp. right) multiplication. An alternate way to view the induced action of G on $L^\infty(G/\Gamma)\overline{\otimes}M$ is to consider the action of $\Gamma \curvearrowright^{R\otimes\theta} L^\infty(G)\overline{\otimes}M$. Fixing a Borel section $\psi : G/\Gamma \to G$ then gives an isomorphism $\Psi : L^\infty(G/\Gamma)\overline{\otimes}M \to (L^\infty(G)\overline{\otimes}M)^{(R\otimes\theta)(\Gamma)}$ by $\Psi(f)(g) = \theta_{\psi(\Gamma)\alpha(g^{-1},g\Gamma)}(f(g\Gamma))$. Under this isomorphism, the induced action of G on $L^\infty(G/\Gamma)\overline{\otimes}M$ translates to the action $G \curvearrowright^{L\otimes\mathrm{id}} (L^\infty(G)\overline{\otimes}M)^{(R\otimes\theta)(\Gamma)}$ (see, e.g., [Zim84, Section 4.2]).

An action of a group on a von Neumann algebra is ergodic if the fixed point subalgebra is $\mathbb{C}$. Note that from the isomorphism given by Ψ above, we have that the action $\Gamma \curvearrowright^{\theta} M$ is ergodic if and only if the induced action $G \curvearrowright^{\tilde{\theta}} L^\infty(G/\Gamma)\overline{\otimes}M$ is ergodic.

Suppose $H < G$ is a closed subgroup such that ΓH is dense in G. For $O \subset G$ a non-empty open subset we set $\Gamma_O = \Gamma \cap OH$, which is non-empty since ΓH is dense in G. For $x \in M$ we set $\mathcal{K}_x(O) = \overline{\mathrm{co}}\{\theta_\gamma(x) \mid \gamma \in \Gamma_O\}$, and $\tilde{\mathcal{K}}_x(O) = \overline{\mathrm{co}}\{\theta_{\gamma^{-1}}(x) \mid \gamma \in \Gamma_O\}$. For each $g \in G$ we let $\kappa_x(g)$ (resp. $\tilde{\kappa}_x(g)$) be the element of minimal $\|\cdot\|_2$ in $\cap_{O\in\mathcal{N}(g)}\mathcal{K}_x(O)$ (resp. $\cap_{O\in\mathcal{N}(g)}\tilde{\mathcal{K}}_x(O)$), where $\mathcal{N}(g)$ denotes the set of open neighborhoods of g.

Proposition 3.1. *Using the notation above, if the induced action $H \curvearrowright L^\infty(G/\Gamma)\overline{\otimes}M$ is ergodic, then $\kappa_x(g) = \tau(x)$ for all $g \in G$.*

Proof. Using the isomorphism $(L^\infty(G/\Gamma)\overline{\otimes}M)^H \cong (L^\infty(G)\overline{\otimes}M)^{H\times\Gamma}$ as described above, ergodicity of the H action is equivalent to ergodicity of the $H \times \Gamma$ action on $L^\infty(G)\overline{\otimes}M$, where the action of H is given by $h \mapsto L_h \otimes \mathrm{id}$, and the action of Γ is given by $\gamma \mapsto R_\gamma \otimes \theta_\gamma$.

Note that $\kappa_x : G \to M$ is a bounded Borel map. Indeed, we have $\|\kappa_x(g)\| \leq \|x\|$ for all $g \in G$, hence κ_x is bounded. If $\{g_n\}_{n\in\mathbb{N}}$ is a countable dense subset of G, then for each $O \in \mathcal{N}(e)$ we may consider the simple function $\kappa_{x,O} : G \to M$ given by setting $\kappa_{x,O}(g)$ to be the unique element of minimal norm in $\mathcal{K}_x(g_j O)$, where j is the smallest natural number such that $g \in g_j O$. Taking a sequence $O_n \in \mathcal{N}(e)$ such that $\cap O_n = \{e\}$ we then have that κ_{x,O_n} converges pointwise in the strong operator topology to κ_x, hence κ_x is Borel.

For $g \in G$, $h \in H$, $\gamma \in \Gamma$, and $O \in \mathcal{N}(g)$, we have $\theta_\gamma(\mathcal{K}_x(O)) = \mathcal{K}_x(\gamma O h)$, hence it follows that $\theta_\gamma(\kappa_x(g)) = \kappa_x(\gamma g h)$.

Thus, $\kappa_x \in (L^\infty(G)\overline{\otimes}M)^{H\times\Gamma} = \mathbb{C}$, and so $\kappa_x(g) = \tau(x)$ for almost every $g \in G$. As $\{g \in G \mid \kappa_x(g) = \tau(x)\}$ is closed, it then follows that $\kappa_x(g) = \tau(x)$ for all $g \in G$. □

We remark that when $H \lhd G$ is normal, then G/H acts on $(L^\infty(G/\Gamma)\overline{\otimes}M)^H$ and the above proposition gives an identification between $(L^\infty(G/\Gamma)\overline{\otimes}M)^H$ and the G/H-algebra as defined in [CP13, Section 4].

In the sequel we will need to consider convex combinations of the form $\theta_{\gamma^{-1}}(x)$ for $\gamma \in \Gamma_O$. In the case when $H < G$ is normal we have $\kappa_x(g) = \tilde{\kappa}_x(g)$, and the above proposition suffices. For the general case we have the following argument:

Proposition 3.2. *Using the notation above, if the induced action $H \curvearrowright L^\infty(G/\Gamma)\overline{\otimes}M$ is ergodic, then $\tilde{\kappa}_x(g) = \tau(x)$ for all $g \in G$.*

Proof. Let $\varepsilon > 0$ be given. We claim that there exists $U \in \mathcal{N}(e)$ such that for all $\gamma \in \Gamma_U$ we have $\|\theta_\gamma(\tilde{\kappa}_x(e)) - \tilde{\kappa}_x(e)\|_2 = \|\tilde{\kappa}_x(e) - \theta_{\gamma^{-1}}(\tilde{\kappa}_x(e))\|_2 < \varepsilon$.

Indeed, from the definition of $\tilde{\kappa}_x(e)$ there exists $O \in \mathcal{N}(e)$ such that if y is the element of minimal norm in $\tilde{\mathcal{K}}_x(O)$ then $\|\tilde{\kappa}_x(e) - y\|_2 < \varepsilon/2$. We may write y as a convex combination $y = \sum_i \alpha_i \theta_{\gamma_i^{-1}}(x)$ where $\gamma_i = g_i h_i$ with $g_i \in O$, and $h_i \in H$. If no such U existed, then there would exist a sequence $\tilde{\gamma}_k = \tilde{g}_k \tilde{h}_k$ with $\tilde{h}_k \in H$, and $\tilde{g}_k \to e$ such that $\|y - \theta_{\tilde{\gamma}_k^{-1}}(y)\|_2 \geq \varepsilon/4$.

However, $\theta_{\tilde{\gamma}_k^{-1}}(y) = \sum_i \alpha_i \theta_{\tilde{\gamma}_k^{-1}\gamma_i^{-1}}(x)$, and since $\tilde{g}_k \to e$, for each i there exists large enough k so that $\gamma_i \tilde{\gamma}_k = g_i(h_i \tilde{g}_k h_i^{-1}) h_i \tilde{h}_k \in OH$. It then follows from uniqueness of y that $\|y - \theta_{\tilde{\gamma}_k^{-1}}(y)\|_2 \to 0$, proving the claim.

From the claim it then follows that $\tilde{\kappa}_x(e)$ is the element of minimal $\|\cdot\|_2$ in $\cap_{O \in \mathcal{N}(e)} \mathcal{K}_{\tilde{\kappa}_x(e)}(O)$, and it then follows from Proposition 3.1 that $\tilde{\kappa}_x(e) = \tau(\tilde{\kappa}_x(e)) = \tau(x)$.

For $\gamma \in \Gamma$, $h \in H$, and $O \subset G$ a non-empty open set, we have $\tilde{\mathcal{K}}_x(\gamma O h) = \tilde{\mathcal{K}}_{\theta_{\gamma^{-1}}(x)}(O)$, hence it follows that $\tilde{\kappa}_x(\gamma h) = \tilde{\kappa}_{\theta_{\gamma^{-1}}(x)}(e) = \tau(\theta_{\gamma^{-1}}(x)) = \tau(x)$. Since ΓH is dense in G and since $\{g \in G \mid \tilde{\kappa}_x(g) = \tau(x)\}$ is closed, it then follows that $\tilde{\kappa}_x(g) = \tau(x)$ for all $g \in G$. $\square$

4 Irreducible lattices in higher-rank groups

We fix the following notation throughout this section. Let A be a finite non-empty set, and for each $\alpha \in A$, let k_α be a local field and let $\mathbb{G}_\alpha$ be a non-trivial connected semi-simple group defined over k_α. Set $G_\alpha = \mathbb{G}_\alpha(k)$, and for any subset $B \subset A$ denote $G_B = \prod_{\alpha \in B} G_\alpha$. We denote G_A by G. The rank of G is given by $\mathrm{rank}(G) = \sum_{\alpha \in A} \mathrm{rank}_{k_\alpha}(G_\alpha)$.

We suppose $\Gamma < G$ is an irreducible lattice, i.e., for every $B \subset A$, $B \neq \emptyset$, $B \neq A$, the subgroup $(\Gamma \cap G_B) \cdot (\Gamma \cap G_{A \setminus B})$ is of infinite index in Γ.

Lemma 4.1. *Suppose for each $\alpha \in A$ the group $\mathbb{G}_\alpha$ is simply connected, k_α-isotropic, and almost k_α-simple. Suppose that M is a finite von Neumann algebra with normal faithful trace τ, and $\theta : \Gamma \to \mathrm{Aut}(M, \tau)$ is a trace preserving action of Γ on M. Then the induced action $G \curvearrowright L^\infty(G/\Gamma)\overline{\otimes}M$ is irreducible, i.e., $G_\alpha \curvearrowright L^\infty(G/\Gamma)\overline{\otimes}M$ is ergodic for each $\alpha \in A$.*

Proof. If $|A| = 1$, then this follows from ergodicity of the induced action, thus we may assume $|A| > 1$. Fix $\alpha \in A$ and let $B = A \setminus \{\alpha\}$. Then, as remarked above, we have an identification between $(L^\infty(G/\Gamma)\overline{\otimes}M)^{G_\alpha}$ and the G_B-algebra, $M_0 \subset M$. As G_B is a product of groups with the Howe-Moore property [HM79, Theorem 5.2] it follows from [CP13, Proposition 4.1 and Lemma 4.3] that the G_B-algebra is trivial and hence $(L^\infty(G/\Gamma)\overline{\otimes}M)^{G_\alpha} = \mathbb{C}$. $\square$

For each $\alpha \in A$ we fix a maximal k_α-split torus $\mathbb{S}_\alpha$ in $\mathbb{G}_\alpha$ and set $S = \prod_{\alpha\in S} \mathbb{S}_\alpha(k_\alpha)$. We also fix $P < G$ a minimal parabolic subgroup containing S and let $V < P$ be its unipotent radical. Let $\overline{P}$ be the opposite parabolic and by $\overline{V}$ its unipotent radical. We fix another parabolic subgroup $P_0 \lneq G$ such that $P < P_0$. (Note that if $\mathrm{rank}(G) \geq 2$ and if $\mathbb{G}_\alpha$ is simply connected for each $\alpha \in A$, then we have that G is generated by all such subgroups [Mar91, Theorem I.2.3.1].) We let V_0 be the unipotent radical of P_0, and we let $\overline{P_0}$ and $\overline{V_0}$ be the corresponding opposite subgroups.

We let R_0 be the reductive component of P_0 containing S so that $P_0 = R_0 \ltimes V_0$, and we set $\overline{L_0} = R_0 \cap \overline{V}$. We then have that $\overline{L_0}$ normalizes $\overline{V_0}$, and $\overline{L_0} \cap \overline{V_0} = \{e\}$ so that we have an identification $\overline{V} = \overline{V_0} \rtimes \overline{L_0}$. For the convenience of the reader we include the example when $G = \mathrm{SL}_n(\mathbb{R})$ (see, e.g., [Mar91, Section II.3]).

Example 4.2. *In the case when $G = \mathrm{SL}_n(\mathbb{R})$ we may consider S the subgroup of diagonal matrices, P the subgroup of upper triangular matrices, and V the subgroup of upper triangular matrices whose diagonal entries are 1. In this case $\overline{P}$ is the subgroup of lower triangular matrices, and $\overline{V}$ is the subgroup of lower triangular matrices whose diagonal entries are 1. If $1 \leq j_1 < j_2 < \cdots < j_k = n$, then P_0 could consist of the block triangular matrices of determinant 1 of the form*

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ 0 & A_{22} & \cdots & A_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_{kk} \end{pmatrix},$$

where A_{hh} is a square matrix of order $j_h - j_{h-1}$ (here we assume $j_0 = 0$). In this case, V_0 consists of the block triangular matrices of the form

$$\begin{pmatrix} E_1 & A_{12} & \cdots & A_{1k} \\ 0 & E_2 & \cdots & A_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & E_k \end{pmatrix},$$

where E_h denotes the $j_h - j_{h-1}$ identity matrix, and R_0 consists of the block diagonal matrices of the form

$$\begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_{kk} \end{pmatrix}.$$

We then have that $\overline{L_0}$ is the subgroup of R_0 whose block matrices A_{hh} are lower triangular with diagonal entries equal to 1.

The mapping $\overline{V} \to G/P$ given by $v \mapsto vP$ gives rise to a measure space isomorphism [Mar91, Lemma IV.2.2], so that the action of $\overline{V}$ on G/P transforms to left multiplication on $\overline{V}$, while the action of R on G/P transforms to the action induced by conjugation on $\overline{V}$. As $\overline{V} = \overline{V_0} \rtimes \overline{L_0}$ we then have a measure space isomorphism between G/P and $\overline{V_0} \times \overline{L_0}$, and hence we obtain a corresponding isomorphism between von Neumann algebras $L^\infty(G/P) \cong L^\infty(\overline{V_0})\overline{\otimes}L^\infty(\overline{L_0})$. Moreover, the natural projectiion map G/P to G/P_0 gives rise to an embedding $L^\infty(G/P_0) \subset L^\infty(G/P)$ which corresponds under the above isomorphism to the natural embedding $L^\infty(\overline{V_0}) \subset L^\infty(\overline{V_0})\overline{\otimes}L^\infty(\overline{L_0})$. Note that if $h \in \overline{V_0} \rtimes \mathcal{Z}(R_0)$, then $\sigma_h(f) = f$ for all $f \in L^\infty(\overline{L_0})$.

At the heart of Margulis' factor theorem is his use of contracting automorphisms to show that if $f \in L^\infty(\overline{V_0})\overline{\otimes}L^\infty(\overline{L_0})$, then the Γ-invariant von Neumann subalgebra generated by f contains the essential range of f when viewed as function from $\overline{V_0}$ to $L^\infty(\overline{L_0})$. This is a consequence of the next lemma, which is implicit in Margulis' proof. We include a proof for completeness.

Lemma 4.3. *[Mar91, Lemma IV.2.7] Suppose that for each $\alpha \in A$, $\mathbb{G}_\alpha$ is simply connected, k_α-isotropic, and almost k_α-simple. If $E \subset \overline{V_0}$ has positive measure, then there exist sequences $\{\gamma_n\} \subset \Gamma$ and $\{h_n\} \subset \overline{V_0} \rtimes \mathcal{Z}(R_0)$ such that $\gamma_n h_n^{-1} \to e$, and $\nu(\gamma_n E) \to 1$.*

Proof. By [Mar91, Lemmas II.3.1, IV.2.5] there exists $s \in \mathcal{Z}(R_0)$ such that $\mathrm{Int}s^{-1}{}_{|\overline{V_0}}$ and $\mathrm{Int}s_{|V_0}$ are contracting. Since $\Gamma < G$ is irreducible, s acts ergodically on G/Γ by Moore's ergodicity theorem. Thus, for almost every $v \in \overline{V_0}$ we have that $\{\Gamma v s^{-n} \mid n \in \mathbb{N}\}$ is dense [Mar91, Lemma IV.2.3]. We denote by F_1 the set of such $v \in \overline{V_0}$. (Lemma IV.2.3 in [Mar91] actually considers only $\overline{V}$, however, as $s \in \mathcal{Z}(R_0)$ the proof holds for $\overline{V_0}$ the same).

As $\mathrm{Int}s^{-1}{}_{|\overline{V_0}}$ is contracting, for almost every $v \in \overline{V_0}$ we have $\nu(s^n v^{-1} E) \to 1$. We denote by F_2 the set of such $v \in \overline{V_0}$. We fix $v_0 \in F_1 \cap F_2$. Since $v_0 \in F_1$ there exists a sequence $\{\gamma_j\} \subset \Gamma$, and a subsequence $\{s^{-n_j}\}$ such that $\gamma_j v_0 s^{-n_j} \to e$. Hence

$$\lim_{j\to\infty} \nu(\gamma_j E) = \lim_{j\to\infty} \nu(s^{n_j} v_0^{-1} E) = 1.$$

Since $h_n = v_0 s^{-n_j} \in \overline{V_0} \rtimes \mathcal{Z}(R_0)$, this finishes the proof. □

Suppose now that M is a finite factor with normal faithful trace τ, and $\pi : \Gamma \to \mathcal{U}(M)$ is a homomorphism such that $\pi(\Gamma)'' = M$. We set

$$\mathcal{B} = L^\infty(G/P)\overline{\otimes}\mathcal{B}(L^2M) \cap \{\sigma^0_\gamma \otimes J\pi(\gamma)J \mid \gamma \in \Gamma\}',$$

which we view as the space of essentially bounded Γ-equivariant functions from G/P to $\mathcal{B}(L^2M)$, where the former has the Γ-action given by left multiplication, and the Γ-action on the latter is given by conjugation by $J\pi(\gamma)J$. Note that $\mathcal{B}$ is injective [Zim77, Theorem 5.2].

We have the following operator algebraic consequence of Lemma 4.3:

Lemma 4.4. *Suppose that for each $\alpha \in A$, $\mathbb{G}_\alpha$ is simply connected, k_α-isotropic, and almost k_α-simple. Using the notation above, suppose $x = x^* \in L^\infty(\overline{V_0})\overline{\otimes}L^\infty(\overline{L_0})\overline{\otimes}\mathcal{B}(L^2M)$ and suppose $x_0 \in L^\infty(\overline{L_0})\overline{\otimes}\mathcal{B}(L^2M)$ is in the $\overline{V_0}$-essential range of x, then there exists $y = y^* \in \mathcal{B}$ such that $yP_{\hat{1}} \in L^\infty(\overline{L_0})\overline{\otimes}\mathcal{B}(L^2M)$, and $P_{\hat{1}}yP_{\hat{1}} = P_{\hat{1}}x_0P_{\hat{1}}$.*

Proof. For each $n \in \mathbb{N}$ we let $E_n \subset \overline{V_0}$ be a positive measure subset such that $x(v_n) - x_0 \to 0$ strongly whenever $v_n \in E_n$. From Lemma 4.3 there exists sequences $\{\gamma_n\} \subset \Gamma$ and $\{h_n\} \subset \overline{V_0} \rtimes \mathcal{Z}(R_0)$ such that $\gamma_n h_n^{-1} \to e$, and $\nu(\gamma_n E_n) \to 1$.

Hence, for all $\xi \in L^2(\overline{V_0} \times \overline{L_0})$ and $\eta \in L^2M$ we have

$$\begin{aligned}\|1_{\gamma_n E_n}(\sigma_{\gamma_n}(x) - x_0)\xi \otimes \eta\| &\leq \|1_{\gamma_n E_n}(\sigma_{\gamma_n}(x - x_0))\xi \otimes \eta\| + \|1_{\gamma_n E_n}(\sigma_{\gamma_n h_n^{-1}}(x_0) - x_0)\xi \otimes \eta\| \\ &= (\int_{\gamma_n E_n} \|(\sigma_{\gamma_n}(x - x_0))\eta\|_2^2 |\xi|^2)^{1/2} \\ &\quad + \|1_{\gamma_n E_n}(\sigma_{\gamma_n h_n^{-1}}(x_0) - x_0)\xi \otimes \eta\| \to 0.\end{aligned}$$

Therefore, $1_{\gamma_n E_n}(\sigma_{\gamma_n}(x) - x_0) \to 0$ strongly. Since $\nu(\gamma_n E_n) \to 1$, we then have that $1_{\gamma_n E_n} \to 1$ strongly, and hence $\sigma_{\gamma_n}(x) \to x_0$ strongly.

We let $y \in \mathcal{B}$ be a weak operator topology cluster point of the set $\{\pi(\gamma_n)x\pi(\gamma_n^{-1})\}$. It then follows that $yP_{\hat{1}}$ is a weak operator topology cluster point of

$$\begin{aligned}\{\pi(\gamma_n)x\pi(\gamma_n^{-1})P_{\hat{1}}\} &= \{(\pi(\gamma_n)(J\pi(\gamma_n)J))(J\pi(\gamma_n^{-1})J)x(J\pi(\gamma_n)J)P_{\hat{1}}\} \\ &= \{(\pi(\gamma_n)(J\pi(\gamma_n)J))\sigma_{\gamma_n}(x)P_{\hat{1}}\}.\end{aligned}$$

Since $\sigma_{\gamma_n}(x) \to x_0$ strongly, we then have that $yP_{\hat{1}}$ is a weak operator topology cluster point of $\{(\pi(\gamma_n)(J\pi(\gamma_n)J))x_0P_{\hat{1}}\} \subset L^\infty(\overline{L_0})\overline{\otimes}\mathcal{B}(L^2M)$, thus $y \in \mathcal{B}$, and $yP_{\hat{1}} \in L^\infty(\overline{L_0})\overline{\otimes}\mathcal{B}(L^2M)$.

Similarly, $P_{\hat{1}}yP_{\hat{1}}$ is a weak operator topology cluster point of

$$\{P_{\hat{1}}\pi(\gamma_n)x\pi(\gamma_n^{-1})P_{\hat{1}}\} = \{P_{\hat{1}}\sigma_{\gamma_n}(x)P_{\hat{1}}\}$$

and hence $P_{\hat{1}}yP_{\hat{1}} = P_{\hat{1}}x_0P_{\hat{1}}$. □

We remark that the case $P_0 = P$ in the previous lemma is also of interest. Indeed, in this case we have $\overline{L_0} = \{e\}$, and hence the y produced above satisfies $yP_{\hat{1}} \in \mathcal{B}(L^2M)$. In fact, it is not hard to see in this case that we actually have $y \in M \subset \mathcal{B}$ (see the proof of Lemma 5.2 below), and $y - \pi(\gamma_n)x\pi(\gamma_n) \to 0$ weakly. This is similar to Lemma 3.1 in [CP13], from which followed the key rigidity properties used in that work.

Theorem 4.5. *Suppose* $\mathrm{rank}(G) \geq 2$, *and for each $\alpha \in A$ the group $\mathbb{G}_\alpha$ has no k_α-anisotropic factors, then either π is induced from a character on $\mathcal{Z}(\Gamma)$, or else M is injective.*

Proof. By Margulis' Arithmeticity Theorem [Mar91, Theorem IX.1.11], passing to a commensurate lattice we may assume that each $\mathbb{G}_\alpha$ is simply connected, k_α-isotropic, and almost k_α-simple.

Suppose that $x = x^* \in \mathcal{B}$. Fix $P < G$ a minimal parabolic subgroup and $P < P_0 \lneq G$. Using the decomposition described above we may consider $x \in L^\infty(\overline{V_0})\overline{\otimes}L^\infty(\overline{L_0})\overline{\otimes}\mathcal{B}(L^2M)$. Let $x_0 \in L^\infty(\overline{L_0})\overline{\otimes}\mathcal{B}(L^2M)$ be in the $\overline{V_0}$-essential range of x. By Lemma 4.4 there exists $y = y^* \in \mathcal{B}$ such that $yP_{\hat{1}} \in L^\infty(\overline{L_0})\overline{\otimes}\mathcal{B}(L^2M)$, and $P_{\hat{1}}yP_{\hat{1}} = P_{\hat{1}}x_0P_{\hat{1}}$.

By Proposition 4.1 the action $G\curvearrowright L^\infty(G/\Gamma)\overline{\otimes}M$ is irreducible, and hence by Moore's ergodicity theorem the action restricted to $\overline{V_0} \rtimes \mathcal{Z}(R_0)$ is ergodic (this group is non-compact since $P_0 \neq G$). By Proposition 3.2, for every open neighborhood $O \subset G$ of e, setting $\Gamma_O = \Gamma \cap O(\overline{V_0} \rtimes \mathcal{Z}(R_0))$ we have $\tau(x) \in \overline{\mathrm{co}}\{\pi(\gamma^{-1})x\pi(\gamma) \mid \gamma \in \Gamma_O\}$.

If π is not induced from a character on $\mathcal{Z}(\Gamma)$ there exists $\gamma_0 \in \Gamma \setminus \mathcal{Z}(\Gamma)$ such that $\alpha_0 = \tau(\pi(\gamma_0)) \neq 0$. Thus, for each open neighborhood $O \subset G$ of e there exists $0 \leq \alpha_i \leq 1$, $\sum_{i=1}^n \alpha_i = 1$, and $\gamma_i \in \Gamma_O$ such that

$$\begin{aligned}[]
[\sigma^0_{\gamma_0} \otimes \alpha_0, P_{\hat{1}}x_0P_{\hat{1}}] &= [\sigma^0_{\gamma_0} \otimes \alpha_0, P_{\hat{1}}yP_{\hat{1}}] \\
&\sim \sum_{i=1}^n \alpha_i P_{\hat{1}}[\sigma^0_{\gamma_0} \otimes J\pi(\gamma_i^{-1}\gamma_0\gamma_i)J, y]P_{\hat{1}},
\end{aligned}$$

where the approximation above is in the strong operator topology. We may write $\gamma_i = g_ih_i$ where $g_i \in O$, and $h_i \in \overline{V_0} \rtimes \mathcal{Z}(R_0)$, and since $yP_{\hat{1}}, P_{\hat{1}}y \in L^\infty(\overline{L_0})\overline{\otimes}\mathcal{B}(L^2M)$, we have that $\sigma_{h_i}(yP_{\hat{1}}) = yP_{\hat{1}}$, and $\sigma_{h_i}(P_{\hat{1}}y) = P_{\hat{1}}y$. Taking O to be a small enough neighborhood, we then have the strong operator topology approximations $\sigma_{\gamma_i}(yP_{\hat{1}}) \sim yP_{\hat{1}}$, and $\sigma_{\gamma_i}(P_{\hat{1}}y) \sim P_{\hat{1}}y$.

It follows that we then have the weak operator topology approximation

$$\begin{aligned}[]
[\sigma^0_{\gamma_0} \otimes \alpha_0, P_{\hat{1}}x_0P_{\hat{1}}] &\sim \sum_{i=1}^n \alpha_i P_{\hat{1}}[\sigma^0_{\gamma_0} \otimes J\pi(\gamma_i^{-1}\gamma_0\gamma_i)J, (\sigma^0_{\gamma_i} \otimes 1)y(\sigma^0_{\gamma_i^{-1}} \otimes 1)]P_{\hat{1}} \\
&= \sum_{i=1}^n \alpha_i P_{\hat{1}}(\sigma^0_{\gamma_i} \otimes 1)[\sigma^0_{\gamma_i^{-1}\gamma_0\gamma_i} \otimes J\pi(\gamma_i^{-1}\gamma_0\gamma_i)J, y](\sigma^0_{\gamma_i^{-1}} \otimes 1)P_{\hat{1}} = 0.
\end{aligned}$$

Hence, we conclude that $[\sigma^0_{\gamma_0} \otimes \alpha_0, P_{\hat{1}}x_0P_{\hat{1}}] = 0$, and since $\alpha_0 \neq 0$ we then have that $[\sigma^0_{\gamma_0} \otimes 1, P_{\hat{1}}x_0P_{\hat{1}}] = 0$.

Hence $\sigma_{\gamma_0}(P_{\hat{1}}x_0P_{\hat{1}}) = P_{\hat{1}}x_0P_{\hat{1}}$, and since $\tau(\pi(\gamma\gamma_0\gamma^{-1})) = \tau(\pi(\gamma_0))$ the same argument shows that $\sigma_\gamma(P_{\hat{1}}x_0P_{\hat{1}}) = P_{\hat{1}}x_0P_{\hat{1}}$ for all $\gamma \in \langle\langle\gamma_0\rangle\rangle$, the normal closure of γ_0. Since $\gamma_0 \notin \mathcal{Z}(\Gamma)$, Margulis' normal subgroup theorem shows that $\langle\langle\gamma_0\rangle\rangle$ is finite index in Γ and hence acts ergodically on $L^\infty(G/P)$. Thus, we have $P_{\hat{1}}x_0P_{\hat{1}} \in P_{\hat{1}}\mathcal{B}(L^2M)P_{\hat{1}} = \mathbb{C}P_{\hat{1}}$.

As x_0 was an arbitrary element in the essential range of x, we conclude that $P_{\hat{1}}xP_{\hat{1}} \in L^\infty(\overline{V_0})\overline{\otimes}\mathbb{C}P_{\hat{1}}$, and, as x was an arbitrary self-adjoint element, we then have $P_{\hat{1}}\mathcal{B}P_{\hat{1}} \subset$

$L^\infty(\overline{V_0})\overline{\otimes}\mathbb{C}P_{\hat{1}}$. If $a, b \in M$ and $z \in \mathcal{B}$, we then have $\langle z\hat{a}, \hat{b}\rangle = \langle (b^*za)\hat{1}, \hat{1}\rangle \in L^\infty(\overline{V_0})$, and hence it follows that $\mathcal{B} \subset L^\infty(\overline{V_0})\overline{\otimes}\mathcal{B}(L^2M) = L^\infty(G/P_0)\overline{\otimes}\mathcal{B}(L^2M)$.

However, P_0 was an arbitrary parabolic such that $P < P_0 \lneq G$. Since $\mathrm{rank}(G) \geq 2$, G is generated by all such subgroups [Mar91, Theorem I.3.2.1] and we therefore have $\mathcal{B} \subset \mathcal{B}(L^2M)$. Thus, $\mathcal{B} = \mathcal{B}(L^2M) \cap \{J\pi(\gamma)J \mid \gamma \in \Gamma\}' = M$, and hence M is injective. □

Proof of Theorem A. Since $\mathcal{Z}(G) = \{e\}$, we have that $\mathcal{Z}(\Gamma) = \{e\}$. Thus, if π does not extend to an isomorphism $L\Gamma \xrightarrow{\sim} M$, then by Theorem 4.5 we have that M is injective. As G has property (T) so does Γ [Kaž67], and hence we must then have that $\pi(\Gamma)$ is precompact in the strong operator topology [CJ85, Rob93]. Hence $M = \pi(\Gamma)''$ must be type I, and since it is a factor it must be finite dimensional. □

5 Irreducible lattices in products

The techniques developed above for irreducible lattices in semi-simple groups can be adapted to give similar results for irreducible lattices in products in a spirit similar to Bader and Shalom's results for normal subgroups [BS06] (see also [Cre13]).

Throughout this section G_1 and G_2 will denote non-compact second countable locally compact groups, $G = G_1 \times G_2$, and $p_i : G \to G_i$ the projection onto G_i. We also fix $\Gamma < G$ an irreducible lattice, i.e., $p_i(\Gamma) < G_i$ is dense for $i = 1, 2$. We fix M a finite factor with normal faithful trace τ, and $\pi : \Gamma \to \mathcal{U}(M)$ a homomorphism such that $\pi(\Gamma)'' = M$.

Analogous to Margulis' Lemma (Lemma 4.3 above), we have the following lemma which is adapted from Proposition 2.4 in [CS12]:

Lemma 5.1. *Suppose $\mu \in \mathrm{Prob}(G_1)$ is absolutely continuous with respect to Haar measure, and let $G_1 \curvearrowright (B, \eta)$ be the corresponding boundary action. Then for any $E \subset B$, such that $\eta(E) > 0$, there exists a sequence $\gamma_n \in \Gamma$ such that $p_2(\gamma_n) \to e$, and $\eta(p_1(\gamma_n)E) \to 1$.*

Proof. Fix $K_1 < G_1$ a compact subset with non-empty interior, $K_2 \subset G_2$ a compact neighborhood of the identity, and consider $F = (K_1 \times K_2)\Gamma \subset G/\Gamma$. If we denote by m the G-invariant probability measure on G/Γ, then by Kakutani's random ergodic theorem [Kak51] we have that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} 1_F(\omega_n^{-1} \cdots \omega_1^{-1} z) = m(F) > 0$$

for m-almost every $z \in G/\Gamma$, and $\mu^{\mathbb{N}}$-almost every $\{\omega_n\} \in G_1^{\mathbb{N}}$. Thus, for some $z_0 \in K_1 \times K_2$ we have that for $\mu^{\mathbb{N}}$-almost every $\{\omega_n\} \in G_1^{\mathbb{N}}$ the sequence $\{\omega_n^{-1}\omega_{n-1}^{-1} \cdots \omega_1^{-1}\}$ intersects $(K_1 \times K_2)\Gamma z_0^{-1}$ infinitely often.

If $E \subset B$ such that $\eta(E) > 0$ then

$$\mu^{\mathbb{N}}(\{\{\omega_n\} \in G_1^{\mathbb{N}} \mid \lim_{n\to\infty} \eta(\omega_n^{-1} \cdots \omega_1^{-1} p_1(z_0)E) = 1\}) = \eta(p_1(z_0)E) > 0.$$

Hence, there exists $\{\omega_n\} \in G_1^{\mathbb{N}}$ such that $\{\omega_n^{-1}\omega_{n-1}^{-1}\cdots\omega_1^{-1}\}$ intersects $(K_1 \times K_2)\Gamma z_0^{-1}$ infinitely often, and $\lim_{n\to\infty} \eta(\omega_n^{-1}\cdots\omega_1^{-1}p_1(z_0)E) = 1$.

We may therefore choose $h_j \in K_1 \times K_2$, $\gamma_j \in \Gamma$, and a subsequence $\{\omega_{n_j}\}$, such that $\omega_{n_j}^{-1}\cdots\omega_1^{-1} = h_j\gamma_j z_0^{-1}$. We then have $p_2(\gamma_j) = p_2(h_j^{-1}z_0) \in K_2^2$, and

$$\lim_{j\to\infty} \eta(p_1(\gamma_j)E) = \lim_{j\to\infty} \eta(p_1(h_j)\omega_{n_j}^{-1}\cdots\omega_1^{-1}z_0E) = 1.$$

As K_2 was an arbitrary compact neighborhood of the identity, a diagonalization argument then gives the result. □

For products, we may strengthen Lemma 4.4 to give the following:

Lemma 5.2. *Suppose $\mu \in \mathrm{Prob}(G_1)$ is absolutely continuous with respect to Haar measure, and let $G_1 \curvearrowright (B,\eta)$ be the corresponding boundary action. Let $G_2 \curvearrowright (Y,\nu)$ be a quasi-invariant action, and set*

$$\mathcal{B} = (L^\infty(B,\eta)\overline{\otimes}L^\infty(Y,\nu)\overline{\otimes}\mathcal{B}(L^2M)) \cap \{\sigma_\gamma^0 \otimes J\pi(\gamma)J \mid \gamma \in \Gamma\}'.$$

If $x \in \mathcal{B}$ and $x_0 \in L^\infty(Y,\nu)\overline{\otimes}\mathcal{B}(L^2M)$ is in the B-essential range of x, then there exists $y \in \mathcal{B} \cap (L^\infty(Y,\nu)\overline{\otimes}\mathcal{B}(L^2M))$ such that $P_{\hat{1}}yP_{\hat{1}} = P_{\hat{1}}x_0P_{\hat{1}}$.

Proof. Let $x \in \mathcal{B}$ and $x_0 \in L^\infty(Y,\nu)\overline{\otimes}\mathcal{B}(L^2M)$ be as above. Using Lemma 5.1, and an argument identical to the beginning of the proof of Lemma 4.4, we may obtain a sequence $\{\gamma_n\} \subset \Gamma$ such that $P_2(\gamma_n) \to e$, and $\sigma_{\gamma_n}(x) \to x_0$ in the strong operator topology.

Let y be any weak operator topology cluster point of $\{\pi(\gamma_n)x\pi(\gamma_n^{-1})\}_{n\in\mathbb{N}}$. Again, just as in the proof of Lemma 4.4, we see that $yP_{\hat{1}} \in \mathcal{B}\cap(L^\infty(Y,\nu)\overline{\otimes}\mathcal{B}(L^2M))$, and $P_{\hat{1}}yP_{\hat{1}} = P_{\hat{1}}x_0P_{\hat{1}}$.

To see that, in fact, $y \in \mathcal{B} \cap (L^\infty(Y,\nu)\overline{\otimes}\mathcal{B}(L^2M))$, we fix $\gamma \in \Gamma$. Then

$$\begin{aligned} yP_{\widehat{\pi(\gamma)}} &= y\pi(\gamma)P_{\hat{1}}\pi(\gamma) \\ &= J\pi(\gamma^{-1})J(J\pi(\gamma)JyJ\pi(\gamma^{-1})J)P_{\hat{1}}\pi(\gamma) \\ &= J\pi(\gamma^{-1})J\sigma_{\gamma^{-1}}(yP_{\hat{1}})\pi(\gamma) \in L^\infty(Y,\nu)\overline{\otimes}\mathcal{B}(L^2M). \end{aligned}$$

Since $\gamma \in \Gamma$ was arbitrary, we then have $y = y(\vee_{\gamma\in\Gamma}P_{\widehat{\pi(\gamma)}}) \in L^\infty(Y,\nu)\overline{\otimes}\mathcal{B}(L^2M)$. □

The replacement for Proposition 3.2 above is Theorem 4.4 from [CP13]. Combining this with the previous two lemmas, we then obtain the following result:

Theorem 5.3. *Suppose that G_1 and G_2 are non-compact second countable locally compact groups, and $\Gamma < G_1 \times G_2$ is an irreducible lattice. Suppose that any trace preserving ergodic action of G_i on a finite von Neumann algebra must be properly outer when restricted to Γ. If M is a finite factor and $\pi : \Gamma \to \mathcal{U}(M)$ is a homomorphism such that $\pi(\Gamma)'' = M$, then either M is injective, or else π extends to an isomorphism $L\Gamma \xrightarrow{\sim} M$.*

Proof. For $i = 1, 2$, take $\mu_i \in \text{Prob}(G_i)$ absolutely continuous with respect to Haar measure and let $G_i \curvearrowright (B_i, \eta_i)$ be the corresponding boundary actions.

We let

$$\mathcal{B} = L^\infty(B_1 \times B_2, \eta_1 \times \eta_2) \overline{\otimes} \mathcal{B}(L^2 M) \cap \{\sigma^0(\gamma) \otimes J\pi(\gamma)J \mid \gamma \in \Gamma\}'.$$

The action $G \curvearrowright (B_1 \times B_2, \eta_1 \times \eta_2)$ is amenable [Fur63], and hence $\mathcal{B}$ is injective [Zim77, Theorem 5.2].

Suppose that π does not extend to an isomorphism $L\Gamma \xrightarrow{\sim} M$, and fix $x \in \mathcal{B}$. If we let $x_0 \in L^\infty(B_2, \eta_2) \overline{\otimes} \mathcal{B}(L^2 M)$ be in the B_1-essential range of x, then by Lemma 5.2 there exists $y \in \mathcal{B} \cap (L^\infty(B_2, \eta_2) \overline{\otimes} \mathcal{B}(L^2 M))$ such that $P_{\hat{1}} y P_{\hat{1}} = P_{\hat{1}} x_0 P_{\hat{1}}$.

By [CP13, Theorem 4.4] we have that $y \in M$, and hence $P_{\hat{1}} x_0 P_{\hat{1}} \in \mathbb{C} P_{\hat{1}}$. As x_0 was an arbitrary element in the essential range of x, it then follows that $P_{\hat{1}} x P_{\hat{1}} \in L^\infty(B_1, \eta_1) \overline{\otimes} \mathbb{C} P_{\hat{1}}$. As x was arbitrary, we then have $P_{\hat{1}} \mathcal{B} P_{\hat{1}} \subset L^\infty(B_1, \eta_1) \overline{\otimes} \mathbb{C} P_{\hat{1}}$, and since $M \subset \mathcal{B}$, for all unitaries $a, b \in \mathcal{U}(M)$ we have $P_{\hat{a}} \mathcal{B} P_{\hat{b}} = a P_{\hat{1}} \mathcal{B} P_{\hat{1}} b^* \subset L^\infty(B_1, \eta_1) \overline{\otimes} \mathcal{B}(L^2 M)$.

Hence, $\mathcal{B} \subset L^\infty(B_1, \eta_1) \overline{\otimes} \mathcal{B}(L^2 M)$, and by symmetry we also have $\mathcal{B} \subset L^\infty(B_2, \eta_2) \overline{\otimes} \mathcal{B}(L^2 M)$. Thus $\mathcal{B} \subset \mathcal{B}(L^2 M) \cap \{J\pi(\gamma)J \mid \gamma \in \Gamma\}' = M$, and hence, $M = \mathcal{B}$ is injective. □

Proof of Theorem B. By Theorem 5.3 it is enough to show that if M is injective, then it must be finite dimensional. As G_1 has property (T), and M is injective, for any compact open subset $O \subset G_2$, if we set $\Gamma_O = \Gamma \cap (G_1 \times O)$, then $\pi(\Gamma_O)$ is precompact in $\| \cdot \|_2$. The result then follows from Proposition 6.1 in [CP13]. □

Acknowledgments

The author would like to thank Darren Creutz for numerous helpful conversations regarding this work, and for his collaboration on the articles [CP12, CP13]. The author is supported by NSF Grant DMS-1201565.

References

[Bek07] Bachir Bekka, *Operator-algebraic superridigity for* $\mathrm{SL}_n(\mathbb{Z})$*,* $n \geq 3$, Invent. Math. **169** (2007), no. 2, 401–425.

[BS06] Uri Bader and Yehuda Shalom, *Factor and normal subgroup theorems for lattices in products of groups*, Invent. Math. **163** (2006), no. 2, 415–454.

[CJ85] A. Connes and V. Jones, *Property T for von Neumann algebras*, Bull. London Math. Soc. **17** (1985), no. 1, 57–62.

[Con76] A. Connes, *Classification of injective factors. Cases* II_1, II_∞, III_λ, $\lambda \neq 1$, Ann. of Math. (2) **104** (1976), no. 1, 73–115.

[Con80] ______, *A factor of type* II_1 *with countable fundamental group*, J. Operator Theory **4** (1980), no. 1, 151–153.

[CP12] Darren Creutz and Jesse Peterson, *Stabilizers of ergodic actions of lattices and commensurators*, arXiv:1303.3949, 2012.

[CP13] ______, *Character rigidity for lattices and commensurators*, arXiv:1311.4513, 2013.

[Cre13] Darren Creutz, *Stabilizers of actions of lattices in products of groups*, arXiv:1305.3648, 2013.

[CS12] Darren Creutz and Yehuda Shalom, *A normal subgroup theorem for commensurators of lattices*, preprint, 2012, to appear in Groups, Geometry, and Dynamics.

[Dix81] Jacques Dixmier, *Von Neumann algebras*, North-Holland Mathematical Library, vol. 27, North-Holland Publishing Co., Amsterdam, 1981.

[DM12] Artem Dudko and Konstantin Medynets, *Finite factor representations of higman-thompson groups*, arXiv:1212.1230, 2012, to appear in Groups, Geometry, and Dynamics.

[Fur63] Harry Furstenberg, *A Poisson formula for semi-simple Lie groups*, Ann. of Math. (2) **77** (1963), 335–386.

[HM79] Roger E. Howe and Calvin C. Moore, *Asymptotic properties of unitary representations*, J. Funct. Anal. **32** (1979), no. 1, 72–96.

[Jon00] V. F. R. Jones, *Ten problems*, Mathematics: frontiers and perspectives, Amer. Math. Soc., Providence, RI, 2000, pp. 79–91.

[Kak51] Shizuo Kakutani, *Random ergodic theorems and Markoff processes with a stable distribution*, Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950 (Berkeley and Los Angeles), University of California Press, 1951, pp. 247–261.

[Kaž67] D. A. Každan, *On the connection of the dual space of a group with the structure of its closed subgroups*, Funkcional. Anal. i Priložen. **1** (1967), 71–74.

[Mar75] G. A. Margulis, *Non-uniform lattices in semisimple algebraic groups*, Lie groups and their representations (Proc. Summer School on Group Representations of the Bolyai János Math. Soc., Budapest, 1971), Halsted, New York, 1975, pp. 371–553.

[Mar78] ______, *Factor groups of discrete subgroups and measure theory*, Funktsional. Anal. i Prilozhen. **12** (1978), no. 4, 64–76.

[Mar79] ______, *Finiteness of quotient groups of discrete subgroups*, Funktsional. Anal. i Prilozhen. **13** (1979), no. 3, 28–39.

[Mar91] ______, *Discrete subgroups of semisimple Lie groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 17, Springer-Verlag, Berlin, 1991.

[PT13] Jesse Peterson and Andreas Thom, *Character rigidty for special linear groups*, arXiv:1303.4007, 2013, to appear in J. Reine Angew. Math.

[Rob93] A. Guyan Robertson, *Property* (T) *for* II_1 *factors and unitary representations of Kazhdan groups*, Math. Ann. **296** (1993), no. 3, 547–555.

[Sha00] Yehuda Shalom, *Rigidity of commensurators and irreducible lattices*, Invent. Math. **141** (2000), no. 1, 1–54.

[SvN50] I. E. Segal and John von Neumann, *A theorem on unitary representations of semisimple Lie groups*, Ann. of Math. (2) **52** (1950), 509–517.

[SZ94] Garrett Stuck and Robert J. Zimmer, *Stabilizers for ergodic actions of higher rank semisimple groups*, Ann. of Math. (2) **139** (1994), no. 3, 723–747.

[Tho64] Elmar Thoma, *Über positiv-definite Klassenfunctionen abzählbarer Gruppen*, Math. Z. **84** (1964), 389–402.

[Zim77] Robert J. Zimmer, *Hyperfinite factors and amenable ergodic actions*, Invent. Math. **41** (1977), no. 1, 23–31.

[Zim80] ———, *Strong rigidity for ergodic actions of semisimple Lie groups*, Ann. of Math. (2) **112** (1980), no. 3, 511–529.

[Zim84] ———, *Ergodic theory and semisimple groups*, Monographs in Mathematics, vol. 81, Birkhäuser Verlag, Basel, 1984.

Department of Mathematics, Vanderbilt University, 1326 Stevenson Center, Nashville, TN 37240, U.S.A.