arXiv:1303.3949v2 [math.DS] 24 Sep 2013

Stabilizers of Ergodic Actions of Lattices and Commensurators

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24 September 2013

Abstract

We prove that any ergodic measure-preserving action of an irreducible lattice in a semisimple group, with finite center and each simple factor having rank at least two, either has finite orbits or has finite stabilizers. The same dichotomy holds for many commensurators of such lattices.

The above are derived from more general results on groups with the Howe-Moore property and property (T). We prove similar results for commensurators in such groups and for irreducible lattices (and commensurators) in products of at least two such groups, at least one of which is totally disconnected.

1   Introduction

A groundbreaking result in the theory of lattices in semisimple groups is the Margulis Normal Subgroup Theorem [Mar79],[Mar91]: any nontrivial normal subgroup of an irreducible lattice in a center-free higher-rank semisimple group has finite index. In the case of real semisimple Lie groups, Stuck and Zimmer [SZ94] generalized this result to ergodic measure-preserving actions of such lattices: any irreducible ergodic measure-preserving action of a semisimple real Lie group, each simple factor having rank at least two, is either essentially free or essentially transitive; and any ergodic measure-preserving action of an irreducible lattice in such a semisimple real Lie group either has finite orbits or has finite stabilizers.

More recently, Bader and Shalom [BS06] proved a Normal Subgroup Theorem for irreducible lattices in products of locally compact groups: any infinite normal subgroup of an irreducible integrable lattice in a product of nondiscrete just noncompact locally compact second countable compactly generated groups, not both isomorphic to $\mathbb{R}$, has finite index.

Our purpose is to extend this dichotomy for ergodic measure-preserving actions to irreducible lattices and commensurators of lattices in semisimple groups, each factor having higher-rank, and more generally to lattices in products of at least two groups with the Howe-Moore property and property (T).

The Stuck-Zimmer result follows from an Intermediate Factor Theorem, a generalization of the Factor Theorem of Margulis, due to Zimmer [Zim82] and Nevo-Zimmer [NZ99b]. A

key step in the work of Bader-Shalom is a similar Intermediate Factor Theorem for product groups which they use to show: any irreducible ergodic measure-preserving action of a product of two locally compact second countable just noncompact groups with property (T) is either essentially free or essentially transitive. One of the main ingredients in our work is an Intermediate Factor Theorem for relatively contractive actions which we will discuss presently.

The methods of Bader-Shalom do not easily yield the same result for actions of irreducible lattices in products of groups. The major issue is that inducing an action of a lattice may yield an action of the ambient group which is not irreducible but the Bader-Shalom (and Stuck-Zimmer) Intermediate Factor Theorems only apply to irreducible actions. The same issue arises when attempting to apply the Stuck-Zimmer methods to lattices in semisimple groups with p-adic parts.

Our techniques are a generalization of the Normal Subgroup Theorem for Commensurators due to the first author and Shalom [CS12],[Cre11]: if Λ is a dense commensurator of a lattice in a locally compact second countable group that is not a compact extension of an abelian group such that Λ does not infinitely intersect any noncocompact normal subgroup then any infinite normal subgroup of Λ contains the lattice up to finite index; the commensurability classes of infinite normal subgroups of such a commensurator are in a one-one onto correspondence with the commensurability classes of open normal subgroups of the relative profinite completion.

The difficulty for lattices does not arise using our techniques as we do not need to induce the action of the lattice to the ambient group, but rather analyze the action directly by treating the lattice as a commensurator in a proper subproduct. For this, we require precisely the object that is the obstruction in the Stuck-Zimmer approach: a totally disconnected factor. In this sense, our methods complement those of Stuck and Zimmer and combining results we are able to handle all S-arithmetic lattices. Our methods also lead to results on actions of commensurators and allow us to prove the corresponding generalization of the Normal Subgroup Theorem of Bader-Shalom to actions of lattices (provided one group in the product is totally disconnected). For this generalization, we impose the requirement of the ambient groups having the Howe-Moore property as the measurable analogue of just noncompactness.

1.1   Main Results

We now state the main results of the paper. Recall that an action is weakly amenable when the corresponding equivalence relation is amenable.

Theorem (Theorem 5.1 and Corollary 8.1). *Let G be a noncompact nondiscrete locally compact second countable group with the Howe-Moore property. Let $\Gamma < G$ be a lattice and let $\Lambda < G$ be a countable dense subgroup such that Λ contains and commensurates Γ and such that Λ has finite intersection with every compact normal subgroup of G.*

Then any ergodic measure-preserving action of Λ either has finite stabilizers or the restriction of the action to Γ is weakly amenable.

If, in addition, G has property (T) then any ergodic measure-preserving action of Λ either has finite stabilizers or the restriction of the action to Γ has finite orbits.

The result on weak amenability applies to all noncompact simple Lie groups–even those without higher-rank–and also to automorphism groups of regular trees, both of which have the Howe-Moore property [HM79], [LM92]. One consequence of this is that any ergodic measure-preserving action of the commensurator on a nonatomic probability space, which is strongly ergodic when restricted to the lattice, has finite stabilizers.

The additional assumption that the ambient group have property (T) allows us to conclude that any weakly amenable action of the lattice, which will also have property (T), has, in fact, finite orbits. This also accounts for the requirement in the Stuck-Zimmer Theorem that each simple factor has higher-rank.

Treating lattices in products of groups, at least one of which is totally disconnected, as commensurators of lattices sitting in proper subproducts, we obtain the following generalization of the Bader-Shalom Theorem to actions:

Theorem (Theorem 9.1). *Let G be a product of at least two simple nondiscrete noncompact locally compact second countable groups with the Howe-Moore property, at least one of which has property (T), at least one of which is totally disconnected and such that every connected simple factor has property (T). Let $\Gamma < G$ be an irreducible lattice.*
Then any ergodic measure-preserving action of Γ either has finite orbits or has finite stabilizers.

Specializing to Lie groups:

Theorem (Corollary 10.5). *Let G be a semisimple Lie group (real or p-adic or both) with no compact factors, trivial center, at least one factor with rank at least two and such that each real simple factor has rank at least two. Let $\Gamma < G$ be an irreducible lattice. Then any ergodic measure-preserving action of Γ on a nonatomic probability space is essentially free.*

In particular, we obtain examples of groups without property (T) having uncountably many subgroups that admit only essentially free actions:

Theorem (Corollary 10.9). *Let $\mathbf{G}$ be a simple algebraic group defined over a global field K such that $\mathbf{G}$ has v-rank at least two for some place v and has v_∞-rank at least two for every infinite place v_∞. Then every nontrivial ergodic measure-preserving action of $\mathbf{G}(K)$ is essentially free.*

One consequence of this fact is that results from the theory of orbit equivalence, which often require that the actions in question be essentially free, apply to all actions of such groups. For example, since any nonamenable group cannot act freely and give rise to the hyperfinite II_1 equivalence relation [Dye59],[Zim84]:

Corollary. *Let $\mathbf{G}$ be a simple algebraic group defined over a global field K such that $\mathbf{G}$ has v-rank at least two for some place v and has v_∞-rank at least two for every infinite place v_∞. Then there is no nontrivial homomorphism of $\mathbf{G}(K)$ to the full group of the hyperfinite II_1 equivalence relation.*

The above result also holds if we replace the hyperfinite II_1 equivalence relation with any measure preserving equivalence relation which is treeable, or more generally which has the Haagerup property (see [Jol05] and [Pop06] Theorem 5.4). More generally, if $\mathbf{G}$ is a simple

algebraic group defined over $\mathbb{Q}$ and S is a set of primes as in the statement of Corollary 10.6 then any homomorphism of $\mathbf{G}(\mathbb{Z}_S)$ into the full group of the hyperfinite II_1 equivalence relation is precompact (when S is finite this also follows from Robertson [Rob93]). It seems plausible that the above result still holds if we replace the full group of the hyperfinite II_1 equivalence relation with the unitary group of the hyperfinite II_1 factor (see Bekka [Bek07] for results in this direction).

1.2   Stabilizers of Actions and Invariant Random Subgroups

The results described above can be suitably interpreted in terms of invariant random subgroups. Invariant random subgroups are conjugation-invariant probability measures on the space of closed subgroups and naturally arise from the stabilizer subgroups of measure-preserving actions. This notion was introduced in [AGV12] where it is shown that, conversely, every invariant random subgroup arises in this way ([AGV12] Proposition 13, see also Section 3 below). From this perspective the Stuck-Zimmer Theorem [SZ94] then states that semisimple real Lie groups, with each factor having higher-rank, and their irreducible lattices, admit no nonobvious invariant random subgroups and the Bader-Shalom result states the same for irreducible invariant random subgroups of products of nondiscrete locally compact second countable groups with property (T). Our results can likewise be interpreted in this context.

The study of stabilizers of actions dates back to the work of Moore, [AM66] Chapter 2, and Ramsay, [Ram71] Section 9 (see also Adams and Stuck [AS93] Section 4). Bergeron and Gaboriau [BG04] observed that invariant random subgroups behave similarly to normal subgroups and this topic has attracted much recent attention: [ABB+11], [AGV12], [Bow12], [GS12], [Gri11], [Ver11], [Ver12].

Our work, like that of Stuck and Zimmer, rules out the existence of nonobvious invariant random subgroups for certain groups. This stands in stark contrast to nonabelian free groups, which admit a large family of invariant random subgroups [Bow12]. Even simple groups can admit large families of nonfree actions: Vershik showed that the infinite alternating group admits many such actions [Ver12]. Another class of examples can be found by considering the commutator subgroup of the topological full group of Cantor minimal systems, which were shown to be simple by Matsui [Mat06] (and more recently to also be amenable by Juschenko and Monod [JM12]).

The main contribution to the theory we make here is introducing a technique, based on joinings, that allows us to formulate meaningful definitions of notions such as containment and commensuration for invariant random subgroups that extend the usual notions for subgroups:

Definition (Definition 3.7). Two invariant random subgroups are commensurate when there exists a joining such that almost surely the intersection has finite index in both.

The joinings technique may be of independent interest and should allow for more general definitions of properties of invariant random subgroups akin to those of subgroups. We use this definition to formulate a one-one correspondence:

Theorem (Theorem 10.1). *Let G be a semisimple Lie group (real or p-adic or both) with finite center where each simple factor has rank at least two. Let $\Gamma < G$ be an irreducible*

lattice and let $\Lambda < G$ be a countable dense subgroup such that Λ contains and commensurates Γ and such that Λ has finite intersection with every proper subfactor of G.

Then any ergodic measure-preserving action of Λ either has finite stabilizers or the restriction of the action to Γ has finite orbits.

Moreover, the commensurability classes of infinite ergodic invariant random subgroups of Λ are in one-one, onto correspondence with the commensurability classes of open ergodic invariant random subgroups of the relative profinite completion $\Lambda /\!/ \Gamma$.

1.3 Relatively Contractive Maps

Strongly approximately transitive (SAT) actions, introduced by Jaworski in [Jaw94], [Jaw95], are the extreme opposite of measure-preserving actions: an action of a group G on a probability space (X, ν), with ν quasi-invariant under the G-action, such that for any measurable set B of less than full measure there exists a sequence $g_n \in G$ which "contracts" B, that is $\nu(g_n B) \to 0$.

We introduce a relative version of this property, akin to relative measure-preserving, by saying that a G-equivariant map $\pi : (X, \nu) \to (Y, \eta)$ between G-spaces with quasi-invariant probability measures is relatively contractive when it is "contractive over each fiber" (see Section 4 for a precise definition). This is a generalization of the notion of proximal maps, which can also be thought of as "relatively boundary maps" in the context of stationary dynamical systems.

Following [FG10], we say a continuous action of a group G on a compact metric space X with quasi-invariant Borel probability measure $\nu \in P(X)$ is contractible when for every $x \in X$ there exists $g_n \in G$ such that $g_n \nu \to \delta_x$ in weak*. Furstenberg and Glasner [FG10] showed that an action is SAT if and only if every continuous compact model is contractible.

We generalize this to the relative case and obtain that a G-space is a relatively contractive extension of a point if and only if it is SAT. For this reason, we adopt the somewhat more descriptive term contractive to refer to such spaces.

Contractive spaces are the central dynamical concept in the proof of the amenability half of the Normal Subgroup Theorem for Commensurators [CS12], [Cre11] and have been studied in the context of stationary dynamical systems by Kaimanovich [Kai02]. Jaworski introduced the notion as a stronger form of the approximate transitivity property of Connes and Woods [CW85] to study the Choquet-Deny property on groups and showed that Poisson boundaries are contractive. The main benefit contractive spaces offer over boundaries is greater flexibility in that one need not impose a measure on the group.

We show that relatively contractive maps are essentially unique, which is crucial for the Intermediate Contractive Factor Theorem:

Theorem (Theorem 4.39). *Let $\Gamma < G$ be a lattice in a locally compact second countable group and let Λ contain and commensurate Γ and be dense in G.*

Let (X, ν) be a contractive stationary G-space and (Y, η) be a measure-preserving G-space. Let $\pi : (X \times Y, \nu \times \eta) \to (Y, \eta)$ be the natural projection map from the product space with the diagonal action.

Let (Z, ζ) be a Λ-space such that there exist Γ-maps $\varphi : (X \times Y, \nu \times \eta) \to (Z, \zeta)$ and $\rho : (Z, \zeta) \to (Y, \eta)$ with $\rho \circ \varphi = \pi$. Then φ and ρ are Λ-maps and (Z, ζ) is Λ-isomorphic to

a G-space and over this isomorphism the maps φ and ρ become G-maps.

We will actually need a stronger version of the Intermediate Factor Theorem (Theorem 4.41) which can be viewed as a "piecewise" or groupoid version in line with the virtual groups of Mackey [Mac66].

We also show that relatively contractive is orthogonal to relatively measure-preserving: any map which is both relatively contractive and relatively measure-preserving is necessarily an isomorphism. Extending this, we prove:

Theorem (Theorem 4.29 and Corollary 4.30). *Let (X, ν) be a contractive G-space and (Y, η) be a G-space. Then there is at most one joining such the projection to X is relatively measure-preserving.*

In particular, if (Y, η) is a measure-preserving G-space then any joining such that the projection to X is relatively measure-preserving is the independent joining.

Relatively contractive maps also allow us to answer a question of Shalom regarding the behavior of contractive actions restricted to lattices:

Theorem (Theorem 4.37). *Let G be a locally compact second countable group and $\Gamma < G$ a lattice. Let (X, ν) be a contractive stationary G-space. Then the restriction of the G-action to Γ makes (X, ν) a contractive Γ-space.*

1.4 Outline

We now outline the structure of the paper. In the next section, we recall basic facts and definitions about lattices and commensurators, group actions on measure spaces, the Howe-Moore property and semisimple groups and their lattices, and state precisely the theorems of Stuck and Zimmer, Bader and Shalom, and the first author and Shalom, mentioned above.

In Section 3, we discuss invariant random subgroups and explain the joinings technique used to define commensurability for invariant random subgroups.

We introduce the definition of relatively contractive maps in Section 4 and prove a series of facts about these maps. These results culminate in the proof of the Intermediate Contractive Factor Theorem and along the way recover most known results on contractive spaces.

Having then introduced the two main ingredients, in Section 5 we prove that the restriction of the action of a commensurator to a lattice is weakly amenable, provided the ambient satisfies appropriate hypotheses. This is a key step that all our further results follow from. In Section 6, we rephrase the results of Section 5 in terms of invariant random subgroups in order to prove the one-one correspondence.

In Section 7, we explain how the Howe-Moore property ensures that the ambient group satisfies the necessary conditions on its actions to apply the results from Sections 5 and 6. In Sections 8 and 9, we derive the consequences for lattices and commensurators in Howe-Moore groups with property (T). We conclude with Section 10, specializing to the case of semisimple groups.

1.5   Acknowledgments

2   Preliminaries

2.1   Commensuration

2.1.1   Commensurators of Lattices

Definition 2.1. Let G be a locally compact second countable group. A subgroup $\Gamma < G$ is a **lattice** when it is discrete and there exists a fundamental domain for G/Γ with finite Haar measure. A lattice is **irreducible** when the projection modulo any noncocompact closed normal subgroup is dense.

Definition 2.2. Let A and B be subgroups of a group G. Then A and B are **commensurate** when $A \cap B$ has finite index in both A and B.

Definition 2.3. Let G be a group. A subgroup $\Lambda < G$ **commensurates** another subgroup $\Gamma < G$ when for every $\lambda \in \Lambda$ the group $\Gamma \cap \lambda\Gamma\lambda^{-1}$ has finite index in both Γ and $\lambda\Gamma\lambda^{-1}$. When $\Gamma < \Lambda$ is a subgroup of Λ we will write

$$\Gamma <_c \Lambda$$

to mean that Γ is a commensurated subgroup of Λ.

Definition 2.4. Let $\Gamma < G$ be a lattice in a locally compact second countable group. Then

$$\mathrm{Comm}_G(\Gamma) = \{g \in G : [\Gamma : \Gamma \cap g\Gamma g^{-1}] < \infty \text{ and } [g\Gamma g^{-1} : \Gamma \cap g\Gamma g^{-1}] < \infty\}$$

is the **commensurator** of Γ in G.

2.1.2   The Relative Profinite Completion

We recall the construction and some basic facts about the relative profinite completion of commensurated subgroups. This construction has appeared in the study of commensurated subgroups [Sch80], [Tza00], [Tza03], [CM09] and the reader is referred to [SW09] and [Cre11] for more information and proofs of the following basic facts.

Definition 2.5. Let $\Gamma <_c \Lambda$ be countable groups with Λ commensurating a subgroup Γ. Define the map $\tau : \Lambda \to \mathrm{Symm}(\Lambda/\Gamma)$ to be the natural mapping of Λ to the symmetry group of the coset space. Endow $\mathrm{Symm}(\Lambda/\Gamma)$ with the topology of pointwise convergence. Then $\mathrm{Symm}(\Lambda/\Gamma)$ is a Polish group since Λ is countable but in general is not locally compact.

The **relative profinite completion** of Λ over Γ is

$$\Lambda /\!\!/ \Gamma := \overline{\tau(\Lambda)}.$$

Theorem 2.6. *Let $\Gamma <_c \Lambda$. The relative profinite completion $\Lambda /\!\!/ \Gamma$ is a totally disconnected locally compact group and $\overline{\tau(\Gamma)}$ is a compact open subgroup.*

Proposition 2.1.1. *Let $\Gamma <_c \Lambda$. Then $\Lambda /\!\!/ \Gamma$ is compact if and only if $[\Lambda : \Gamma] < \infty$. In particular, $\Lambda /\!\!/ \Gamma$ is finite if and only if it is compact.*

Proposition 2.1.2. *Let $\Gamma <_c \Lambda$. Then $\Lambda /\!\!/ \Gamma$ is discrete if and only if there exists $\Gamma_0 < \Gamma$ such that $[\Gamma : \Gamma_0] < \infty$ and $\Gamma_0 \lhd \Lambda$.*

Proposition 2.1.3. *Let $\Gamma <_c \Lambda$. Then $\tau(\Lambda) \cap \overline{\tau(\Gamma)} = \tau(\Gamma)$ and $\tau^{-1}\big(\overline{\tau(\Gamma)}\big) = \Gamma$.*

Proposition 2.1.4. *Let H be a totally disconnected locally compact group and K be a compact open subgroup of H. Define $\tau_{H,K} : H \to \mathrm{Symm}(H/K)$ as before (K is necessarily commensurated by H). Then $\tau_{H,K}$ is a continuous open map with closed range.*

Moreover $H /\!\!/ K$ is isomorphic to $H/\ker(\tau_{H,K})$ and in fact $\ker(\tau_{H,K})$ is the largest normal subgroup of H that is contained in K.

Proposition 2.1.5. *Let $B < A$ be any countable groups. Let H be a locally compact totally disconnected group and $K < H$ a compact open subgroup. Let $\varphi : A \to H$ be a homomorphism such that $\varphi(A)$ is dense in H and $\varphi^{-1}(K) = B$.*

Then $B <_c A$ and $B /\!\!/ A$ is isomorphic to $H /\!\!/ K$.

2.2   Group Actions on Measure Spaces

Throughout the paper, we will always assume groups are locally compact second countable and that measure spaces are standard probability spaces (except when otherwise stated).

2.2.1   Free and Transitive Actions

Definition 2.7. A group G **acts on** a space X when there is a map $G \times X \to X$, written gx, such that $g(hx) = (gh)x$. For $\nu \in P(X)$ a Borel probability measure on X, we say that ν is **quasi-invariant** when the G-action preserves the measure class and **invariant** or **measure-preserving** when G preserves ν.

We write $G \curvearrowright (X, \nu)$ and refer to (X, ν) as a G-**space** when G acts on X and $\nu \in P(X)$ is quasi-invariant and the action map $G \times X \to X$ is Haar $\times \nu$-measurable.

Definition 2.8. Let $G \curvearrowright (X, \nu)$. The **stabilizer** subgroups are written

$$\mathrm{stab}_G(x) = \{g \in G : gx = x\}$$

and when $\Gamma < G$ is a subgroup we write

$$\mathrm{stab}_\Gamma(x) = \{\gamma \in \Gamma : \gamma x = x\} = \mathrm{stab}_G(x) \cap \Gamma$$

for the stabilizer of x when the action is restricted to Γ.

Definition 2.9. $G \curvearrowright (X, \nu)$ is **essentially transitive** when for some $x \in X$ the orbit Gx is conull in X.

Note that an essentially transitive ergodic action of a countable group means the space is atomic (hence, in the case of measure-preserving actions, finite).

Definition 2.10. $G \curvearrowright (X, \nu)$ is **essentially free** when for almost every $x \in X$ the stabilizer group $\mathrm{stab}_G(x)$ is trivial.

Definition 2.11. $G \curvearrowright (X, \nu)$ has **finite stabilizers** when for almost every x the stabilizer subgroup $\mathrm{stab}_G(x)$ is finite.

Definition 2.12. $G \curvearrowright (X, \nu)$ has **finite orbits** when for almost every x the orbit Gx is finite.

Definition 2.13. $G \curvearrowright (X, \nu)$ is **ergodic** when every G-invariant measurable set is either null or conull.

Definition 2.14. $G \curvearrowright (X, \nu)$ is **irreducibly ergodic** or simply **irreducible** when every nontrivial normal subgroup of G acts ergodically on (X, ν).

2.2.2   Factor Maps

Definition 2.15. Let G be a locally compact second countable group and $\pi : (X, \nu) \to (Y, \eta)$ a measurable map such that $\pi_* \nu = \eta$ and $\pi(gx) = g\pi(x)$ for all $g \in G$ and almost every $x \in X$. Such a map π is a G-**map** of G-spaces.

Definition 2.16. Given a measurable map $\pi : X \to Y$ the **push-forward map** $\pi_* : P(X) \to P(Y)$, mentioned above, is defined by $(\pi_* \nu)(B) = \nu(\pi^{-1}(B))$ for $B \subseteq Y$ measurable.

In order to relativize properties of G-spaces to G-maps, we will need to focus on the behavior of the disintegration measures over a G-map. Recall that:

Definition 2.17. Let $\pi : (X, \nu) \to (Y, \eta)$ be a G-map of G-spaces. Then there exist almost surely unique measures $D_\pi(y)$, called the **disintegration measures**, such that $D_\pi(y)$ is supported on $\pi^{-1}(y)$ and $\int D_\pi(y) \, d\eta(y) = \nu$.

Of course, the disintegration measures correspond to the conditional expectation at the level of the function algebras: if $\pi : (X, \nu) \to (Y, \eta)$ then the algebra $\{f \circ \pi : f \in L^\infty(Y, \eta)\}$ is a subalgebra of $L^\infty(X, \nu)$ and the disintegration measures define the conditional expectation to this subalgebra.

2.2.3   Joinings

We will make use of the concept of joinings of G-spaces both to study relatively contractive maps and to work with invariant random subgroups. The reader is referred to [Gla03] for more information.

Definition 2.18. Let (X, ν) and (Y, η) be G-spaces. Let $\alpha \in P(X \times Y)$ such that $(\mathrm{pr}_X)_* \alpha = \nu$, $(\mathrm{pr}_Y)_* \alpha = \eta$ and α is quasi-invariant under the diagonal G-action. The space $(X \times Y, \alpha)$ with the diagonal G-action is called a **joining** of (X, ν) and (Y, η).

Definition 2.19. A joining α of G-spaces is G-**invariant** when α is G-measure-preserving under the diagonal action.

Definition 2.20. Let (X, ν) and (Y, η) be G-spaces. The space $(X \times Y, \nu \times \eta)$ with the diagonal G-action is the **independent joining** of (X, ν) and (Y, η).

Proposition 2.2.1. *Let $\alpha \in P(X \times Y)$ be a joining of the G-spaces (X, ν) and (Y, η). Consider the projection $p : X \times Y \to Y$. The disintegration of α over η via p is of the form $D_p(y) = \alpha_y \times \delta_y$ for some $\alpha_y \in P(X)$ almost surely.*

Definition 2.21 ([Gla03] Definition 6.9)**.** Let (X, ν), (Y, η) and (Z, ζ) be G-spaces and let α be a joining of (X, ν) and (Y, η) and β be a joining of (Y, ν) and (Z, ζ). Let $\alpha_y \in P(X)$ and $\beta_y \in P(Z)$ be the projections of the disintegrations of α and β over η. The measure $\rho \in P(X \times Z)$ by

$$\rho = \int_Y \alpha_y \times \beta_y \ d\eta(y)$$

is the **composition** of α and β.

Proposition 2.2.2 ([Gla03] Proposition 6.10)**.** *The composition of two joinings is a joining. If two joinings are G-invariant then so is their composition.*

2.2.4   Induced Actions

We recall now the construction of the induced action from a lattice to the ambient group, see, e.g., [Zim84].

Let $\Gamma < G$ be a lattice in a locally compact second countable group and let (X, ν) be a Γ-space. Take a fundamental domain F for G/Γ such that $e \in F$. Let $m \in P(F)$ be the Haar measure of G restricted to F and normalized to be a probability measure on F. Define the cocycle $\alpha : G \times F \to \Gamma$ by $\alpha(g, f) = \gamma$ such that $gf\gamma \in F$ and observe that such a γ is unique so this is well-defined. Note that $\alpha(gh, f) = \alpha(h, f)\alpha(g, hf\alpha(h, f))$ meaning α is indeed a cocycle. We also remark that $\alpha(f, e) = e$ for $f \in F$ and that $\alpha(\gamma, e) = \gamma^{-1}$ for $\gamma \in \Gamma$. Consider now the action of G on $F \times X$ given by

$$g \cdot (f, x) = (gf\alpha(g, f), \alpha(g, f)^{-1}x)$$

and observe that the measure $m \times \nu$ is quasi-invariant under this action. So $(F \times X, m \times \nu)$ is a G-space.

Also consider the Γ-action on $(G \times X, \mathrm{Haar} \times \nu)$ given by

$$\gamma \cdot (g, x) = (g\gamma^{-1}, \gamma x)$$

and observe that this is quasi-invariant as well. Since the Γ-action on G/Γ is proper the space of Γ-orbits of $G \times X$ under that action is well-defined and we denote it by $G \times_\Gamma X$ and write elements as $[g, x]$. Define a G-action on $G \times_\Gamma X$ by $h \cdot [g, x] = [hg, x]$.

Define the map $\tau : F \times X \to G \times_\Gamma X$ by $\tau(f, x) = [f, x]$ and the map $\rho : G \times_\Gamma X \to F \times X$ by $\rho([g, x]) = (g\alpha(g, e), \alpha(g, e)^{-1}x)$. Observe that ρ is well-defined since $\alpha(g\gamma, e) = \gamma^{-1}\alpha(g, e)$.

Clearly, $\tau(\rho([g,x])) = [g,x]$ and $\rho(\tau(f,x)) = (f,x)$ so these maps invert one another. Moreover,

$$\tau(g \cdot (f,x)) = [gf\alpha(g,f), \alpha(g,f)^{-1}x] = [gf,x] = g \cdot [f,x] = g \cdot \tau(f,x)$$

and similarly, $\rho(h \cdot [g,x]) = h \cdot \rho([g,x])$ so τ and ρ are inverse G-isomorphisms of $(F \times X, m \times \nu)$ and $(G \times_\Gamma X, \alpha)$ where $\alpha = \tau_*(m \times \nu)$.

These isomorphisms show that the construction defined is independent of the fundamental domain chosen and we define the **induced action to G of $\Gamma \curvearrowright (X, \nu)$** to be the G-space $(G \times_\Gamma X, \alpha)$.

2.2.5 Induced Maps

Let $\Gamma < G$ be a lattice in a locally compact second countable group. Let $\pi : (X, \nu) \to (Y, \eta)$ be a Γ-map of Γ-spaces. Fix a fundamental domain F for G/Γ and $m \in P(F)$ as above. Define the map $\Phi : (F \times X, m \times \nu) \to (F \times Y, m \times \eta)$ by $\Phi(f, x) = (f, \pi(x))$. Then

$$\Phi(g \cdot (f,x)) = (gf\alpha(g,f), \pi(\alpha(g,f)^{-1}x)) = (gf\alpha(g,f), \alpha(g,f)^{-1}\pi(x)) = g \cdot \Phi(f,x)$$

so Φ is a G-map of G-spaces. Let $\Pi : (G \times_\Gamma X, \alpha) \to (G \times_\Gamma Y, \beta)$ be the image of Φ over the canonical isomorphisms defined above for the induced actions.

The G-map Π between the induced G-spaces is referred to as the **induced G-map from the Γ-map π**.

2.2.6 Continuous Compact Models

We will need a basic fact about the existence of compact models. This result does not appear explicitly in the literature but the proof is essentially contained in [Zim84].

Definition 2.22. Let (X, ν) be a (measurable) G-space. A compact metric space X_0 and fully supported Borel probability measure $\nu_0 \in P(X_0)$ is a **continuous compact model** of (X, ν) when G acts continuously on X_0 and there exists a G-equivariant measure space isomorphism $(X, \nu) \to (X_0, \nu_0)$.

Definition 2.23. Let $\pi : (X, \nu) \to (Y, \eta)$ be a measurable G-map of (measurable) G-spaces. Let X_0 and Y_0 be compact metric spaces on which G acts continuously and let $\pi_0 : X_0 \to Y_0$ be a continuous G-equivariant map. Let $\nu_0 \in P(X_0)$ and $\eta_0 \in P(Y_0)$ be fully supported Borel probability measures such that $(\pi_0)_*\nu_0 = \eta_0$. The map and spaces $\pi_0 : (X_0, \nu_0) \to (Y_0, \eta_0)$ is a **continuous compact model** for the G-map π and G-spaces (X, ν) and (Y, η) when there exist G-equivariant measure space isomorphisms $\Phi : (X, \nu) \to (X_0, \nu_0)$ and $\Psi : (Y, \eta) \to (Y_0, \eta_0)$ such that the resulting diagram commutes: $\pi = \Psi^{-1} \circ \pi_0 \circ \Phi$.

Lemma 2.2.3 (Varadarjan [Var63]). *Let G be a locally compact second countable group and $\pi : (X, \nu) \to (Y, \eta)$ a G-map of G-spaces. Then there exists a continuous compact model for π.*

Proof. Let $\mathcal{X}$ be a countable collection of functions in $L^\infty(X, \nu)$ that separates points and let $\mathcal{Y}$ be a countable collection in $L^\infty(Y, \eta)$ that separates points. Let $\mathcal{F} = \mathcal{X} \cup \{f \circ \pi : f \in \mathcal{Y}\}$.

Let B be the unit ball in $L^\infty(G, \text{Haar})$ which is a compact metric space in the weak* topology (as the dual of L^1).

Define $X_{00} = \prod_{f \in \mathcal{F}} B$ and $Y_{00} = \prod_{f \in \mathcal{Y}} B$, both of which are compact metric spaces using the product topology. Define $\pi_{00} : X_{00} \to Y_{00}$ to be the restriction map: for $f \in \mathcal{Y}$ take the fth coordinate of $\pi_{00}(x_{00})$ to be the $(f \circ \pi)^{th}$ coordinate of x_{00}. Then π_{00} is continuous.

Define the map $\Phi : X \to X_{00}$ by $\Phi(x) = (\varphi_f(x))_{f \in \mathcal{F}}$ where $(\varphi_f(x))(g) = f(gx)$. Then Φ is an injective map (since $\mathcal{F}$ separates points). Observe that $(\varphi_f(hx))(g) = f(ghx) = (\varphi_f(x))(gh)$. Consider the G-action on X_{00} given by the right action on each coordinate. Then G acts on X_{00} continuously (and likewise on Y_{00} continuously) and Φ is G-equivariant. Similarly, define $\Psi : Y \to Y_{00}$ by $\Psi(y) = (\psi_f(y))_{f \in \mathcal{Y}}$ where $(\psi_f(y))(g) = f(gy)$.

Let $X_0 = \overline{\Phi(X)}$, let $\nu_0 = \Phi_*\nu$, let $Y_0 = \overline{\Psi(Y)}$, let $\eta_0 = \Psi_*\eta$ and let π_0 be the restriction of π_{00} to X_0. Then $\Phi : (X, \nu) \to (X_0, \nu_0)$ and $\Psi : (Y, \eta) \to (Y_0, \eta_0)$ are G-isomorphisms. Since $(\psi_f(\pi(x)))(g) = f(g\pi(x)) = f \circ \pi(gx) = (\varphi_{f \circ \pi}(x))(g)$, $\pi_0(X_0) = Y_0$ and $\Psi^{-1} \circ \pi_0 \circ \Phi = \pi$. $\qquad\square$

2.2.7 Stationary Spaces

Definition 2.24. Let G be a locally compact second countable group and $\mu \in P(G)$ a Borel probability measure on G such that the support of μ generates G. A G-space (X, ν) is μ-**stationary** when $\mu * \nu = \nu$. A μ-stationary G-space is referred to as a (G, μ)-**space**.

The main fact about stationary spaces we need is the random ergodic theorem, due to Kakutani [Kak51] in the measure-preserving case and to Kifer [Kif86] in the general case.

Theorem 2.25 (The Random Ergodic Theorem). *Let G be a locally compact second countable group, $\mu \in P(G)$ a probability measure on G such that the support of μ generates G and (X, ν) an ergodic (G, μ)-space. Then for $f \in L^1(X, \nu)$ and $\mu^{\mathbb{N}}$-almost every $(\omega_1, \omega_2, \omega_3, \ldots) \in G^{\mathbb{N}}$,*

$$\frac{1}{N} \sum_{n=1}^{N} f(\omega_n \omega_{n-1} \cdots \omega_2 \omega_1 x) \to \int f \, d\nu$$

where the convergence is both ν-almost everywhere and in $L^1(X, \nu)$.

2.2.8 Poisson Boundaries

Poisson boundaries were introduced by Furstenberg [Fur63] and have several important properties, some of which we mention in the following sections. The Poisson boundary for G and μ is a (G, μ)-space. We include the definition for completeness but will not need to study the structure of such spaces. The reader is referred to [BS06] and [Cre11] for more information.

Definition 2.26. Let G be a locally compact second countable group and $\mu \in P(G)$ a Borel probability measure on G such that the support of μ generates G. Let $T : G^{\mathbb{N}} \to G^{\mathbb{N}}$ be defined by $T(w_1, w_2, w_3, \ldots) = (w_1 w_2, w_3, \ldots)$. The space of T-ergodic components of $(G^{\mathbb{N}}, \mu^{\mathbb{N}})$ is the Poisson boundary for G and μ.

2.3 Equivalence Relations

Definition 2.27. Let $G \curvearrowright (X, \nu)$ be a measure-preserving action. The **equivalence relation** generated by the action is given by R where xRy when there exists $g \in G$ such that $gx = y$.

Definition 2.28. The equivalence relation $R_{G \curvearrowright (X,\nu)}$ is **amenable** when there exists a sequence of nonnegative function $\rho_n : R \to \mathbb{R}$ such that

- $\rho_{n,x} \in \ell^1([x])$ for almost every x where $[x]$ is the equivalence class of x and $\rho_{n,x}(y) = \rho_n(x, y)$ is defined on $y \in [x]$

- $\|\rho_{n,x}\|_{\ell^1} = 1$

- there exists a Borel R-invariant conull set X_0 such that $\|\rho_{n,x} - \rho_{n,y}\|_{\ell^1} \to 0$ as $n \to \infty$ for all xRy with $x, y \in X_0$

Lemma 2.3.1. *Let $G \curvearrowright (X, \nu)$ be a measure-preserving action such that the associated equivalence relation is amenable. Then the equivalence relation associated to almost every G-ergodic component of (X, ν) is amenable.*

Proof. The definition of amenable equivalence relation is formulated solely in terms of the behavior on orbits which of course each lie in a single ergodic component. $\square$

2.4 Weak Amenability of Actions

Definition 2.29 (Zimmer)**.** Let $G \curvearrowright (X, \nu)$ be a measure-preserving action. Let E be a separable Banach space and write E_1^* for the unit ball in the dual of E. Let $\alpha : G \times X \to \mathrm{Iso}(E)$ be a cocycle. Denote the dual cocycle α^* by $\alpha^*(g, x) = (\alpha(g, x)^{-1})^*$. Let $A_x \subseteq E_1^*$ be a closed convex nonempty set for almost every x such that $\alpha^*(g, x)A_{gx} = A_x$. Consider the space

$$F(X, \{A_x\}) = \{\varphi : X \to E_1^* \mid \varphi(x) \in A_x \text{ for a.e. } x\}.$$

This is a closed convex compact subset of $L^\infty(X, E_1^*)$ which is G-invariant under the α^*-twisted action. Such a space $A = F(X, \{A_x\})$ is called an **affine G-space over** (X, ν).

Definition 2.30. The cocycle α is called **orbital** when $\alpha(g, x) = e$ for all $g \in \mathrm{stab}_G(x)$ for almost every x. The affine G-space A is called an **orbital affine G-space** when α is orbital.

Definition 2.31. $G \curvearrowright (X, \nu)$ is **amenable** when for every affine G-space A over (X, ν) there exists an α^*-invariant function $f : X \to E_1^*$ such that $f(x) \in A_x$ for almost every x: α^*-invariant means $f(gx) = \alpha^*(g, x)f(x)$. $G \curvearrowright (X, \nu)$ is **weakly amenable** when that condition holds for all orbital affine G-spaces over (X, ν).

Theorem 2.32 (Zimmer [Zim77])**.** *The equivalence relation generated by the action $G \curvearrowright (X, \nu)$ is amenable if and only if the action is weakly amenable.*

Theorem 2.33 (Connes-Feldman-Weiss [CFW81], Stuck-Zimmer [SZ94])**.** *Let G be a group with property (T) and $G \curvearrowright (X, \nu)$ be an ergodic measure-preserving action. Then $G \curvearrowright (X, \nu)$ is weakly amenable if and only if $G \curvearrowright (X, \nu)$ is essentially transitive.*

Proposition 2.4.1. *Let Γ be a discrete countable group with property (T) and $\Gamma \curvearrowright (X, \nu)$ a measure-preserving action (not necessarily ergodic) that is weakly amenable. Then almost every Γ-orbit is finite.*

Proof. Since the action is weakly amenable, the associated equivalence relation is weakly amenable by Theorem 2.32. By Lemma 2.3.1 then the equivalence relation on almost every ergodic component is amenable. Therefore the action on almost every ergodic component is weakly amenable by Theorem 2.32 and so by Theorem 2.33 the action on almost every ergodic component is essentially transitive. Since Γ is discrete, for an ergodic component (Y, η) with $y \in Y$ such that $\eta(\Gamma y) = 1$ it follows that $\eta(\gamma y) > 0$ for some $\gamma \in \Gamma$ and then by the invariance of η that $\eta(\gamma y)$ is constant and nonzero over Γ hence Γy must be a finite set. As this holds for almost every ergodic component then almost every Γ-orbit in X must be finite (though the size of each orbit can vary over the ergodic components). $\square$

We will make use of the following standard facts due to Zimmer [Zim84]:

Proposition 2.4.2 (Zimmer [Zim84])**.** *Let G be a locally compact second countable group. Then G acts amenably on any of its Poisson Boundaries.*

Proposition 2.4.3 (Zimmer [Zim84])**.** *Let G act amenably on (X, ν) and $H < G$ be any closed subgroup. Then H acts amenably on (X, ν).*

Proposition 2.4.4 (Zimmer [Zim84])**.** *Let G act amenably on (X, ν) and let (Y, η) be any G-space. Then G acts amenably on $(X \times Y, \nu \times \eta)$ (with the diagonal action).*

Proposition 2.4.5 (Stuck-Zimmer [SZ94])**.** *Let G act amenably on (Y, η) and A be an affine G-space over a G-space (X, ν). Then there exist measurable G-maps $Y \times X \to A \to X$ composing to the natural projection to X.*

We include a proof since the above statement does not appear explicitly in the literature (though it is proved implicitly in [SZ94]).

Proof. Let $p : A \to X$ be the G-map given by the projection to X: for $(x, f) \in X \times_{\alpha^*} E_1^*$ let $p(x, f) = x$ (here α is the cocycle defining the action on A). Define the cocycle $\beta : G \times (Y \times X) \to \text{Iso}(E)$ by $\beta(g, y, x) = \alpha(g, x)$ and set $C_{y,x} = A_x$. Then $\beta^*(g, y, x)C_{gy,gx} = \alpha^*(g, x)A_{gx} = A_x = C_{y,x}$ making $F(Y \times X, \{C_{y,x}\})$ an affine G-space over $(Y \times X, \eta \times \nu)$.

By Proposition 2.4.4, G acts amenably on $(Y \times X, \eta \times \nu)$ hence there exists a fixed point $\pi : Y \times X \to E_1^*$ such that $\pi(y, x) \in C_{y,x} = A_x$ for almost every y, x and such that $\beta^*(g, y, x)\pi(gy, gx) = \pi(y, x)$ for all $g \in G$ and almost every y, x. Note that $p \circ \pi : Y \times X \to X$ is the natural projection and therefore we have constructed the necessary maps. $\square$

2.5   The Howe-Moore Property

The Howe-Moore property was first proven for simple Lie groups by Howe and Moore [HM79]. The reader is referred to [Zim84].

Definition 2.34. A group G has the **Howe-Moore property** when for every irreducible unitary representation of G on a Hilbert space the matrix coefficients vanish at infinity.

The following is Theorem 3.6 in [Sch84]:

Theorem 2.35 (Schmidt [Sch84])**.** *A group has the Howe-Moore property if and only if every ergodic measure-preserving action is mixing.*

A consequence of Rothman's work on connected Howe-Moore groups and minimally almost periodic groups is:

Theorem 2.36 (Rothman [Rot80])**.** *A simple connected group with the Howe-Moore property is a Lie group.*

We also make use of the following well-known facts:

Proposition 2.5.1. *Any proper closed normal subgroup of a Howe-Moore group is compact.*

Proposition 2.5.2. *Any proper open subgroup of a Howe-Moore group is compact.*

2.6   Properties of Lie and Algebraic Groups

We note the following well-known facts [HM79], [Kaz67], [Zim84], [Mar91] and [Ben04].

Theorem 2.37. *Simple real and p-adic Lie groups have the Howe-Moore property.*

Theorem 2.38. *Higher-rank simple real and p-adic Lie groups have property (T).*

Theorem 2.39. *Irreducible lattices in higher-rank semisimple real and p-adic Lie groups (and combinations thereof) have dense commensurators.*

Definition 2.40. Let $\mathbf{G}$ be a semisimple algebraic group over $\mathbb{Q}$. Let S be a finite set of primes. Denote by $\mathbb{Z}_S$ the S-integers: $\mathbb{Z}$ adjoin $^1/_p$ for each $p \in S$. Then $\mathbf{G}(\mathbb{Z}_S)$ is a lattice in $\prod_{p \in S \cup \{\infty\}} \mathbf{G}(\mathbb{Q}_p)$ where $\mathbb{Q}_\infty = \mathbb{R}$ (theorems of Borel and Harish-Chandra and of Borel).

Let G be a group such that $\phi : \prod_{p \in S \cup \{\infty\}} \mathbf{G}(\mathbb{Q}_p) \to G$ is an onto homomorphism with compact kernel. Let $\Gamma < G$ such that $\Gamma \cap \phi(\mathbf{G}(\mathbb{Z}_S))$ has finite index in both Γ and $\phi(\mathbf{G}(\mathbb{Z}_S))$. Then Γ is a lattice in G and is called S-**arithmetic**.

Theorem 2.41 (Margulis [Mar91])**.** *Let G be a higher-rank semisimple Lie group (real and p-adic factors allowed) with no compact factors and $\Gamma < G$ an irreducible lattice. Then Γ is S-arithmetic.*

Theorem 2.42 (Margulis [Mar91])**.** *Let K be a global field, let V be the set of places (inequivalent valuations) and let V_∞ the infinite places (archimedean valuations in the case of a number field). Write K_v for the completion of k over a valuation $v \in V$. Let $\mathbf{G}$ be a simple algebraic group defined over K. Then $\mathbf{G}(K_v)$ has the Howe-Moore property for every $v \in V$ where $\mathbf{G}(K_v)$ is noncompact. If $\mathbf{G}$ has v-rank at least two then $\mathbf{G}(K_v)$ has property (T) for every $v \in V$.*

2.7   The Stuck-Zimmer Theorem

Theorem 2.43 (Stuck-Zimmer [SZ94])**.** *Let G be a connected semisimple Lie group with finite center such that each simple factor of G has $\mathbb{R}$-rank at least two. Then any faithful, irreducible, finite measure-preserving action of G is either essentially free or essentially transitive.*

Theorem 2.44 (Stuck-Zimmer [SZ94])**.** *Let G be a connected semisimple Lie group with finite center such that each simple factor of G has $\mathbb{R}$-rank at least two. Let $\Gamma < G$ be an irreducible lattice in G. Then any finite measure-preserving action of Γ either has finite orbits or has finite stabilizers.*

We remark that this is a generalization of Margulis' theorem since one can always construct Bernoulli shift actions with the normal subgroup being the stabilizer almost surely.

2.8   The Normal Subgroup Theorem for Commensurators of Lattices

Theorem 2.45 (Creutz-Shalom [CS12])**.** *Let G be a compactly generated locally compact group that is not a compact extension of an abelian group. Let Γ be an integrable lattice in G and let $\Lambda < G$ be dense in G such that $\Gamma <_c \Lambda$. Assume that for every closed normal subgroup M of G that is not cocompact that $M \cap \Lambda$ is finite. Then for any infinite normal subgroup $N \lhd \Lambda$ it holds that $[\Gamma : \Gamma \cap N] < \infty$.*

Moreover, there is a one-one, onto correspondence between commensurability classes of infinite normal subgroups of Λ and commensurability classes of open normal subgroups of the relative profinite completion $\Lambda /\!\!/ \Gamma$.

We remark that the requirement that Γ be integrable can be dropped if the ambient group G is known to have property (T) since integrability is used only in the proof to handle the case when Γ does not have property (T) (see [CS12]). Likewise, the compact generation assumption can be dropped since property (T) implies it.

3   Invariant Random Subgroups

Invariant random subgroups are the natural context for the presentation of some of our results. We recall here the definition and a basic construction, introduce a definition of commensurability for invariant random subgroups and explain how the technique employed can be used to define relationships between invariant random subgroups.

Definition 3.1. Let G be a group. The space of closed subgroups $S(G)$ is a compact topological space (with the Chabauty topology) and G acts on it by conjugation. An **invariant random subgroup** of G is a probability measure $\nu \in P(S(G))$ that is invariant under the conjugation action.

3.1   Measure-Preserving Actions

Let G be a group and $G \curvearrowright (X, \nu)$ be a measure-preserving action. Then the mapping $x \mapsto \mathrm{stab}_G(x)$ sending each point to its stabilizer subgroup defines a Borel map $X \to S(G)$

([AM66] Chapter 2, Proposition 2.3). Let η be the pushforward of ν under this map. Observe that $\mathrm{stab}_G(gx) = g\,\mathrm{stab}_G(x)\,g^{-1}$ so the mapping is a G-map and therefore η is an invariant measure on $S(G)$. Hence $G \curvearrowright (X, \nu)$ gives rise in a canonical way to an invariant random subgroup of G defined by the stabilizer subgroups.

3.2 Invariant Random Subgroups Always Arise From Actions

In fact the converse of this is also true: any invariant random subgroup can be realized as the stabilizer subgroups of some measure-preserving action:

Theorem 3.2 (Abert-Glasner-Virág [AGV12]). *Let $\eta \in P(S(G))$ be an invariant random subgroup of a countable group G. Then there exists a measure-preserving G-space (X, ν) such that η is the invariant random subgroup that arises from the stabilizers of the action.*

In our setting, we consider invariant random subgroups of nondiscrete locally compact groups and so we need to generalize the result of Abert, Glasner and Virág to the locally compact case. We make use of the Gaussian action construction: for a separable Hilbert space H one can associate a probability space (Y_H, ν_H) and an embedding $\rho : H \to L^2(Y_H, \nu_H)$ such that for any orthogonal $T : H \to K$ between Hilbert spaces there is a measure-preserving map $V_T : (Y_H, \nu_H) \to (Y_K, \nu_K)$ such that $\rho(T(\xi)) = \rho(\xi) \circ V_T^{-1}$ and that for $T : H \to K$ and $S : K \to L$, $V_S \circ V_T = V_{S \circ T}$ almost everywhere for each fixed pair S, T. The reader is referred to Schmidt [Sch96] for details.

Theorem 3.3. *Let G be a locally compact second countable group. Given an invariant random subgroup $(S(G), \eta)$ there exists a measure-preserving G-space (X, ν) such that the G-equivariant mapping $x \mapsto \mathrm{stab}_G(x)$ pushes ν to η.*

Proof. Decompose $S(G)$ into into the conjugation invariant Borel sets $S_1 = \{H < G : H \text{ is cocompact in } G\}$ and $S_2 = S(G) \setminus S_1$. For each $H \in S_1$ let (Y_H, η_H) be $Y_H = G/H$ and η_H the Haar measure normalized to be a probability measure. For each $H \in S_2$ let (Y_H, η_H) be the Gaussian probability space corresponding to $L^2(G/H)$. Let $Y = ((Y_H, \eta_H))_{H \in S(G)}$ be the field of measure spaces just constructed.

Define the cocycle $\alpha : G \times S(G) \to Y$ such that $\alpha(g, H) \in \mathrm{Aut}(Y_H, Y_{gHg^{-1}})$ as follows: for $H \in S_1$ define $\alpha(g, H)(kH) = kg^{-1}(gHg^{-1})$ and for $H \in S_2$ define $\alpha(g, H)$ to be the measure-preserving isomorphism from Y_H to $Y_{gHg^{-1}}$ induced by the orthogonal operator $T_{g,H}$ given by $(T_{g,H}f)(kgHg^{-1}) = f(kgH)$. For each $g, h \in G$, the cocycle identity holds almost everywhere by the nature of the Gaussian construction. Define the measure space

$$(X, \nu) = \left(\bigsqcup Y_H, \int \eta_H \, d\eta(H) \right)$$

equipped with the measure-preserving cocycle action of G coming from α. By Mackey's point realization [Mac62] (see Appendix B of [Zim84]), as G is locally compact second countable by removing a null set we may assume, the cocycle identity holds everywhere.

For each fixed $H \in S(G)$ the map $g \mapsto \alpha(g, H)$ defines an action of the normalizer $N_G(H)$ of H in G modulo H on Y_H which is essentially free (Proposition 1.2 in [AEG94]). For $g \in G$ and $(H, x) \in X$ we see that $g(H, x) = (gHg^{-1}, \alpha(g, H)x)$ and therefore $(H, x) = g(H, x)$

if and only if $g \in N_G(H)$ and $\alpha(g, H)x = x$ hence if and only if $g \in H$. That is to say, $\mathrm{stab}_G(H, x) = H$ for almost every (H, x). Therefore the G-action on (X, ν) gives rise to the invariant random subgroup η as required. $\square$

3.3   Ergodic Invariant Random Subgroups

Definition 3.4. An invariant random subgroup $\nu \in P(S(G))$ is **ergodic** when ν is an ergodic measure.

We remark that ergodic invariant random subgroups are precisely the same as the extremal invariant measures in the (weak*) compact convex set of invariant random subgroups.

Proposition 3.3.1. *Let G be a locally compact second countable group. Given an ergodic invariant random subgroup $(S(G), \eta)$ there exists an ergodic measure-preserving G-space (X, ν) such that the G-equivariant mapping $x \mapsto \mathrm{stab}_G(x)$ pushes ν to η.*

Proof. Let (Z, ζ) be the G-action constructed in Theorem 3.3 such that $z \mapsto \mathrm{stab}_G(z)$ pushes ζ to η. Consider the ergodic decomposition $\pi : (Z, \zeta) \to (W, \rho)$. Then G acts trivially on (W, ρ) and almost every fiber $(\pi^{-1}(w), D_\pi(w))$ is an ergodic G-space. Observe that

$$\int_W \mathrm{stab}_* D_\pi(w) \ d\rho(w) = \mathrm{stab}_* \int_W D_\pi(w) \ d\rho(w) = \mathrm{stab}_* \zeta = \eta.$$

Since η is ergodic, it is extremal in the set of invariant random subgroups. The above convex combination of invariant random subgroups must then almost surely be constantly equal to η. That is, $\mathrm{stab}_* D_\pi(w) = \eta$ for ρ-almost every $w \in W$. Let (X, ν) be one such fiber. Then (X, ν) is an ergodic G-space with the required properties. $\square$

3.4   Compact and Open Invariant Random Subgroups

Definition 3.5. An invariant random subgroup $\nu \in P(S(G))$ is a **finite (compact) invariant random subgroup** when ν is supported on the finite (compact) subgroups of G and is an **infinite (noncompact) invariant random subgroup** when it is supported on the infinite (noncompact) subgroups of G.

We remark that in the case of ergodic invariant random subgroups, infinite is equivalent to not finite.

Definition 3.6. An invariant random subgroup $\nu \in P(S(G))$ is an **open invariant random subgroup** when ν is supported on the open subgroups of G.

3.5   Commensurate Invariant Random Subgroups

Recall that two subgroups are commensurate when their intersection has finite index in each. We introduce a definition of commensurability for invariant random subgroups that generalizes this notion to invariant random subgroups. We remark that one regains the usual definition in the case when the invariant random subgroups are point masses.

Definition 3.7. Let G be a group and η_1 and η_2 be invariant random subgroups of G. If there exists a G-invariant joining $\alpha \in P(S(G) \times S(G))$ of η_1 and η_2 such that for α-almost every $H, L \in S(G) \times S(G)$ the subgroup $H \cap L$ has finite index in both H and L then η_1 and η_2 are **commensurate invariant random subgroups**.

Theorem 3.8. *The property of being commensurate is an equivalence relation on the space of invariant random subgroups.*

Proof. Let η_1, η_2, η_3 be invariant random subgroups of G such that η_1 and η_2 are commensurate and η_2 and η_3 are commensurate. Let α be a joining of η_1 and η_2 and β be a joining of η_2 and η_3 witnessing the commensuration. Let D be the disintegration of α over η_2. Then for almost every $K \in S(G)$ we have that $D(K) = \alpha_K \times \delta_K$ for some $\alpha_K \in P(S(G))$ and likewise the disintegration of β over η_2 is of the form $\delta_K \times \beta_K$ for some $\beta_K \in P(S(G))$.

Let $\rho \in P(S(G) \times S(G))$ be the composition of the joinings α and β (see Glasner [Gla03]):

$$\rho = \int_{S(G)} \alpha_K \times \beta_K \ d\eta_2(K).$$

Then ρ is a joining of η_1 and η_3 (Proposition 2.2.2) and for ρ-almost every (H, L) we have that for η_2-almost every K the subgroup $H \cap K$ has finite index in H and K and the subgroup $K \cap L$ has finite index in both K and L. Then $H \cap K \cap L$ has finite index in H, K and L and so $H \cap L$ has finite index in H and L (that is, commensurability is an equivalence relation on subgroups). Therefore ρ makes η_1 and η_3 commensurate invariant random subgroups. $\qquad \square$

Definition 3.9. Let G be a group. The **commensurability classes of invariant random subgroups** of G are the classes of invariant random subgroups equivalent under commensuration.

3.6   Relationships Between Invariant Random Subgroups

We now outline an approach to defining the usual properties of subgroups of groups for invariant random subgroups similar to the above. The joinings technique used for commensuration should allow, in general, the definition of the usual properties of subgroups for invariant random subgroups. We point out a few now but opt not to include all the details here, and instead leave them to the reader, as we will not make explicit use of these notions in what follows.

Definition 3.10. Let $\nu, \eta \in P(S(G))$ be invariant random subgroups of G. Then ν is a **subgroup** of η, written $\nu < \eta$, when there exists a G-invariant joining α of ν and η such that for α-almost every $(H, L) \in S(G) \times S(G)$, H is a subgroup of L.

Definition 3.11. A subgroup ν of an invariant random subgroup η has **finite index** when there exists a G-invariant joining of ν and η such that for almost every $(H, L) \in S(G) \times S(G)$, H has finite index in L.

Theorem 3.12. *The relation $<$ of being a subgroup is a transitive relation on invariant random subgroups that is antisymmetric–if $\nu < \eta$ and $\eta < \nu$ then $\nu = \eta$–and if $\{\nu_\alpha\}_{\alpha \in I}$ is a decreasing (respectively increasing) net of invariant random subgroups, then there exists a weak*-limit ν_∞ such that $\nu_\infty < \nu_\alpha$ (respectively $\nu_\infty > \nu_\alpha$) for all α.*

Proof. Let ν, η, ζ be invariant random subgroups such that $\nu < \eta$ and $\eta < \zeta$. Let α be a joining of ν and η such that for α-almost every (H, L), $H < L$, and let β be a joining of η and ζ such that for β-almost every (L, K), $L < K$. Take $\rho = \alpha \circ \beta$ to be the composition joining of α and β which is a joining of ν and ζ. Then for ρ-almost every (H, L) we have that $H < K$ and $K < L$ for η-almost every K so $H < L$. Hence the relation is transitive.

Let $d : G \times G \to \mathbb{R}$ be a metric on G compatible with the Haar measure algebra. Extend d to a distance function on $S(G) \times G$ as usual: $d(H, g) = \inf\{d(h, g) : h \in H\}$. Given an invariant random subgroup ν, define the functions $\varphi_{n,\nu} : G \to [0, 1]$ for each $n \in \mathbb{N}$ by

$$\varphi_{n,\nu}(g) = \nu\{H \in S(G) : d(H, g) < \frac{1}{n}\}.$$

Note that $\varphi_{n,\nu}$ is a bounded Borel function since d is compatible with the topology of G. Let η be an invariant random subgroup such that $\nu < \eta$ and let α be a joining witnessing this fact. Then for α-almost every (H, L) we have that $H < L$ and therefore for any $g \in G$, $d(H, g) \geq d(L, g)$. Therefore

$$\varphi_{n,\eta}(g) = \eta\{L \in S(G) : d(L, g) < \frac{1}{n}\} \geq \nu\{H \in S(G) : d(H, g) < \frac{1}{n}\} = \varphi_{n,\nu}(g)$$

for each n.

Moreover, if $\nu < \eta$ and $\nu \neq \eta$ then for a joining ρ witnessing $\nu < \eta$ we have that $\rho\{(H, L) \in S(G) \times S(G) : H < L, H \neq L\} > 0$, and hence by considering a countable dense subset of G we see that there must be some $g \in G$ and $n \in \mathbb{N}$ such that $\rho\{(H, L) \in S(G) \times S(G) : H < L,\ d(H, g) \geq \frac{1}{n},$ and $d(L, g) < \frac{1}{n}\} > 0$. Thus for this g and n we have $\varphi_{n,\eta}(g) > \varphi_{n,\nu}(g)$. Therefore if $\nu < \eta$ and $\eta < \nu$ then in fact $\nu = \eta$.

Now let $\{\nu_\alpha\}_{\alpha \in I}$ be a decreasing net of invariant random subgroups. Since $P(S(G))$ is compact, there exists a subnet I_0 such that $\{\nu_\beta\}_{\beta \in I_0}$ has a weak*-limit point ν_∞. If I_1 is another subnet such that $\{\nu_\alpha\}_{\alpha \in I_1}$ has a weak*-limit point η_∞, then for each $\alpha \in I_1$ there exists $\beta \in I_0$ such that $\alpha < \beta$ and hence there exists a joining $\rho_{\beta,\alpha}$ of ν_β and ν_α such that for $\rho_{\beta,\alpha}$-almost every (H, L) we have $H < L$. If we take ρ_∞ an accumulation point of $\{\rho_{\beta,\alpha}\}$ then we have that ρ_∞ is a joining of η_∞ and ν_∞ such that for ρ_∞-almost every (H, L) we have $H < L$. Therefore $\eta_\infty < \nu_\infty$, and by symmetry we have $\nu_\infty < \eta_\infty$. Hence $\eta_\infty = \nu_\infty$ from above, and therefore $\{\eta_\alpha\}_{\alpha \in I}$ has a unique accumulation point showing that the limit exists.

Now $\nu_\beta < \nu_\alpha$ for $\alpha < \beta$ so for every pair $\alpha < \beta$ there exists a joining $\rho_{\beta,\alpha}$ of ν_β and ν_α such that for $\rho_{\beta,\alpha}$-almost every (H, L) we have $H < L$. Hold α fixed and let $\rho_{\infty,\alpha}$ be a weak*-accumulation point of $\{\rho_{\beta,\alpha}\}_{\beta > \alpha}$. Then $\rho_{\infty,\alpha}$ is a joining of ν_∞ and ν_α witnessing that $\nu_\infty < \nu_\alpha$. Hence $\nu_\infty < \nu_\alpha$ for all $\alpha \in I$.

The case of increasing nets follows from a similar argument. $\qquad\square$

Theorem 3.13. *The property of having finite index is a transitive relation on invariant random subgroups.*

4   Relatively Contractive Maps

We now introduce the notion of relatively contractive maps and develop the machinery needed to study actions of commensurators and lattices. We first spend some time developing basic facts about relatively contractive maps which we then use to recover most known results about contractive actions. We also take a short detour to study joinings of contractive spaces and show that relatively contractive is indeed the opposite of relatively measure-preserving in some very strong senses.

   We will always assume the group G is locally compact second countable in what follows.

Definition 4.1 (Jaworski [Jaw94]). A G-space (X, ν) is **contractive**, also called **SAT (strongly approximately transitive)**, when for all measurable sets $B \subseteq X$ of less than full measure and all $\epsilon > 0$ there exists $g \in G$ such that

$$\nu(gB) < \epsilon.$$

4.1   Conjugates of Disintegration Measures

The principal notion in formulating the idea of relatively contractive maps is to "conjugate" the disintegration measures. For a G-map of G-spaces $\pi : (X, \nu) \to (Y, \eta)$, the disintegration of ν over η can be summarized as saying that for almost every $y \in Y$ there is a unique measure $D_\pi(y) \in P(X)$ such that $D_\pi(y)$ is supported on the fiber over y and $\int_Y D_\pi(y) \, d\eta(y) = \nu$.

   For $g \in G$ and $y \in Y$, we have that $D_\pi(gy)$ is supported on the fiber over gy, that is, on $\pi^{-1}(gy) = g\pi^{-1}(y)$, and that for any Borel $B \subseteq X$, we have that $gD_\pi(y)(B) = D_\pi(y)(g^{-1}B)$ meaning that $gD_\pi(y)$ is supported on $g\pi^{-1}(y)$. Therefore we can formulate the following:

Definition 4.2. Let $\pi : (X, \nu) \to (Y, \rho)$ be a G-map of G-spaces. The **conjugated disintegration measure** over π at a point $y \in Y$ by the group element $g \in G$ is

$$D_\pi^{(g)}(y) = g^{-1}D_\pi(gy).$$

   The preceding discussion shows that $D_\pi^{(g)}(y)$ is supported on $g^{-1}g\pi^{-1}(y) = \pi^{-1}(y)$. Hence:

Proposition 4.1.1. *Let $\pi : (X, \nu) \to (Y, \eta)$ be a G-map of G-spaces and fix $y \in Y$. The conjugated disintegration measures*

$$\mathcal{D}_y = \{g^{-1}D_\pi(gy) : g \in G\}$$

are all supported on $\pi^{-1}(y)$.

   Another approach to the conjugates of disintegration measures is to observe that:

Proposition 4.1.2. *Let $\pi : (X, \nu) \to (Y, \eta)$ be a G-map of G-spaces. For any $g \in G$ then $\pi : (X, g^{-1}\nu) \to (Y, g^{-1}\eta)$ is also a G-map of G-spaces. Let $D_\pi : Y \to P(X)$ be the disintegration of ν over η. Then $D_\pi^{(g)}$ is the disintegration of $g^{-1}\nu$ over $g^{-1}\eta$.*

Proof. To see that π maps $(X, g^{-1}\nu)$ to $(Y, g^{-1}\eta)$ follows from π being G-equivariant.

We have already seen that $g^{-1}D_\pi(gy)$ is supported on $\pi^{-1}(y)$ so to prove the proposition it remains only to show that $\int g^{-1}D_\pi(gy)\, dg^{-1}\eta(y) = g^{-1}\nu$. This is clear as

$$\int_Y g^{-1}D_\pi(gy)\, dg^{-1}\eta(y) = g^{-1}\int_Y D_\pi(gg^{-1}y)\, d\eta(y)$$
$$= g^{-1}\int_Y D_\pi(y)\, d\eta(y) = g^{-1}\nu$$

since D_π disintegrates ν over η. $\square$

A basic fact we will need in what follows is that the conjugated disintegration measures are mutually absolutely continuous to one another (over a fixed point y of course, as y varies they have disjoint supports):

Proposition 4.1.3. *Let* $\pi : (X,\nu) \to (Y,\eta)$ *be a G-map of G-spaces. For almost every y the set*

$$\mathcal{D}_y = \{g^{-1}D_\pi(gy) : g \in G\}$$

is a collection of mutually absolutely continuous probability measures supported on $\pi^{-1}\{y\}$.

Proof. For $g \in G$ write

$$A_g = \{y \in Y : D_\pi(y) \text{ and } g^{-1}D_\pi(gy) \text{ are not in the same measure class}\}.$$

Then A_g is a Borel set for each $g \in G$ since $D_\pi : Y \to P(X)$ is a Borel map and the equivalence relation on $P(X)$ given by $\alpha \sim \beta$ if and only if α and β is in the same measure class is Borel.

Since $g^{-1}D_\pi(gy)$ is the disintegration of $g^{-1}\nu$ over $g^{-1}\eta$ and $g^{-1}\nu$ is in the same measure class as ν, Lemma 4.1.4 (following the proof) gives that $\eta(A_g) = 0$ for each $g \in G$. Therefore

$$\eta(\bigcup_{g \in G_0} A_g) = 0$$

where the union is taken over a countable dense subset G_0 (the existence of such a subset is a consequence of the second countability of G). When G is itself countable the claim is then proven.

Suppose now that there is some g such that $\eta(A_g) > 0$. Take a continuous compact model for π via Lemma 2.2.3. Define the sets, for $g \in G$ and $\epsilon > 0$ and $f \in C(X)$ with $f \geq 0$,

$$A_{g,\epsilon,f} = \{y \in Y : D_\pi(y)(f) = 0, D_\pi^{(g)}(y)(f) \geq \epsilon\}.$$

These sets are Borel since $y \mapsto D_\pi(y)(f)$ is Borel. Now $A_g = \bigcup_{\epsilon > 0, f} A_{g,\epsilon,f}$ and since $\eta(A_g) > 0$, (taking a countable sequence $\epsilon \to 0$ and a countable dense set of $C(X)$) there is some $\epsilon > 0$ and $f \in C(X)$ with $f \geq 0$ such that

$$\eta(A_{g,\epsilon,f}) > 0.$$

But now for fixed $\epsilon > 0$ and $f \in C(X)$ with $f \geq 0$ we see that

$$g^{-1}\nu(f) \geq \int_{A_{g,\epsilon,f}} D_\pi^{(g)}(y)(f)\, dg^{-1}\eta(y) \geq g^{-1}\eta(A_{g,\epsilon,f})\epsilon > 0$$

by the quasi-invariance of η.

Consider the function $F : G \to \mathbb{R}$ given by

$$F(h) = \int_{A_{g,\epsilon,f}} D_\pi^{(h)}(y)(f) \; dh^{-1}\eta(y) = h^{-1}\nu(\mathbb{1}_{\pi^{-1}(A_{g,\epsilon,f})}f).$$

Then $F(g) > 0$ by the above. Now F is continuous since $f \in C(X)$ and $G \curvearrowright X$ continuously. Hence there is some open neighborhood U of g in G such that $F(u) > 0$ for all $u \in U$.

For $g_0 \in G_0$, however, we know that $g_0^{-1}\nu(f) = 0$ and so, as $f \geq 0$, then $F(g_0) = 0$. But $U \cap G_0 \neq \emptyset$ since G_0 is dense and U is open, leading to a contradiction. Hence when G is locally compact second countable the claim also holds. $\square$

Lemma 4.1.4. *Let (X, ν) be a probability space and $\pi : (X, \nu) \to (Y, \pi_*\nu)$ a measurable map to a probability space. Let α be a probability measure in the same measure class as ν. Let $D(y)$ denote the disintegration of ν over $\pi_*\nu$ via π and let $D'(y)$ denote the disintegration of α over $\pi_*\alpha$ via π. Then for almost every $y \in Y$, $D(y)$ and $D'(y)$ are in the same measure class.*

Proof. Since α and ν are in the same measure class, the Radon-Nikodym derivative $\frac{d\alpha}{d\nu}$ exists and is in $L^1(X, \nu)$. Likewise, $\pi_*\alpha$ and $\pi_*\nu$ are in the same measure class so $\frac{d\pi_*\nu}{d\pi_*\alpha}$ exists in $L^1(X, \pi_*\nu)$.

For $y \in Y$, define the measure α_y by, for $B \subseteq X$ measurable,

$$\alpha_y(B) = \int_B \frac{d\alpha}{d\nu}(x) \; dD(y)(x) \; \frac{d\pi_*\nu}{d\pi_*\alpha}(y).$$

Note that the Radon-Nikodym derivatives are always positive so these are positive measures. Also $\alpha_y(X) = 1$ since

$$\frac{d\pi_*\alpha}{d\pi_*\nu}(y) = \int_X \frac{d\alpha}{d\nu}(x) \; dD(y)(x)$$

which can be verified directly (using the uniqueness of the Radon-Nikodym derivative).

Now the support of α_y is contained in the support of $D(y)$ which is contained in $\pi^{-1}(y)$, hence α_y is supported on $\pi^{-1}(y)$. For $B \subseteq X$ measurable,

$$\begin{aligned}
\int_Y \alpha_y(B) \; d\pi_*\alpha(y) &= \int_Y \int_B \frac{d\alpha}{d\nu}(x) \; dD(y)(x) \; \frac{d\pi_*\nu}{d\pi_*\alpha}(y) \; d\pi_*\alpha(y) \\
&= \int_Y \int_X \mathbb{1}_B(x)\frac{d\alpha}{d\nu}(x) \; dD(y)(x) \; d\pi_*\nu(y) \\
&= \int_X \mathbb{1}_B(x)\frac{d\alpha}{d\nu}(x) \; d\nu(x) \\
&= \int_X \mathbb{1}_B(x) \; d\alpha(x) = \alpha(B).
\end{aligned}$$

Therefore, by uniqueness of disintegration, $D'(y) = \alpha_y$ for almost every y.

Suppose that $D(y)(B) = 0$ for some y and some measurable $B \subseteq X$. Then

$$\alpha_y(B) = \int_B \frac{d\alpha}{d\nu}(x) \ dD(y)(x) \ \frac{d\pi_*\nu}{d\pi_*\alpha}(y) = 0$$

since $D(y)(B) = 0$. So α_y is absolutely continuous with respect to $D(y)$.

Therefore $D'(y)$ is absolutely continuous with respect to $D(y)$ for almost every $y \in Y$. The symmetric argument (reversing the roles of ν and α) shows that $D(y)$ is also absolutely continuous with respect to $D(y)$ almost everywhere. $\square$

4.2 Definition of Relatively Contractive Maps

We now define relatively contractive factor maps, which are the counterpart of relatively measure-preserving factor maps.

4.2.1 Relatively Measure-Preserving

We first recall the definition of relative measure-preserving:

Definition 4.3. Let $\pi : (X, \nu) \to (Y, \eta)$ be a G-map of G-spaces. Then π is **relatively measure-preserving** when for almost every $y \in Y$ the disintegration map D_π is G-equivariant: $D_\pi(gy) = gD_\pi(y)$.

In terms of conjugating disintegration measures, relative measure-preserving means that $D_\pi^{(g)}(y) = D_\pi(y)$ almost surely.

We also remark that a G-space (X, ν) is measure-preserving if and only if the map from (X, ν) to the trivial (one-point) space is relatively measure-preserving (the disintegration over the trivial space is $D^{(g)}(0) = g^{-1}D(g \cdot 0) = g^{-1}D(0) = g^{-1}\nu$).

4.2.2 Relatively Contractive

Definition 4.4. Let $\pi : (X, \nu) \to (Y, \eta)$ be a G-map of G-spaces. We say π is **relatively contractive** when for almost every $y \in Y$ and any measurable $B \subseteq X$ with $D_\pi(y)(B) < 1$ and any $\epsilon > 0$ there exists $g \in G$ such that $g^{-1}D_\pi(gy)(B) < \epsilon$.

This is also stated as saying that (X, ν) is a **relatively contractive extension** or **contractive extension** of (Y, η) or that (Y, η) is a **relatively contractive factor** or just a **contractive factor** of (X, ν).

We have the following easy reformulation of the above definition:

Proposition 4.2.1. *A G-map $\pi : (X, \nu) \to (Y, \eta)$ of G-spaces is relatively contractive if and only if for almost every y and any measurable $B \subseteq Y$ with $D_\pi(y)(B) > 0$ we have*

$$\sup_{g \in G} D_\pi^{(g)}(y)(B) = 1.$$

4.2.3   Contractive Extensions of a Point

We now show that contractive can be defined in terms of relatively contractive extensions of a point (just as measure-preserving can be defined as being a relatively measure-preserving extension of a point).

Theorem 4.5. *A G-space (X, ν) is contractive if and only if it is a relatively contractive extension of a point.*

Proof. In the case where $(Y, \eta) = 0$ is the trivial one point system, the disintegration measure is always ν and so being a relatively contractive extension reduces to the definition of contractive: $g^{-1}D_\pi(g \cdot 0) = g^{-1}\nu$ for all $g \in G$ since $g \cdot 0 = 0$ and therefore $\sup_g D_\pi^{(g)}(0)(B) = 1$ implies $\sup_g g^{-1}D_\pi(0)(B) = 1$ so $\sup_g g^{-1}\nu(B) = 1$ for all measurable B with $\nu(B) > 0$. □

4.3   The Algebraic Characterization

Generalizing Jaworksi [Jaw94], we characterize relatively contractive maps algebraically:

Theorem 4.6. *Let $\pi : (X, \nu) \to (Y, \rho)$ be a G-map of G-spaces. Then π is relatively contractive if and only if the map $f \mapsto D_\pi^{(g)}(y)(f)$ is an isometry between $L^\infty(X, D_\pi(y))$ and $L^\infty(G, \mathrm{Haar})$ for almost every $y \in Y$ (here $D_\pi^{(g)}(y)(f)$ is a function of g).*

Proof. Assume π is relatively contractive. Take y in the measure one set where the disintegration measures are relatively contractive. Let f be any simple function $f = \sum a_n \mathbb{1}_{B_n}$ with $B_n \subseteq \pi^{-1}(y)$. Choose N such that $|a_N| = \max_n |a_n| = \|f\|_\infty$. For $\epsilon > 0$ choose $g \in G$ such that $D_\pi^{(g)}(y)(B_N) > 1 - \epsilon$. Then $D_\pi^{(g)}(y)(B_N^C) < \epsilon$ and since the B_n are disjoint then $D_\pi^{(g)}(y)(B_n) < \epsilon$ for $n \neq N$. This means that

$$\left| D_\pi^{(g)}(y)(f) - a_N \right| = \left| \sum_n a_n D_\pi^{(g)}(y)(B_n) - a_N \right| \leq \sum_{n \neq N} |a_n|\epsilon + |a_N||1 - \epsilon - 1| = \epsilon \sum_n |a_n|$$

and since $\epsilon > 0$ was arbitrary then $\sup_g |D_\pi^{(g)}(y)(f)| = |a_N| = \|f\|$. As simple functions are uniformly dense in $L^\infty(X, D_\pi(y))$ and the map is a contraction this proves one direction.

Conversely, assume the map is an isometry for almost every y. For such a y, let $B \subseteq \pi^{-1}(y)$ with $D_\pi(y)(B) > 0$ and then $1 = \|\mathbb{1}_B\|_\infty = \sup_g D_\pi^{(g)}(y)(B)$ so π relatively contractive. □

Note that π is relatively measure-preserving if and only if the map that would be isometric for relatively contractive, $f \mapsto D_\pi^{(g)}(y)(f)$, is simply the map $f \mapsto D_\pi(y)(f)$ which is the projection to the "constants" on each fiber.

We remark that in effect there is a zero-one law for relatively contractive extensions. Namely, if $\pi : (X, \nu) \to (Y, \eta)$ is a G-map of ergodic G-spaces then the set of y such that $D_\pi^{(g)}(y)$ induces an isometry $L^\infty(X, D_\pi(y)) \to L^\infty(G, \mathrm{Haar})$ has either measure zero or measure one. This follows from the fact that the set of such y must be G-invariant and hence follows by ergodicity: if $D_\pi^{(g)}(y)$ induces an isometry then for any $h \in G$ and $f \in L^\infty(X, \nu)$

$$\sup_{g \in G} \left| D_\pi^{(g)}(hy)(f) \right| = \sup_{g \in G} \left| D_\pi^{(gh)}(y)(h \cdot f) \right| = \sup_{g \in G} \left| D_\pi^{(g)}(y)(h \cdot f) \right| = \|h \cdot f\| = \|f\|.$$

Specializing to the case of a contractive extension of the trivial one point system we obtain:

Corollary 4.7 (Jaworski [Jaw94]). *A G-space (X, ν) is contractive if and only if the mapping $L^\infty(X, \nu) \to L^\infty(G, \mathrm{Haar})$ by $f \mapsto g\nu(f)$ is an isometry.*

One can also characterize relatively contractive maps in terms of convex combinations of measures:

Theorem 4.8. *A G-map of G-spaces $\pi : (X, \nu) \to (Y, \eta)$ is relatively contractive if and only if for almost every $y \in Y$ the space of absolutely continuous measures $L_1^1(\pi^{-1}(y), D_\pi(y)) \subseteq \overline{\mathrm{conv}} \, \mathcal{D}_y$.*

Proof. An immediate consequence of Theorem 4.12 (in the following subsection). □

Specializing to the one point system:

Corollary 4.9 (Jaworski [Jaw94]). *A G-space (X, ν) is contractive if and only if the space of absolutely continuous measures $L_1^1(X, \nu) \subseteq \overline{\mathrm{conv}} \, G\nu$.*

4.4 Relatively Contractible Spaces

Definition 4.10 (Furstenberg-Glasner [FG10]). A continuous compact model (X_0, ν_0) of a G-space (X, ν) is **contractible** when for every $x \in X_0$ there exists $g_n \in G$ such that $g_n \nu_0 \to \delta_x$ in weak*.

Definition 4.11. Let $\pi : (X, \nu) \to (Y, \eta)$ be a G-map of G-spaces. A continuous compact model $\pi_0 : (X_0, \nu_0) \to (Y_0, \eta_0)$ for this map is **relatively contractible** when for η_0-almost every $y \in Y_0$ and every $x \in X_0$ such that $\pi_0(x) = y$ there exists a sequence $g_n \in G$ such that $D_{\pi_0}^{(g_n)}(y) \to \delta_x$ in weak*.

Theorem 4.12. *Let $\pi : (X, \nu) \to (Y, \eta)$ be a G-map of G-spaces. Then π is relatively contractive if and only if every continuous compact model of π is relatively contractible.*

Proof. Recall that a continuous compact model for π means compact models for X and Y such that $G \curvearrowright X$ and $G \curvearrowright Y$ are continuous and the map π is continuous (Lemma 2.2.3).

Assume that π is relatively contractive. By Theorem 4.6, there is a measure one set of y such that $f \mapsto D_\pi^{(g)}(y)(f)$ is an isometry between $L^\infty(X, D_\pi(y))$ and $L^\infty(G, \mathrm{Haar})$. Fix y in that set and let $x \in X$ such that $\pi(x) = y$. Choose $f_n \in C(X)$ such that $0 \le f_n \le 1$, $\|f_n\| = 1$ and $f_n \to \mathbb{1}_{\{x\}}$ (possible since $C(X)$ separates points) and such that $f_{n+1} \le f_n$. Since π is relatively contractive, $\sup_g D_\pi^{(g)}(y)(f_n) = 1$ for each n. Choose $g_n \in G$ such that

$$1 - \frac{1}{n} < D_\pi^{(g_n)}(y)(f_n)$$

and observe then that, since $f_{n+1} \le f_n$,

$$1 - \frac{1}{n+1} < D_\pi^{(g_{n+1})}(y)(f_{n+1}) \le D_\pi^{(g_{n+1})}(y)(f_n)$$

and therefore $\lim_{m\to\infty} D_\pi^{(g_m)}(f_n) = 1$ for each fixed n.

Now $P(X)$ is compact so there exists a limit point $\zeta \in P(X)$ such that $\zeta = \lim_j D_\pi^{(g_{n_j})}(y)$ along some subsequence. Now $\zeta(f_n) = 1$ for each n by the above and $f_n \to \mathbb{1}_{\{x\}}$ is pointwise decreasing so by bounded convergence $\zeta(\{x\}) = \lim \zeta(f_n) = 1$. This means that for almost every y, the conclusion holds for all $x \in \pi^{-1}(y)$.

For the converse, first consider any continuous compact model such that for almost every $y \in Y$ and every $x \in \pi^{-1}(y)$ there exists a sequence $\{g_n\}$ such that $D_\pi^{(g_n)}(y) \to \delta_x$. Let $f \in C(X)$. Then the supremum of f on $\pi^{-1}(y)$ is attained at some $x \in \pi^{-1}(y)$ since $\pi^{-1}(y)$ is a closed, hence compact, set. Take g_n such that $g_n^{-1} D_\pi(g_n y) \to \delta_x$. Then $g_n^{-1} D_\pi(g_n y)(f) \to f(x) = \|f\|_{L^\infty(\pi^{-1}(y))}$. Hence for $f \in C(X)$ the map is an isometry.

Now assume that for every continuous compact model for π and for almost every y and every $x \in \pi^{-1}(y)$ there is a sequence $g_n \in G$ such that $g_n^{-1} D_\pi(g_n y) \to \delta_x$.

Suppose that π is not relatively contractive. Then there exists a measurable set $A \subseteq X$ with $\nu(A) > 0$ and $1 > \delta > 0$ such that

$$B = \{y \in Y : D_\pi(y)(A) > 0 \text{ and } \sup_g D_\pi^{(g)}(y)(A) \leq 1 - \delta\} > 0$$

has $\eta(B) > 0$.

Fix $\epsilon > 0$. Let $\psi_n \in C_c(G)$ be an approximate identity (ψ_n are nonnegative continuous functions with $\int \psi_n dm = 1$ where m is a Haar measure on G such that the compact supports of the ψ_n are a decreasing sequence and $\cap_n \text{supp } \psi_n = \{e\}$; the reader is referred to [FG10] Corollary 8.7). Define $f_n = \mathbb{1}_A * \psi_n = \int_G \mathbb{1}_A(hx)\psi_n(h) \, dm(h)$. Then the f_n are G-continuous functions by [FG10] Lemma 8.6.

By Proposition 4.4.1 (below),

$$\lim_n \|\mathbb{1}_A * \psi_n\|_{L^\infty(X, D_\pi(y))} = 1$$

for all $y \in B$.

There then exists a set $B_1 \subseteq B$ with $\eta(B_1) > \eta(B) - \epsilon$ and $N \in \mathbb{N}$ such that for all $y \in B_1$ and all $n \geq N$, $\|\mathbb{1}_A * \psi_n\|_{L^\infty(X, D_\pi(y))} > 1 - \epsilon$. Let V be a compact set neighborhood of the identity in G such that $|\eta(B_1 \cap h^{-1}B_1) - \eta(B_1)| < \epsilon$ for all $h \in V$ (possible as the G-action is continuous on the algebra of measurable sets). Choose $n \geq N$ such that the support of $\psi = \psi_n$ is contained in V.

Set $f = \mathbb{1}_A * \psi$. Since f is G-continuous there exists a continuous compact model on which $f \in C(X)$ by [FG10] Theorem 8.5. Hence, for almost every $y \in Y$,

$$\sup_g D_\pi^{(g)}(y)(f) = \|f\|_{L^\infty(X, D_\pi(y))}.$$

Removing a null set from B_1, then for all $y \in B_1$ there exists $g_y \in G$ such that

$$D_\pi^{(g_y)}(y)(f) > \|f\|_{L^\infty(X, D_\pi(y))} - \epsilon > 1 - 2\epsilon.$$

Observe that

$$(1 - 2\epsilon)\eta(B_1) \leq \int_{B_1} D_\pi^{(g_y)}(f) \, d\eta(y)$$

$$= \int_{B_1} \int_X f(g_y^{-1}x) \, dD_\pi(g_yy)(x) \, d\eta(y)$$

$$= \int_{B_1} \int_X \int_G \mathbb{1}_A(hg_y^{-1}x)\psi(h) \, dm(h) \, dD_\pi(g_yy) \, d\eta(y)$$

$$= \int_G \int_{B_1} D_\pi(g_yy)(g_yh^{-1}A) \, d\eta(y)\psi(h) \, dm(h)$$

$$= \int_G \int_{hB_1} D_\pi(g_yh^{-1}y)(g_yh^{-1}A) \, dh\eta(y)\psi(h) \, dm(h)$$

$$= \int_G \int_{hB_1} D_\pi^{(g_yh^{-1})}(y)(A) \, dh\eta(y)\psi(h) \, dm(h)$$

$$\leq \int_G \int_{hB_1} \sup_g D_\pi^{(g)}(y)(A) \, dh\eta(y)\psi(h) \, dm(h)$$

$$= \int_G \Big(\int_{hB_1 \setminus B_1} \sup_g D_\pi^{(g)}(y)(A) \, dh\eta(y)$$

$$+ \int_{hB_1 \cap B_1} \sup_g D_\pi^{(g)}(y)(A) \, dh\eta(y) \Big)\psi(h) \, dm(h)$$

$$\leq \int_G \Big(h\eta(hB_1 \setminus B_1) + (1-\delta)h\eta(hB_1 \cap B_1) \Big)\psi(h) \, dm(h)$$

$$= \int_G \Big(h\eta(hB_1) - \delta h\eta(hB_1 \cap B_1) \Big)\psi(h) \, dm(h)$$

$$= \eta(B_1) - \delta \int_G \eta(B_1 \cap h^{-1}B_1)\psi(h) \, dm(h).$$

Now the support of ψ is contained in V and $|\eta(B_1 \cap h^{-1}B_1) - \eta(B_1)| < \epsilon$ for all $h \in V$. Therefore

$$-2\epsilon\eta(B_1) \leq -\delta \int_G (\eta(B_1) - \epsilon)\psi(h) \, dm(h) = -\delta\eta(B_1) + \delta\epsilon.$$

Hence

$$\delta\eta(B_1) \leq \epsilon(2\eta(B_1) + \delta).$$

Then

$$\delta\eta(B) \leq \delta(\eta(B_1) + \epsilon) \leq 2\epsilon(\eta(B_1) + \delta) \leq 2\epsilon(\eta(B) + \delta).$$

Since δ is fixed and this holds for all $\epsilon > 0$, $\eta(B) = 0$ contradicting that π is not relatively contractive. $\qquad \square$

Specializing to the case of a relatively contractive extension of a point, we obtain as a consequence the result of Furstenberg and Glasner mentioned above:

Corollary 4.13 (Furstenberg-Glasner [FG10]). *A G-space is contractive if and only if every continuous compact model of the space is contractible.*

The following fact was used in the above proof and is step by step equivalent to the proof of [FG10] Proposition 8.8 but relativized over a G-map:

Proposition 4.4.1. *Let* $\pi : (X, \nu) \to (Y, \eta)$ *be a G-map of G-spaces. Let $\psi_n \in C_c(G)$ be an approximate identity (the ψ_n are nonnegative continuous functions with decreasing compact supports V_n such that $\cap V_n = \{e\}$ and $\int \psi_n dm = 1$ for m a Haar measure on G). Then for any measurable set $A \subseteq X$ and almost every $y \in Y$ such that $D_\pi(y)(A) > 0$,*

$$\lim_n \|\mathbb{1}_A * \psi_n\|_{L^\infty(X, D_\pi(y))} = 1.$$

Proof. Take a continuous compact model for π. Let

$$B = \{y \in Y : D_\pi(y)(A) > 0\}.$$

Fix $\delta > 0$ and choose $\epsilon_y > 0$ for each $y \in B$ such that $\epsilon_y < \frac{1}{4}\delta D_\pi(y)(A)$.

For each $y \in B$, let $C_y \subseteq A \subseteq U_y$ such that C_y is closed and U_y is open and $D_\pi(y)(U_y \setminus C_y) < \epsilon_y$ (possible since $D_\pi(y)$ is regular). Let V_y be a symmetric compact neighborhood of the identity in G such that $D_\pi(y)(hC_y \triangle C_y) < \epsilon_y$ and such that $hC_y \subseteq U_y$ for all $h \in V_y$ (possible since the G-action is continuous). Then $\mathbb{1}_{C_y}(hx) = 0$ for all $x \notin U_y$ and $h \in V_y$.

Let $N_y \in \mathbb{N}$ such that supp $\psi_n \subseteq V_y$ for all $n \geq N_y$. For $n \geq N_y$, set $f_{y,n} = \mathbb{1}_{C_y} * \psi_n$. Then $f_{y,n}(x) = 0$ for $x \notin U_y$. So

$$
\begin{aligned}
D_\pi(y)(f_{y,n}) &= \int_X \int_G \mathbb{1}_{C_y}(hx)\psi_n(h) \, dm(h) \, dD_\pi(y)(x) \\
&= \int_G D_\pi(y)(h^{-1}C_y)\psi_n(h) \, dm(h) \\
&\geq \int_G (D_\pi(y)(C_y) - \epsilon_y)\psi_n(h) \, dm(h) \\
&= D_\pi(y)(C_y) - \epsilon_y.
\end{aligned}
$$

Define

$$E_{y,n} = \{x \in U : f_{y,n}(x) < 1 - \delta\}.$$

Then

$$
\begin{aligned}
D_\pi(y)(C_y) - \epsilon_y &\leq \int_X f_{y,n}(x) \, dD_\pi(y)(x) \\
&= \int_{U_y} f_{y,n}(x) \, dD_\pi(y)(x) \\
&= \int_{E_{y,n}} f_{y,n}(x) \, dD_\pi(y)(x) + \int_{U_y \setminus E_{y,n}} f_{y,n}(x) \, dD_\pi(y)(x) \\
&\leq (1 - \delta)D_\pi(y)(E_{y,n}) + D_\pi(y)(U_y \setminus E_{y,n}) \\
&= D_\pi(y)(U_y) - \delta D_\pi(y)(E_{y,n}).
\end{aligned}
$$

Therefore

$$\delta D_\pi(y)(E_{y,n}) \leq D_\pi(y)(U_y \setminus C_y) + \epsilon_y < 2\epsilon_y$$

Hence $D_\pi(y)(E_{y,n}) < 2\epsilon_y\delta^{-1} < \frac{1}{2}D_\pi(y)(A)$. So, for $x \in U_y \setminus E_{y,n}$, $f_{y,n}(x) \geq 1 - \delta$ and $D_\pi(y)(U_y \setminus E_{y,n}) \geq \frac{1}{2}D_\pi(y)(A) > 0$.

Therefore
$$(\mathbb{1}_A * \psi_n)(x) \geq (\mathbb{1}_{C_y} * \psi(x)) \geq 1 - \delta$$
for all x in a $D_\pi(y)$-positive measure set. Hence for $n \geq N_y$, $\|\mathbb{1}_A * \psi_n\|_{L^\infty(X, D_\pi(y))} \geq 1 - \delta$. As this holds for all $\delta > 0$,
$$\lim_n \|\mathbb{1}_A * \psi_n\|_{L^\infty(X, D_\pi(y))} = 1$$
for all $y \in B$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4.5   Relatively Contractive Maps and Dense Subgroups

In general, the map $g \mapsto D_\pi^{(g)}(y)$ is not continuous (however, it can be shown to be continuous almost everywhere for almost every y) which can be seen by considering an induced action from a lattice to a locally compact second countable group. This fact accounts for the difficulty in the proof of the following statement.

Theorem 4.14. *Let $\pi : (X, \nu) \to (Y, \eta)$ be a relatively contractive G-map of G-spaces. Let G_0 be a countable dense subgroup of G. Then π is a relatively contractive G_0-map.*

Proof. Suppose that π is not G_0-relatively contractive. By the proof of Theorem 4.12, there then exists a continuous compact model for $\pi : X \to Y$, a positive measure set $A \subseteq Y$, a nonnegative continuous function $f \in C(X)$ and $\delta > 0$ such that for all $y \in A$,
$$\sup_{g_0 \in G_0} D_\pi^{(g_0)}(y)(f) \leq \|f\|_{L^\infty(X, D_\pi(y))} - \delta.$$

Let $\epsilon > 0$ such that $\eta(A) > \epsilon$. Since π is G-relatively contractive, there is a conull Borel set Y_{00} such that for every $y \in Y_{00}$, $\sup_g D_\pi^{(g)}(y)(f) = \|f\|_{L^\infty(X, D_\pi(y))}$.

Consider the set
$$E = \{(g, y) \in G \times Y_{00} : D_\pi^{(g)}(y)(f) \geq \|f\|_{L^\infty(X, D_\pi(y))} - \epsilon\}.$$

Since D_π is a Borel map, this is a Borel set. By the von Neumann Selection Theorem (Theorem A.9 in [Zim84]) there then exists a conull Borel set $Y_0 \subseteq Y_{00}$ such that the map $\mathrm{pr}_Y : E \to Y_{00}$ admits a Borel section on Y_0. Choose a Borel section $g_y \in G$ for $y \in Y_0$ such that $D_\pi^{(g_y)}(y) \geq \|f\|_{L^\infty(X, D_\pi(y))} - \epsilon$.

Consider the Borel function $Y \to P(X)$ given by $y \mapsto D_\pi(g_y y)$. By Lusin's Theorem, there exists a measurable set $D \subseteq Y$ with $\eta(D) > 1 - \epsilon$ and a continuous map $F : Y \to P(X)$ such that $F(y) = D_\pi(g_y y)$ for $y \in D$.

For $y \in Y_0$, choose $\{g_n\}$ in G_0 such that $g_n \to g_y$. Then $\|g_y \cdot f - g_n \cdot f\|_\infty \to 0$ since G acts continuously on $C(X)$ and $F(g_y^{-1} g_n y) \to F(y)$ in weak* hence $F(g_y^{-1} g_n y)(g_y \cdot f) \to F(y)(g_y \cdot y)$. Therefore

$$\left| F(y)(g_y \cdot f) - F(g_y^{-1} g_n y)(g_n \cdot f) \right|$$
$$\leq \left| F(y)(g_y \cdot f) - F(g_y^{-1} g_n y)(g_y \cdot f) \right| + \left| F(g_y^{-1} g_n y)(g_y \cdot f) - F(g_y^{-1} g_n y)(g_n \cdot f) \right|$$
$$\leq \left| F(y)(g_y \cdot f) - F(g_y^{-1} g_n y)(g_y \cdot f) \right| + \int_X \left| f(g_y^{-1} x) - f(g_n^{-1} x) \right| \, dF(g_y^{-1} g_n y)(x)$$

$$\leq \left| F(y)(g_y \cdot f) - F(g_y^{-1} g_n y)(g_y \cdot f) \right| + \| g_y \cdot f - g_n \cdot f \|_\infty \to 0.$$

Observe that for $y \in D$,

$$F(y)(g_y \cdot f) = D_\pi(g_y y)(g_y \cdot f) = D_\pi^{(g_y)}(y)(f) > \| f \|_{L^\infty(X, D_\pi(y))} - \epsilon.$$

Consider the set $D'_n = \{ y \in A : g_y^{-1} g_n y \in D \}$. Then for $y \in D'_n$,

$$F(g_y^{-1} g_n y)(g_n \cdot f) = D_\pi(g_y g_y^{-1} g_n y)(g_n \cdot f) = D_\pi^{(g_n)}(y)(f) \leq \| f \|_{L^\infty(X, D_\pi(y))} - \delta.$$

Consider the sets $E_n = D \cap D'_n$. Since $g_n \to g_y$, $\eta(E_n) \to \eta(D \cap A) > 0$. For $y \in E_n$,

$$\left| F(y)(g_y \cdot f) - F(g_y^{-1} g_n y)(g_n \cdot f) \right| \geq \delta - \epsilon.$$

But $\left| F(y)(g_y \cdot f) - F(g_y^{-1} g_n y)(g_n \cdot f) \right| \to 0$ as $n \to \infty$ for every y. This contradiction means that π is relatively contractive for G_0. $\qquad\square$

4.6 Examples of Relatively Contractive Maps

Let (X, ν) and (Y, η) be contractive G-spaces. In general it need not hold that $(X \times Y, \nu \times \eta)$ is contractive (with the diagonal G-action), however:

Theorem 4.15. *Let (X, ν) be a contractive G-space and (Y, η) be a G-space. The map* $\mathrm{pr}_Y : (X \times Y, \nu \times \eta) \to (Y, \eta)$ *is relatively contractive ($X \times Y$ has the diagonal G-action).*

Proof. The disintegration measures $D_\pi(y)$ are supported on $X \times \delta_y$ and have the form $D_\pi(y) = \nu \times \delta_y$. Clearly

$$D_\pi^{(g)}(y) = g^{-1}(\nu \times \delta_{gy}) = g^{-1}\nu \times \delta_y$$

and since (X, ν) is contractive then π is relatively contractive. $\qquad\square$

More generally, the following holds:

Theorem 4.16. *Let $\pi : (X, \nu) \to (Y, \eta)$ be a relatively contractive G-map of G-spaces. Let (Z, ζ) be a G-space. The map $\pi \times \mathrm{id} : (X \times Z, \nu \times \zeta) \to (Y \times Z, \eta \times \zeta)$ is relatively contractive (where $X \times Z$ and $Y \times Z$ have the diagonal G-action).*

Proof. Since the disintegration of the identity is point masses, for almost every $(y, z) \in Y \times X$, $D_{\pi \times \mathrm{id}}^{(g)}(y, z) = D_\pi^{(g)}(y) \times \delta_z$. Then π being relatively contractive implies $\pi \times \mathrm{id}$ is relatively contractive. $\qquad\square$

4.6.1 Inducing Actions and Relatively Contractive Maps

Theorem 4.17. *Let $\Gamma < G$ be a lattice in a locally compact second countable group. Let (X, ν) be a contractive Γ-space and $p : G \times_\Gamma X \to G/\Gamma$ be the G-map that is the natural projection from the induced G-space over (X, ν) to G/Γ. Then p is a relatively contractive G-map.*

Proof. First, assume (X, ν) is a contractive Γ-space. Treat $G \times_\Gamma X$ as $(F \times X, m \times \nu)$ for F a fundamental domain for G/Γ with cocycle $\alpha : G \times F \to \Gamma$. Consider $p : F \times X \to F$ the projection. The disintegration $D_p(f)$ of $m \times \nu$ over m is of the form $D_p(f) = \delta_f \times \nu$. For $g \in G$,

$$D_p^{(g)}(f) = g^{-1} D_p(gf\alpha(g, f)) = g^{-1}(\delta_{gf\alpha(g,f)} \times \nu) = \delta_f \times \alpha(g, f)\nu.$$

Fix $(f_0, x_0) \in F \times X$ and choose $\gamma_n \in \Gamma$ such that $\gamma_n \nu \to \delta_{x_0}$. Set $g_n = \gamma_n^{-1} f_0^{-1}$. Then $\alpha(g_n, f_0) = \gamma_n$ so $D_p^{(g_n)}(f_0) = \delta_{f_0} \times \gamma_n \nu \to \delta_{f_0} \times \delta_{x_0}$ meaning p is relatively contractive. $\qquad \square$

4.6.2 Proximal Maps are Contractive

Proximal maps are the relative version of Poisson boundaries; the reader is referred to Furstenberg and Glasner [FG10] and to Furman [Fur00] for more information.

Definition 4.18. Let (X, ν) be a (G, μ)-space, meaning that (X, ν) is a G-space and that μ is a probability measure on G such that $\mu * \nu = \nu$. A G-map of (G, μ)-spaces $\pi : (X, \nu) \to (Y, \eta)$ is **proximal** when for $\mu^{\mathbb{N}}$-almost every $\omega \in G^{\mathbb{N}}$ the map $\pi : (X, \nu_\omega) \to (Y, \eta_\omega)$ is an isomorphism.

Recall that $\nu_\omega = \lim \omega_1 \cdots \omega_n \nu$ is the limit measure which exists almost surely by the Martingale Convergence Theorem.

Theorem 4.19. *Let $\pi : (X, \nu) \to (Y, \eta)$ be a proximal G-map of (G, μ)-spaces. Then π is relatively contractive.*

Proof. Let $\omega \in G^{\mathbb{N}}$ such that $\pi : (X, \nu_\omega) \to (Y, \eta_\omega)$ is an isomorphism. Let $D_{\pi, \omega}$ be the disintegration of ν_ω over η_ω. Set $g_n = (\omega_1 \cdots \omega_n)^{-1}$. Then $g_n^{-1} \nu \to \nu_\omega$ and $g_n^{-1} \eta \to \eta_\omega$. Now $D_\pi^{(g_n)}$ disintegrates $g_n^{-1} \nu$ over $g_n^{-1} \eta$ and therefore $D_\pi^{(g_n)}(y) \to D_{\pi, \omega}(y)$. Since π is an isomorphism of ν_ω to η_ω for each x in the support of ν_ω, set $y = \pi(x)$ and we have that $D_{\pi, \omega}(y) = \delta_x$. Hence $D_\pi^{(g_n)}(y) \to \delta_x$. Now the union of the supports of the ν_ω is the support of ν which is all of X and therefore π is relatively contractive. $\qquad \square$

Corollary 4.20 (Jaworski [Jaw94],[Jaw95])**.** *Let (X, ν) be a (G, μ)-boundary. Then (X, ν) is a contractive G-space.*

Proof. Since (X, ν) is a boundary the map $(X, \nu) \to 0$ to the trivial system is a proximal map because ν_ω is a point mass almost surely is one of the equivalent definitions of being a boundary. $\qquad \square$

4.7 Factorization of Contractive Maps

We now prove that if a composition of G-maps is relatively contractive then each of the maps is also relatively contractive. This fact will be an important ingredient in the proof of the uniqueness of relatively contractive maps.

Lemma 4.7.1. *Let $\pi : (X, \nu) \to (Y, \eta)$ and $\varphi : (Y, \eta) \to (Z, \rho)$ be G-maps of G-spaces. Then for almost every $z \in Z$,*

$$\pi_* D_{\varphi \circ \pi}(z) = D_\varphi(z) \qquad and \qquad D_{\varphi \circ \pi}(z) = \int_Y D_\pi(y) \, dD_\varphi(z)(y)$$

Proof. Observe that the support of $\pi_* D_{\varphi \circ \pi}(z)$ is

$$\pi((\varphi \circ \pi)^{-1}(z)) = \pi(\pi^{-1}(\varphi^{-1}(z))) = \varphi^{-1}(z).$$

Also for $f \in L^\infty(Y, \eta)$, using the definition of disintegration over $\varphi \circ \pi$,

$$\int_Z \int_Y f(y) \, d\pi_* D_{\varphi \circ \pi}(z)(y) \, d\zeta(z) = \int_Z \int_X f(\pi(x)) \, dD_{\varphi \circ \pi}(z)(x) \, d\zeta(z)$$
$$= \int_X f(\pi(x)) \, d\nu(x) = \int_Y f(y) \, d\pi_* \nu(y) = \int_Y f(y) \, d\eta(y)$$

and therefore by uniqueness of disintegration the first claim is proved.

Similarly, the support of $\int_Y D_\pi(y) \, dD_\varphi(z)(y)$ is

$$\bigcup_{y \in \varphi^{-1}(z)} \pi^{-1}(y) = \pi^{-1}(\varphi^{-1}(z)) = (\varphi \circ \pi)^{-1}(z)$$

and also for $f \in L^\infty(X, \nu)$, using the definition of disintegration,

$$\int_Z \int_X f(x) \, d\Big(\int_Y D_\pi(y) \, dD_\varphi(z)(y) \Big)(x) \, d\zeta(z)$$
$$= \int_Z \int_Y \int_X f(x) \, dD_\pi(y)(x) \, dD_\varphi(z)(y) \, d\zeta(z)$$
$$= \int_Y \int_X f(x) \, dD_\pi(y)(x) \, d\eta(y) = \int_X f(x) \, d\nu(x)$$

and therefore by uniqueness the second claim holds. $\qquad \square$

Theorem 4.21. *Let $\pi : (X, \nu) \to (Y, \eta)$ and $\varphi : (Y, \eta) \to (Z, \rho)$ be G-maps of G-spaces. If $\varphi \circ \pi$ is relatively contractive then both φ and π are relatively contractive.*

Proof. We use Theorem 4.12 and take a continuous compact model for π to do so. First observe, for all $g \in G$ and almost every z, that $\pi_* D_{\varphi \circ \pi}^{(g)}(z) = D_\varphi^{(g)}(z)$. For such z where also $\overline{\text{conv}} \, \{D_{\varphi \circ \pi}^{(g)}(z)\} = P((\varphi \circ \pi)^{-1}(z))$ and every x such that $\varphi(\pi(x)) = z$ there is $g_n \in G$ such that $D_{\varphi \circ \pi}^{(g_n)}(z) \to \delta_x$. Therefore

$$D_\varphi^{(g_n)}(z) = \pi_* D_{\varphi \circ \pi}^{(g_n)}(z) \to \pi_* \delta_x = \delta_{\pi(x)}$$

and so for every y such that $\varphi(y) = z$ the point mass δ_y is a limit point of $D_\varphi^{(g)}(z)$. Hence φ is relatively contractive.

Suppose that π is not relatively contractive. Then, by the proof of Theorem 4.12, there exists a continuous compact model for $\pi : X \to Y$ such that $f \mapsto |D_\pi^{(g)}(y)(f)|$ is not an isometry from $C(X)$ to $L^\infty(G)$ for a positive measure set of $y \in Y$.

Observe that if the map is an isometry on a countable dense set $C_0 \subseteq C(X)$ then for any $f \in C(X)$ there exists $f_n \in C_0$ with $f_n \to f$ in sup norm, hence

$$|D_\pi^{(g)}(y)(f)| = |D_\pi^{(g)}(y)(f - f_n) + D_\pi^{(g)}(f_n)| \geq |D_\pi^{(g)}(y)(f_n)| - \|f - f_n\|_\infty.$$

For $\epsilon > 0$, choose n such that $\|f - f_n\|_\infty < \epsilon$. Then choose g such that $|D_\pi^{(g)}(y)(f_n)| > \|f_n\| - \epsilon$. Then

$$|D_\pi^{(g)}(y)(f)| > \|f_n\| - \epsilon - \epsilon > \|f\| - 3\epsilon$$

and so the map is an isometry for f as well.

Therefore, there is a positive measure set of y such that the map $f \mapsto |D_\pi^{(g)}(y)(f)|$ is not an isometry on C_0. Hence, since C_0 is countable, there is some $f \in C_0$ and a positive measure set of y such that $\sup_g |D_\pi^{(g)}(y)(f)| < \|f\|_{L^\infty(X, D_\pi(y))}$. So there is some $\delta > 0$ and a measurable set $A \subseteq Y$ with $\eta(A) > 0$ such that $\sup_g |D_\pi^{(g)}(y)(f)| < \|f\|_{L^\infty(X, D_\pi(y))} - \delta$ for all $y \in A$. We may assume (by taking a subset) that A is closed. Since η is a Borel measure, it is regular, hence we may assume A is closed (by taking a subset).

Now there exists a positive measure set $B \subseteq Z$ on which $D_\varphi(z)(A) > 0$ for $z \in B$. For $z \in B$ such that z is in the measure one set on which $\varphi \circ \pi$ contracts to point masses,

$$
\begin{aligned}
D_{\varphi \circ \pi}^{(g)}&(z)(f) \\
&= \int_{\varphi^{-1}(z)} D_\pi^{(g)}(y)(f) \, dD_\varphi^{(g)}(z)(y) \\
&= \int_{\varphi^{-1}(z) \cap A} D_\pi^{(g)}(y)(f) \, dD_\varphi^{(g)}(z)(y) + \int_{\varphi^{-1}(z) \setminus A} D_\pi^{(g)}(y)(f) \, dD_\varphi^{(g)}(z)(y) \\
&\leq \int_{\varphi^{-1}(z) \cap A} \|f\|_{L^\infty(X, D_\pi(y))} - \delta \, dD_\varphi^{(g)}(z)(y) + \int_{\varphi^{-1}(z) \setminus A} \|f\|_{L^\infty(X, D_\pi(y))} \, dD_\varphi^{(g)}(z)(y) \\
&\leq \|f\|_{L^\infty(X, D_{\varphi \circ \pi}(z))} - \delta D_\varphi^{(g)}(z)(A).
\end{aligned}
$$

Now for any $x \in (\varphi \circ \pi)^{-1}(z)$, there exists g_n such that $D_{\varphi \circ \pi}^{(g_n)}(z) \to \delta_x$. Hence also $D_\varphi^{(g_n)}(z) \to \delta_{\pi(x)}$. Choose $x \in \pi^{-1}(A) \cap (\varphi \circ \pi)^{-1}(z)$ such that $f(x) = \|f\|_{L^\infty(X, D_{\varphi \circ \pi}(z))}$ (possible since $\pi^{-1}(A) \cap (\varphi \circ \pi)^{-1}(z)$ is closed, hence compact, and f is continuous). Then

$$
\begin{aligned}
f(x) = \lim_n D_{\varphi \circ \pi}^{(g_n)}(z)(f) &\leq \lim_n \|f\|_{L^\infty(X, D_{\varphi \circ \pi}(z))} - \delta D_\varphi^{(g_n)}(z)(A) \\
&= \|f\|_{L^\infty(X, D_{\varphi \circ \pi}(z))} - \delta \delta_{\pi(x)}(A) = \|f\|_{L^\infty(X, D_{\varphi \circ \pi}(z))} - \delta
\end{aligned}
$$

is a contradiction. Hence π is relatively contractive. $\qquad \square$

The above statement is the analogue of one direction of the similar well-known fact about relative measure-preserving:

Theorem 4.22. *Let $\pi : (X, \nu) \to (Y, \eta)$ and $\varphi : (Y, \eta) \to (Z, \zeta)$ be G-maps of G-spaces such that $\varphi \circ \pi : (X, \nu) \to (Z, \zeta)$ is relatively measure-preserving. Then π and φ are both relatively measure-preserving. Conversely, if π and φ are relatively measure-preserving then so is $\varphi \circ \pi$.*

Corollary 4.23. *Any G-factor of a contractive G-space is a contractive G-space. Any G-factor of a measure-preserving G-space is a measure-preserving G-space.*

Proof. Let (X, ν) be a contractive G-space and $\pi : (X, \nu) \to (Y, \eta)$ be a G-map of G-spaces. Take $\varphi : (Y, \eta) \to 0$ to be the G-map to the trivial one-point space. Then $\varphi \circ \pi : (X, \nu) \to 0$

is relatively contractive since (X, ν) is contractive and therefore φ is relatively contractive since its composition with π is and so (Y, η) is contractive. The same argument applied to relative measure-preserving maps shows the second statement. $\square$

4.8   Relatively Measure-Preserving and Relatively Contractive

We now show that relatively measure-preserving extensions are orthogonal to relatively contractive extensions.

Theorem 4.24. *Let $\pi : (X, \nu) \to (Y, \rho)$ be a G-map between G-spaces. If π is both relatively contractive and relatively measure-preserving then π is an isomorphism.*

Proof. Since π is relatively contractive for almost every y there exists a sequence $g_n \in G$ such that $D_\pi^{(g_n)}(y) \to \delta_x$ for some $x \in X$ such that $\pi(x) = y$. Since π is relatively measure-preserving, for almost every y and any $g \in G$ we have $D_\pi^{(g)}(y) = D_\pi(y)$. Therefore for almost every y

$$D_\pi(y) = D_\pi^{(g_n)}(y) \to \delta_x$$

meaning that $D_\pi(y) = \delta_x$. Therefore π must be an isomorphism since D_π are (almost) all point masses. $\square$

Corollary 4.25. *Let (X, ν) be a contractive G-space and $\pi : (X, \nu) \to (Y, \eta)$ be a G-map to a G-space (Y, η). If π is relatively measure-preserving then π is an isomorphism.*

Proof. The mapping $(X, \nu) \to (Y, \eta) \to 0$ where 0 is the one point system is a composition of maps which compose to a relatively contractive map. Therefore each map is relatively contractive. Hence π is both relatively measure-preserving and relatively contractive and is therefore an isomorphism. $\square$

Akin to the previous result, we show that relative contractive and relative measure-preserving are orthogonal in the sense of products.

Corollary 4.26. *Let (X, ν) be a G-space such that $\pi : (X, \nu) \to (Y, \eta)$ is a relatively contractive G-map of G-spaces and $\varphi : (X, \nu) \to (Z, \zeta)$ is a relatively measure-preserving G-map of G-spaces. Then $\pi \times \varphi : (X, \nu) \to (Y \times Z, (\pi \times \varphi)_* \nu)$ by $(\pi \times \varphi)(x) = (\pi(x), \varphi(x))$ is a G-isomorphism.*

Proof. Consider the G-map $\mathrm{pr}_Y \circ (\pi \times \varphi) = \pi$. Since π is relatively contractive then both the projection map to Y and $\pi \times \varphi$ are relatively contractive (Theorem 4.21). Likewise $\mathrm{pr}_Z \circ (\pi \times \varphi) = \varphi$ is relatively measure-preserving so the projection to Z and $\pi \times \varphi$ are relatively measure-preserving. By the previous theorem then $\pi \times \varphi$ is an isomorphism. $\square$

4.9   Uniqueness of Relatively Contractive Maps

Theorem 4.27. *Let (X, ν) be a contractive G-space and (Y, η) be a measure-preserving G-space. Let $\psi : (X \times Y, \nu \times \eta) \to (Y, \eta)$ be the natural projection map (treating $(X \times Y, \nu \times \eta)$ as G-space with the diagonal action). Let $\pi : (X \times Y, \nu \times \eta) \to (Z, \alpha)$ be a G-map of G-spaces and let $\pi' : (X \times Y, \nu \times \eta) \to (Z, \beta)$ be a G-map of G-spaces such that α is in the same*

measure class as β. Let $\varphi : (Z, \alpha) \to (Y, \eta)$ and $\varphi' : (Z, \beta) \to (Y, \eta)$ be G-maps such that $\varphi \circ \pi = \psi$ and $\varphi' \circ \pi' = \psi$. Assume that the disintegrations $D_\varphi(y)$ of α over η via φ and the disintegrations $D_{\varphi'}(y)$ of β over η via φ' have the property that $D_\varphi(y)$ and $D_{\varphi'}(y)$ are in the same measure class almost surely. Then $\pi = \pi'$ almost everywhere, $\varphi = \varphi'$ almost everywhere and $\alpha = \beta$.

Proof. First we consider φ and φ'. Define the Borel set

$$B = \{z \in Z : \varphi(z) \neq \varphi'(z)\}.$$

Then for every $y \in Y$, it holds that $B \cap \varphi^{-1}(y) \cap (\varphi')^{-1}(y) = \emptyset$. Since $D_\varphi(y)$ is in the same measure class as $D_{\varphi'}(y)$ almost everywhere and since $D_{\varphi'}(y)((\varphi')^{-1}(y)) = 1$, for almost every y it holds that

$$D_\varphi(y)(B) = D_\varphi(y)(B \cap \varphi^{-1}(y)) = D_\varphi(y)(B \cap \varphi^{-1}(y) \cap (\varphi')^{-1}(y)) = D_\varphi(y)(\emptyset) = 0.$$

Therefore $\zeta(B) = 0$. Likewise, $\zeta'(B) = 0$. Hence $\varphi = \varphi'$ almost everywhere.

Now we consider π and π'. Suppose that

$$\nu \times \eta(\{(x, y) \in X \times Y : \pi(x, y) \neq \pi'(x, y)\}) > 0.$$

Fix compact models for X, Y and Z and let d be a compatible metric on Z and observe that

$$\{(x, y) \in X \times Y : \pi(x, y) \neq \pi'(x, y)\} = \bigcup_{\delta > 0} \{(x, y) \in X \times Y : d(\pi(x, y), \pi'(x, y)) \geq \delta\}$$

which is a decreasing union and therefore there is some $\delta > 0$ such that

$$\nu \times \eta(\{(x, y) \in X \times Y : d(\pi(x, y), \pi'(x, y)) > \delta\}) > 0.$$

By Fubini's Theorem there is then some $x_0 \in X$ such that

$$A = \{y \in Y : d(\pi(x_0, y), \pi'(x_0, y)) > \delta\}$$

has $\eta(A) > 0$.

Since (X, ν) is contractive, there exists a sequence $\{g_n\}$ in G such that $g_n^{-1}\nu \to \delta_{x_0}$. Observe that for almost every $y \in Y$,

$$D_\varphi^{(g_n)}(y) = \pi_* D_\psi^{(g_n)}(y) = \pi_*(g_n^{-1}(\nu \times \delta_{g_n y})) = \pi_*(g_n^{-1}\nu \times \delta_y) \to \pi_*(\delta_{x_0} \times \delta_y) = \delta_{\pi(x_0, y)}$$

and likewise that

$$D_{\varphi'}^{(g_n)}(y) \to \delta_{\pi'(x_0, y)}.$$

Define the set

$$U = \{z \in Z : d(\pi(x_0, \varphi(z)), z) < \frac{1}{2}\delta\} \cap \varphi^{-1}(A).$$

Note that $U \cap \varphi^{-1}(y)$ is open in $\varphi^{-1}(y)$ for all $y \in A$ since d is compatible. Moreover, for each $y \in A \cap \varphi(U)$, it holds that $\pi(x_0, y)$ is in the interior of U. Observe that for $z \in U$ we have that $d(\pi(x_0, \varphi(z)), z) < \frac{1}{2}\delta$ and

$$d(\pi(x_0, \varphi(z)), \pi'(x_0, \varphi(z))) > \delta$$

(using that $\varphi = \varphi'$) meaning that $d(\pi'(x_0, \varphi(z)), z) > \frac{1}{2}\delta$ and therefore we conclude that $\pi'(x_0, y) \in (\overline{U})^C$ for every $y \in A$. Therefore U is a continuity set for $\delta_{\pi(x_0,y)}$ and $\delta_{\pi'(x_0,y)}$ for all $y \in A$.

Then for almost every $y \in A$,

$$D_\varphi^{(g_n)}(y)(U) \to \delta_{\pi(x_0,y)}(U) = 1$$

and

$$D_{\varphi'}^{(g_n)}(y)(U) \to \delta_{\pi'(x_0,y)}(U) = 0$$

since U is a continuity set.

For $\epsilon > 0$, define the Borel sets

$$A_{\epsilon,n} = \{y \in A : \text{ for all } m \geq n, \, D_\varphi^{(g_m)}(y)(U) > 1 - \epsilon \text{ and } D_{\varphi'}^{(g_m)}(y)(U) < \epsilon\}.$$

The sets $A_{\epsilon,n}$ increase with n for a fixed ϵ and, up to a null set,

$$A = \bigcup_{n=1}^{\infty} A_{\epsilon,n}$$

and therefore, for each $\epsilon > 0$ there exists n such that $\eta(A_{\epsilon,n}) > \eta(A) - \epsilon$.

Now, using that η is measure-preserving, for every $\epsilon > 0$,

$$\begin{aligned}
\alpha(g_n U) &= \int_Y D_\varphi(y)(g_n U) \, d\eta(y) \\
&= \int_{g_n A} D_\varphi(y)(g_n U) \, d\eta(y) \\
&= \int_A D_\varphi^{(g_n)}(y)(U) \, d\eta(y) \\
&\geq \int_{A_{n,\epsilon}} D_\varphi^{(g_n)}(y)(U) \, d\eta(y) \\
&\geq (1 - \epsilon)\eta(A_{n,\epsilon}) \geq (1 - \epsilon)(\eta(A) - \epsilon).
\end{aligned}$$

Similarly,

$$\beta(g_n U) \leq \int_{A_{n,\epsilon}} D_{\varphi'}^{(g_n)}(y)(U) \, d\eta(y) + \eta(A) - \eta(A_{n,\epsilon}) \leq \epsilon\eta(A_{n,\epsilon}) + \epsilon \leq 2\epsilon.$$

By Lemma 4.9.1 (following the proof), then α and β are not in the same measure class, a contradiction. Therefore we conclude that $\pi = \pi'$ almost everywhere and so $\alpha = \pi_*\nu = (\pi')_*\nu = \beta$.   □

Lemma 4.9.1. *Let Z be a compact metric space and $\alpha, \beta \in P(Z)$ be Borel probability measures on it. If there exists $\delta > 0$ such that for every $\epsilon > 0$ there exists a Borel set $B_\epsilon \subseteq Z$ such that $\alpha(B_\epsilon) < \epsilon$ and $\beta(B_\epsilon) > (1 - \epsilon)\delta$ then α is not absolutely continuous with respect to β.*

Proof. Observe that

$$\alpha(\bigcap_{n=1}^{\infty} \bigcup_{m=n+1}^{\infty} B_{2^{-m}}) \leq \limsup_{n \to \infty} \sum_{m=n+1}^{\infty} \alpha(B_{2^{-m}}) \leq \limsup_{n \to \infty} \sum_{m=n+1}^{\infty} 2^{-m} = 0$$

but that

$$\beta(\bigcap_{n=1}^{\infty} \bigcup_{m=n+1}^{\infty} B_{2^{-m}}) \geq \liminf_{n \to \infty} \beta(B_{2^{-n-1}}) = \liminf_{n \to \infty} (1 - 2^{-n-1})\delta = \delta > 0.$$

$\square$

Corollary 4.28 (Creutz-Shalom [CS12])**.** *Let (X, ν) be a contractive G-space and let $\pi :$ $(X, \nu) \to (Y, \eta)$ and $\pi' : (X, \nu) \to (Y, \eta')$ be G-maps of G-spaces such that η and η' are in the same measure class. Then $\pi = \pi'$ almost surely and $\eta = \eta'$.*

Proof. Consider the composition of maps $\varphi \circ \pi : (X, \nu) \to 0$ where $\varphi : (Y, \eta) \to 0$ is the map to the trivial system. Since (X, ν) is contractive, the preceding theorem gives the result. $\square$

4.10   Joinings With Contractive Spaces

Theorem 4.29. *Let (X, ν) be a contractive G-space and let (Y, η) be a G-space. Then there is at most one joining $(X \times Y, \alpha)$ of (X, ν) and (Y, η) such that the projection to X is relatively measure-preserving.*

Proof. Let $f \in L^\infty(Y, \eta)$ and define

$$F(x) = D_{\mathrm{pr}_X}(x)(f \circ \mathrm{pr}_Y).$$

Taking compact models for X and Y such that π is continuous makes clear that F is a bounded Borel function on X. Then for any $g \in G$ we have that, using that D_{pr_X} is relatively measure-preserving,

$$\begin{aligned}
g\nu(F) &= \int_X F(gx) \, d\nu(x) \\
&= \int_X \int_{X \times Y} f(\mathrm{pr}_Y(z, y)) \, dD_{\mathrm{pr}_X}(gx)(z, y) \, d\nu(x) \\
&= \int_X \int_{X \times Y} f(g\mathrm{pr}_Y(z, y)) \, dD_{\mathrm{pr}_X}(x)(z, y) \, d\nu(x) \\
&= \int_{X \times Y} f(g\mathrm{pr}_Y(z, y)) \, d\alpha(z, y) \\
&= \int_{X \times Y} f(gy) \, d(\mathrm{pr}_Y)_* \alpha(y) = \int_Y f(gy) \, d\eta(y) = g\eta(f).
\end{aligned}$$

Suppose now that $(X \times Y, \alpha_1)$ and $(X \times Y, \alpha_2)$ are both joinings such that pr_X is relatively measure-preserving. Fix $f \in L^\infty(Y, \eta)$ and let $F_1(x) = D^1_{\mathrm{pr}_X}(x)(f \circ \mathrm{pr}_Y)$ and $F_2(x) =$

$D^2_{\mathrm{pr}_X}(x)(f \circ \mathrm{pr}_Y)$ where $D^j_{\mathrm{pr}_X}$ is the disintegration of α_j over ν. Set $F(x) = F_1(x) - F_2(x)$. Then F is a bounded Borel function on X and by the above we have that $g\nu(F) = g\nu(F_1) - g\nu(F_2) = g\eta(f) - g\eta(f) = 0$ for all $g \in G$. Since (X, ν) is contractive we also know that $\|F\|_{L^\infty(X,\nu)} = \sup_g |g\nu(F)| = 0$. Therefore $F(x) = 0$ almost surely and so $F_1(x) = F_2(x)$ almost surely. As this holds for all $f \in L^\infty(Y, \eta)$ we conclude that

$$(\mathrm{pr}_Y)_* D^1_{\mathrm{pr}_X}(x) = (\mathrm{pr}_Y)_* D^2_{\mathrm{pr}_X}(x)$$

for almost every $x \in X$. The conclusion now follows from the next Lemma applied to $\mathrm{pr}_X \times \mathrm{pr}_Y$. $\square$

Lemma 4.10.1. *Let (X, ν_1), (Y, η) and (Z, ζ) be probability spaces and let $\nu_2 \in P(X)$. Let $\pi : X \to Y$ and $\varphi : X \to Z$ be measurable maps defined almost everywhere such that $\pi_* \nu_j = \eta$ for both $j = 1, 2$. Let D^j_π denote the disintegration of ν_j over η. Define the map $\pi \times \varphi : X \to Y \times Z$ by $(\pi \times \varphi)(x) = (\pi(x), \varphi(x))$. If $\varphi_* D^1_\pi(y) = \varphi_* D^2_\pi(y)$ for almost every y then $(\pi \times \varphi)_* \nu_1 = (\pi \times \varphi)_* \nu_2$.*

Proof. Observe that for any f a bounded Borel function on $Y \times Z$,

$$
\begin{aligned}
(\pi \times \varphi)_* \nu_j(f) &= \int_X f(\pi(x), \varphi(x)) \, d\nu_j(x) \\
&= \int_Y \int_X f(\pi(x), \varphi(x)) \, dD^j_\pi(y)(x) \, d\eta(y) \\
&= \int_Y \int_X f(y, \varphi(x)) \, dD^j_\pi(y)(x) \, d\eta(y) \\
&= \int_Y \int_Z f(y, z) \, d\varphi_* D^j_\pi(y)(z) \, d\eta(y)
\end{aligned}
$$

and therefore, by the hypothesis that $\varphi_* D^1_\pi = \varphi_* D^2_\pi$, we have that $(\pi \times \varphi)_* \nu_1(f) = (\pi \times \varphi)_* \nu_2(f)$ for all f which means that $(\pi \times \varphi)_* \nu_1 = (\pi \times \varphi)_* \nu_2$. $\square$

Corollary 4.30. *Let (X, ν) be a contractive G-space and let (Y, η) be a measure-preserving G-space. Then any joining of (X, ν) and (Y, η) such that pr_X is relatively measure-preserving is the independent joining.*

Proof. Observe that the independent joining $(X \times Y, \nu \times \eta)$ is a joining and that $D_{\mathrm{pr}_X}(x) = \delta_x \times \eta$. Since (Y, η) is measure-preserving,

$$D_{\mathrm{pr}_X}(gx) = \delta_{gx} \times \eta = (g\delta_x) \times \eta = (g\delta_x) \times (g\eta) = g(\delta_x \times \eta) = gD_{\mathrm{pr}_X}(x)$$

so pr_X is relatively measure-preserving. By the previous theorem then the independent joining is the unique such joining. $\square$

Corollary 4.31. *Let (X, ν) be a G-space such that $\pi : (X, \nu) \to (Y, \eta)$ is a relatively measure-preserving G-map of G-spaces and $\varphi : (X, \nu) \to (Z, \zeta)$ is a relatively contractive G-map of G-spaces where (Y, η) is a contractive G-space and (Z, ζ) is a measure-preserving G-space. Then (X, ν) is G-isomorphic to $(Y \times Z, \eta \times \zeta)$.*

Proof. By Corollary 4.26 the map $\pi \times \varphi$ is a G-isomorphism of (X, ν) with $(Y \times Z, (\pi \times \varphi)_* \nu)$. Now $(\mathrm{pr}_Y)_*(\pi \times \varphi)_* \nu = \pi_* \nu = \eta$ and likewise $(\mathrm{pr}_Z)_*(\pi \times \varphi)_* \nu = \zeta$ so this is a joining of (Y, η) and (Z, ζ). Since π is relatively measure-preserving and $\pi = \mathrm{pr}_X \circ (\pi \times \varphi)$ we have that pr_X is relatively measure-preserving. The previous corollary then says that it is the independent joining. $\qquad\square$

Corollary 4.32. *Let (X, ν) be a contractive G-space and $\pi : (X, \nu) \to (Y, \eta)$ a G-map of G-spaces. Then the only joining of (X, ν) and (Y, η) such that the projection to X is relatively measure-preserving is the joining $(X \times Y, \tilde{\pi}_* \nu)$ where $\tilde{\pi}(x) = (x, \pi(x))$.*

Proof. Let $D(x)$ be the disintegration of $\tilde{\pi}_* \nu$ over ν. Then $D(x)$ is supported on $\{x\} \times Y \cap$ supp $\tilde{\pi}_* \nu = \{(x, \pi(x))\}$. Therefore $D(x) = \delta_{(x, \pi(x))}$. So $D(gx) = \delta_{(gx, \pi(gx))} = \delta_{g(x, \pi(x))} = g\delta_{(x, \pi(x))} = gD(x)$. By the previous theorem this is then the unique joining with projection to X being relatively measure-preserving. $\qquad\square$

More generally:

Theorem 4.33. *Let (X, ν) be a contractive G-space and $\pi : (X, \nu) \to (Y, \eta)$ a G-map of G-spaces. Let $\zeta \in P(X \times Y)$ be a joining of (X, ν) and (Y, η') for some η' absolutely continuous with respect to η such that the projection to X of ζ to ν is relatively measure-preserving. Then $\zeta = \tilde{\pi}_* \nu$ where $\tilde{\pi}(x) = (x, \pi(x))$ and in particular, $\eta' = \eta$.*

Proof. Let D be the disintegration of ζ over ν. Then $D(x) = \delta_x \times \zeta_x$ for some $\zeta_x \in P(Y)$ for almost every x. Note that $D(gx) = gD(x)$ for $g \in G$ since the projection is relatively measure-preserving and therefore $\zeta_{gx} = g\zeta_x$ for all $g \in G$. Let $f \in C(Y)$. Define $F \in L^\infty(X, \nu)$ by

$$F(x) = f(\pi(x)) - \zeta_x(f).$$

Let $\epsilon > 0$ and take $x_0 \in X$ such that $|F(x_0)| > \|F\|_{L^\infty(X, \nu)} - \epsilon$. Since (X, ν) is contractive, there exists $g_n \in G$ such that $g_n \nu \to \delta_{x_0}$. Observe that, using that $\zeta_{gx} = g\zeta_x$,

$$
\begin{aligned}
g_n \nu(F) &= \int_X f(\pi(g_n x)) - \zeta_{g_n x}(f) \ d\nu(x) \\
&= \int_X f(g_n \pi(x)) - g_n \zeta_x(f) \ d\nu(x) \\
&= g_n \eta(f) - g_n \eta'(f)
\end{aligned}
$$

since $\int_X \zeta_x \ d\nu(x) = \eta'$.

Now η' is absolutely continuous with respect to η and $g_n \eta = \pi_* g_n \nu \to \pi_* \delta_{x_0} = \delta_{\pi(x_0)}$. Since (Y, η) is contractive, being a factor of a contractive space, by Corollary 4.28 (the proof of which goes through even when η' is only absolutely continuous with respect to and not necessarily in the same measure class as η), $g_n \eta' \to \delta_{\pi(x_0)}$ also. Therefore

$$g_n \nu(F) = g_n \eta(f) - g_n \eta'(f) \to f(\pi(x_0)) - f(\pi(x_0)) = 0$$

since $f \in C(Y)$. So we have that $\|F\| < \epsilon$. This holds for all $\epsilon > 0$ so $F(x) = 0$ almost surely. As this holds for all $f \in C(Y)$ we then have that $\zeta_x = \delta_{\pi(x)}$ almost surely. This means that $D(x) = \delta_x \times \delta_{\pi(x)} = \delta_{\tilde{\pi}(x)}$ almost surely so $\zeta = \tilde{\pi}_* \nu$ as claimed. Since $\mathrm{proj}_Y \tilde{\pi}_* \nu = \pi_* \nu = \eta$, then $\eta' = \mathrm{proj}_Y \zeta = \eta$. $\qquad\square$

We also obtain a special case of a result of Furstenberg and Glasner. Proposition 3.1 in [FG10] states that there is a unique stationary joining between a G-boundary and an arbitrary G-space; we obtain another proof of this fact when the G-space is measure-preserving:

Corollary 4.34 (Furstenberg-Glasner [FG10]). *Let G be a group and $\mu \in P(G)$ a probability measure on G. Let (B, β) be the (G, μ)-boundary and (X, ν) a measure-preserving G-space. Then the only joining $(B \times X, \alpha)$ of (B, β) and (X, ν) such that $\mu * \alpha = \alpha$ is the independent joining.*

Proof. Let $\pi : G^{\mathbb{N}} \to B$ be the boundary map (see [BS06] section 2), meaning that $\beta_\omega = \lim_n \omega_1 \cdots \omega_n \beta = \delta_{\pi(\omega)}$ $\mu^{\mathbb{N}}$-almost surely and $\pi_* \mu^{\mathbb{N}} = \beta$. Since α is μ-stationary, $\alpha = \int \alpha_\omega \, d\mu^{\mathbb{N}}(\omega)$. Now $(\mathrm{proj}_B)_* \alpha_\omega = \beta_\omega = \delta_{\pi(\omega)}$ and $(\mathrm{proj}_X)_* \alpha_\omega = \nu_\omega = \nu$ since (X, ν) is measure-preserving. Therefore $\alpha_\omega = \delta_{\pi(\omega)} \times \nu$ and since $\pi_* \mu^{\mathbb{N}} = \beta$ then the disintegration of α over β is $D(b) = \delta_b \times \nu$ which is G-equivariant. Hence the projection to B is relatively measure-preserving so the claim follows by the previous corollaries. $\square$

For completeness, we point out an example showing that there can in fact be no joining at all with the projection to the contractive factor being relatively measure-preserving. A concrete example of this can be found in the case when $G = \mathrm{PSL}_3(\mathbb{R})$ in which case the boundaries (under any admissible measure) are of the form G/P, 0 and G/Q where Q is a parabolic subgroup containing P (see Furstenberg's work [Fur63] for details).

Let (Y, η) be G/P, the Poisson boundary of G, let (X, ν) be G/Q, a proper factor of Y and let $\pi : (Y, \eta) \to (X, \nu)$ be the natural factor map. Suppose that $(X \times Y, \alpha)$ is a joining of (X, ν) and (Y, η) such that the projection to X is relatively measure-preserving. Take continuous compact models for the spaces and let $y_0 \in Y$ be arbitrary and set $x_0 = \pi(y_0)$. Since (Y, η) is contractive there exists $g_n \in G$ such that $g_n \eta \to \delta_{y_0}$ and therefore $g_n \nu = \pi_* g_n \eta \to \pi_* \delta_{y_0} = \delta_{x_0}$.

Since the projection to X is relatively measure-preserving,

$$ g_n \alpha = \int_X g_n D_{\mathrm{pr}_X}(x) \, d\nu(x) = \int_X D_{\mathrm{pr}_X}(x) \, dg_n \nu(x) \to D_{\mathrm{pr}_X}(x_0) $$

but on the other hand

$$ (\mathrm{pr}_Y)_* g_n \alpha = g_n \eta \to \delta_{y_0} $$

and therefore $(\mathrm{pr}_Y)_* D_{\mathrm{pr}_X}(x_0) = \delta_{\pi(x_0)}$ meaning that $D_{\mathrm{pr}_X}(x_0) = \delta_{x_0} \times \delta_{\pi(x_0)}$. This yields a map $X \to Y$ which is clearly G-equivariant forcing Y to be a factor of X but this is ruled out by the choice of X and Y.

4.11   The Operator Algebraic Formulation

We pause here to note that properties of relatively contractive maps can be alternatively formulated in the language of operator algebras. Thus, there may be similar phenomena occurring in the noncommutative setting. For example, we can restate Theorem 4.33 as:

Theorem 4.35. *Let (X, ν) be a contractive G-space. If $B \subseteq L^\infty(X, \nu)$ is a G-invariant von Neumann subalgebra and $\Phi : B \to L^\infty(X, \nu)$ is a G-equivariant normal unital (completely) positive map then $\Phi = \mathrm{id}$.*

The above should be compared to Proposition 3 in [Oza07]. The other results in this section can be similarly rephrased; this is another indication of the greater flexibility allowed by contractive actions over boundary actions: the operator algebraic formulations are much more straightforward and easier to work with.

4.12   Relatively Contractive Maps and Finite Index Subgroups

Theorem 4.36. *Let G be a locally compact second countable group and $H < G$ be a finite index subgroup. Let $\pi : (X, \nu) \to (Y, \eta)$ be a relatively contractive G-map of ergodic G-spaces. Then, restricting the actions to H makes π a relatively contractive H-map.*

Proof. Fix continuous compact models of X, Y and π. Let $\ell_1, \ell_2, \ldots, \ell_N$ be a system of representatives for $H \backslash G$. Define the set

$$Q = \{x \in X : \text{ there exists } \{h_n\} \text{ in } H \text{ such that } D_\pi^{(h_n)}(\pi(x)) \to \delta_x \}.$$

By Theorem 4.12, for every $x \in X$ there exists $\{g_n\}$ in G such that $D_\pi^{(g_n)}(\pi(x)) \to \delta_x$. Write $g_n = h_n \ell_{j_n}$ for $h_n \in H$ and $j_n \in \{1, 2, \ldots, N\}$. Since N is finite, there exists a subsequence $\{n_t\}$ along which $g_{n_t} = h_{n_t} \ell$ for some fixed ℓ in the system of representatives. Define the sets, for $j = 1, 2, \ldots, N$,

$$C_j = \{x \in X : \text{ there exists } \{g_n\} \text{ such that } g_n \in H\ell_j \text{ for all } n \text{ and } D_\pi^{(g_n)}(y) \to \delta_x \}.$$

Then $X = \cup_{j=1}^N C_j$ by the above.

Let $x \in C_j$. Then, writing $g_n = h_n \ell_j$, it holds that $\ell_j^{-1} D_\pi^{(h_n)}(\ell_j \pi(x)) = D_\pi^{(h_n \ell_j)}(\pi(x)) \to \delta_x$ and so $D_\pi^{(h_n)}(\pi(\ell_j x)) \to \ell_j \delta_x = \delta_{\ell_j x}$. Therefore $\ell_j x \in Q$ and so we have that

$$\bigcup_{j=1}^N \ell_j C_j \subseteq Q.$$

Since $X = \cup_j C_j$, there is some j such that $\nu(C_j) > 0$. So $\nu(Q) \geq \nu(\ell_j C_j) > 0$ as ν is quasi-invariant.

Now let $x \in Q$ and $h \in H$. There exists $\{h_n\}$ in H such that $D_\pi^{(h_n)}(\pi(x)) \to \delta_x$ and therefore

$$D_\pi^{(h_n h^{-1})}(\pi(hx)) = h D_\pi^{(h_n)}(h^{-1}\pi(hx)) = h D_\pi^{(h_n)}(\pi(x)) \to h\delta_x = \delta_{hx}$$

meaning that $hx \in Q$. Since (X, ν) is G-ergodic, it is also H-ergodic (H being finite index) and therefore $\nu(Q) = 1$ meaning precisely that π is a relatively contractive H-map. $\square$

4.13   Contractive Actions and Lattices

The following is a generalization of Proposition 3.7 in [CS12] (which shows the same result only for Poisson boundaries):

Theorem 4.37. *Let G be a locally compact second countable group and $\Gamma < G$ a lattice. Let (X, ν) be a contractive (G, μ)-space for some $\mu \in P(G)$ such that the support of μ generates G. Then the restriction of the G-action to Γ makes (X, ν) a contractive Γ-space.*

Proof. Let $B \subseteq X$ be a measurable set such that $\nu(B) < 1$. Let U be an open neighborhood of the identity in G such that $\nu(UB) < 1$ where $UB = \{ux : u \in U, x \in B\}$ (the existence of such an open set is a standard consequence of the continuity and the proof is left to the reader). Let $\epsilon > 0$ such that $m(U\Gamma) > \epsilon$ where m is the Haar probability measure on G/Γ. Note that $m(U\Gamma) > 0$ since U is open. Since (X, ν) is contractive, there exists $g \in G$ such that $\nu(gUB) < \epsilon^2$.

By the Random Ergodic Theorem (Theorem 2.25), there exists $Q_0 \subseteq G^{\mathbb{N}}$ with $\mu^{\mathbb{N}}(Q_0) = 1$ such that for all $(\omega_1, \omega_2, \ldots) \in Q_0$,

$$\lim_{N \to \infty} \int \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{gUB}(\omega_n \cdots \omega_1 x) \, d\nu(x) = \nu(gUB).$$

So for all $(\omega_1, \ldots) \in Q_0$,

$$\frac{1}{N} \sum_{n=1}^{N} \nu(\omega_1^{-1} \cdots \omega_n^{-1} gUB) \to \nu(gUB) < \epsilon^2.$$

Therefore, for all $(\omega_1, \ldots) \in Q_0$,

$$\limsup_{N \to \infty} \frac{1}{N} \#\{n \leq N : \nu(\omega_1^{-1} \cdots \omega_n^{-1} gUB) > \epsilon\} \leq \epsilon.$$

Also by the Random Ergodic Theorem, there exists $Q_1 \subseteq G^{\mathbb{N}}$ with $\mu^{\mathbb{N}}(Q_1) = 1$ such that for all $(\omega_1, \ldots) \in Q_1$ and almost every $y \in G/\Gamma$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{gU\Gamma}(\omega_n \cdots \omega_1 y) = m(gU\Gamma) = m(U\Gamma).$$

Therefore, for all $(\omega_1, \ldots) \in Q_1$ and almost every $y \in G/\Gamma$,

$$\liminf_{N \to \infty} \frac{1}{N} \#\{n \leq N : y \in \omega_1^{-1} \cdots \omega_n^{-1} gU\Gamma\} \geq m(U\Gamma).$$

And so we conclude that for all $(\omega_1, \ldots) \in Q_1$ and almost every $h \in G$,

$$\liminf_{N \to \infty} \frac{1}{N} \#\{n \leq N : \text{ there exists } \gamma \in \Gamma \text{ with } h\gamma \in \omega_1^{-1} \cdots \omega_n^{-1} gU \} \geq m(U\Gamma).$$

Combining these facts, for $(\omega_1, \ldots) \in Q_0 \cap Q_1$ and almost every $h \in G$,

$$\liminf_{N \to \infty} \frac{1}{N} \#\{n \leq N : \text{there exists } \gamma \in \Gamma \text{ with } h\gamma \in \omega_1^{-1} \cdots \omega_n^{-1} gU$$
$$\text{and } \nu(\omega_1^{-1} \cdots \omega_n^{-1} gUB) < \epsilon \} \geq m(U\Gamma) - \epsilon.$$

As $m(U\Gamma) > \epsilon$, the above sets are nonempty, so for almost every $h \in G$ there exists $\gamma \in \Gamma$, $u \in U$ and $\omega_1, \ldots, \omega_n \in G$ (for infinitely many choices of n) such that $h\gamma = \omega_1^{-1} \cdots \omega_n^{-1} gu$ and $\nu(\omega_1^{-1} \cdots \omega_n^{-1} gUB) < \epsilon$. Then

$$\nu(h\gamma B) = \nu(\omega_1^{-1} \cdots \omega_n^{-1} guB) \leq \nu(\omega_1^{-1} \cdots \omega_n^{-1} gUB) < \epsilon.$$

This holds for all $0 < \epsilon < m(U\Gamma)$ and so for almost every $h \in G$, there exists $\{\gamma_n\}$ in Γ such that $\nu(h\gamma_n B) \to 0$.

Fix such an $h \in G$ and let $\{\gamma_n\}$ such that $\nu(h\gamma_n B) \to 0$. For each $m \in \mathbb{N}$, let n_m such that $\nu(h\gamma_{n_m} B) < 2^{-m}$. Then

$$\nu\big(h \bigcup_{\ell=1}^{\infty} \bigcup_{m=\ell+1}^{\infty} \gamma_{n_m} B\big) \leq \lim_{\ell \to \infty} \sum_{m=\ell+1}^{\infty} \nu(h\gamma_{n_m} B) \leq \lim_{\ell \to \infty} \sum_{m=\ell+1}^{\infty} 2^{-m} = 0.$$

Since ν is quasi-invariant,

$$\limsup_{m \to \infty} \nu(\gamma_{n_m} B) \leq \limsup_{\ell \to \infty} \nu\big(\bigcup_{m=\ell+1}^{\infty} \gamma_{n_m} B\big) = \nu\big(\bigcap_{\ell=1}^{\infty} \bigcup_{m=\ell+1}^{\infty} \gamma_{n_m} B\big) = 0$$

meaning that (X, ν) is Γ-contractive. $\qquad\qquad\square$

4.14 Inducing Relatively Contractive Maps

Theorem 4.38. *Let $\Gamma < G$ be a lattice in a locally compact second countable group. Let $\pi : (X, \nu) \to (Y, \eta)$ be a Γ-map of Γ-spaces and let $\Pi : G \times_\Gamma X \to G \times_\Gamma Y$ be the induced G-map of G-spaces. Then π is a relatively contractive Γ-map if and only if Π is a relatively contractive G-map.*

Proof. Assume first that Π is relatively contractive. Fix a fundamental domain (F, m) for G/Γ as in the induced action construction (see Section 2.2.4) and let $\alpha : G \times F \to \Gamma$ be the associated cocycle for the G-action on $F \times X$. Let $\Phi : (F \times X, m \times \nu) \to (F \times Y, m \times \eta)$ by $\Phi = \mathrm{id} \times \pi$. Then Φ is isomorphic to Π over the canonical isomorphisms $G \times_\Gamma X \simeq F \times X$ and $G \times_\Gamma Y \simeq F \times Y$ (Section 2.2.5) so Φ is relatively contractive. Consider the disintegration map $D_\Phi : F \times Y \to P(F \times X)$. Observe that for $(f, y) \in F \times Y$

$$D_\Phi(f, y) = \delta_f \times D_\pi(y)$$

since $\Phi = \mathrm{id} \times \pi$ and all the spaces have the product measure. Now consider the conjugates of the disintegration map: for $g \in G$ and $(f, y) \in F \times Y$,

$$\begin{aligned}
D_\Phi^{(g)}(f, y) &= g^{-1} D_\Phi(g(f, y)) = g^{-1} D_\Phi(gf\alpha(g, f), \alpha(g, f)^{-1} y) \\
&= g^{-1}(\delta_{gf\alpha(g,f)} \times D_\pi(\alpha(g, f)^{-1} y)) \\
&= \delta_f \times \alpha(g^{-1}, gf\alpha(g, f))^{-1} D_\pi(\alpha(g, f)^{-1} y) \\
&= \delta_f \times D_\pi^{(\alpha(g,f)^{-1})}(y).
\end{aligned}$$

Now take $r \in L^\infty(X, \nu)$ and define $q(f, x) = r(x)$. Then for $m \times \eta$-almost every (f, y)

$$\|q\|_{L^\infty(F \times X, D_\Phi(f,y))} = \|r\|_{L^\infty(X, D_\pi(y))}$$

and since Φ is relatively contractive, for $m \times \eta$-almost every (f, y) there exists $g_n \in G$ such that

$$D_\Phi^{(g_n)}(f, y)(q) \to \|q\|_{L^\infty(F \times X, D_\Phi(f,y))}.$$

Therefore

$$\delta_f \times D_\pi^{(\alpha(g_n,f)^{-1})}(y)(q) \to \|r\|_{L^\infty(X,D_\pi(y))}$$

and by construction of q then

$$D_\pi^{(\alpha(g_n,f)^{-1})}(y)(r) \to \|r\|_{L^\infty(X,D_\pi(y))}.$$

Hence for η-almost every y there exists a sequence $\gamma_n = \alpha(g_n,f)^{-1} \in \Gamma$ (outside of possibly a measure zero set, which f is chosen is irrelevant) such that

$$D_\pi^{(\gamma_n)}(r) \to \|r\|_{L^\infty(X,D_\pi(y))}$$

which means that π is relatively contractive.

Now assume that π is relatively contractive. Let $x \in X$ and $f \in F$ and set $y = \pi(x)$. As above,

$$D_\Phi^{(g)}(f,y) = \delta_f \times D_\pi^{(\alpha(g,f)^{-1})}(y).$$

Since π is relatively contractive, there exists $\{\gamma_n\}$ such that $D_\pi^{(\gamma_n)}(y) \to \delta_x$. Set $g_n = f\gamma_n f^{-1}$. Then $\alpha(g_n,f) = \gamma_n^{-1}$ and so

$$D_\Phi^{(g_n)}(f,y) = \delta_f \times D_\pi^{(\gamma_n)}(y) \to \delta_{(f,x)}$$

meaning that Π is relatively contractive. $\qquad\square$

4.15   The Intermediate Contractive Factor Theorem

Theorem 4.39. *Let $\Gamma < G$ be a lattice in a locally compact second countable group and let Λ contain and commensurate Γ and be dense in G. Let (X,ν) be a contractive (G,μ)-space (for some $\mu \in P(G)$ such that the support of μ generates G) and (Y,η) be a measure-preserving G-space. Let $\pi : (X \times Y, \nu \times \eta) \to (Y,\eta)$ be the natural projection map from the product space with the diagonal action. Let (Z,ζ) be a Λ-space such that there exist Γ-maps $\varphi : (X \times Y, \nu \times \eta) \to (Z,\zeta)$ and $\rho : (Z,\zeta) \to (Y,\eta)$ with $\rho \circ \varphi = \pi$. Then φ and ρ are Λ-maps and (Z,ζ) is Λ-isomorphic to a G-space and over this isomorphism the maps φ and ρ become G-maps.*

Proof. Write $(W,\rho) = (X \times Y, \nu \times \eta)$. Fix $\lambda \in \Lambda$. Define the maps $\varphi_\lambda : W \to Z$ and $\rho_\lambda : Z \to Y$ by $\varphi_\lambda(w) = \lambda^{-1}\varphi(\lambda w)$ and $\rho_\lambda(z) = \lambda^{-1}\rho(\lambda z)$. Then $\rho_\lambda \circ \varphi_\lambda(w) = \lambda^{-1}\rho(\lambda\lambda^{-1}\varphi(\lambda w)) = \lambda^{-1}\rho(\varphi(\lambda w)) = \lambda^{-1}\pi(\lambda w) = \pi(w)$ since π is Λ-equivariant. Let $\Gamma_0 = \Gamma \cap \lambda^{-1}\Gamma\lambda$. Then for $\gamma_0 \in \Gamma_0$, write $\gamma_0 = \lambda^{-1}\gamma\lambda$ for some $\gamma \in \Gamma$ and we see that $\varphi_\lambda(\gamma_0 w) = \lambda^{-1}\varphi(\lambda\gamma_0 w) = \lambda^{-1}\varphi(\gamma\lambda w) = \lambda^{-1}\gamma\varphi(\lambda w) = \gamma_0\lambda^{-1}\varphi(\lambda w) = \gamma_0\varphi_\lambda(w)$ meaning that φ_λ is Γ_0-equivariant. Likewise ρ_λ is Γ_0-equivariant. Hence φ, φ_λ, ρ and ρ_λ are all Γ_0-equivariant.

Since (X,ν) is a contractive (G,μ)-space and Γ_0 is a lattice in G, by Theorem 4.37, (X,ν) is a contractive Γ_0-space. By Theorem 4.27 we can conclude that $\varphi_\lambda = \varphi$ and that $\rho_\lambda = \rho$ provided we can show that the disintegration measures $D_\rho(y)$ and $D_{\rho_\lambda}(y)$ are in the same measure class for almost every y. Assuming this for the moment, we then conclude that φ is Λ-equivariant since $\varphi_\lambda = \varphi$ for each λ. The σ-algebra of pullbacks of measurable functions on (Z,ζ) form a Λ-invariant sub-σ-algebra of $L^\infty(W,\rho)$ which is therefore also G-invariant

(because Λ is dense in G) and so (Z, ζ) has a point realization as a G-space [Mac62] and likewise φ and ρ as G-maps.

It remains only to show that the disintegration measures have the required property. First note that $D_\rho(y) = \varphi_* D_{\rho \circ \varphi}(y)$ by the uniqueness of the disintegration measure and likewise that $D_{\rho_\lambda}(y) = (\varphi_\lambda)_* D_{\rho_\lambda \circ \varphi_\lambda}(y) = \lambda^{-1} \varphi_* \lambda D_{\rho \circ \varphi}(y) = \lambda^{-1} \varphi_* D_{\rho \circ \varphi}^{(\lambda^{-1})}(\lambda y)$. Now $\rho \circ \varphi = \pi$ is a Λ-map so $D_{\rho \circ \varphi}^{(\lambda^{-1})}(\lambda y)$ is in the same measure class as $D_{\rho \circ \varphi}(\lambda y)$. Therefore $D_{\rho_\lambda}(y)$ is in the same measure class as $\lambda^{-1} \varphi_* D_{\rho \circ \varphi}(\lambda y) = \lambda^{-1} D_\rho(\lambda y)$. Now $\lambda^{-1} D_\rho(\lambda y)$ disintegrates $\lambda^{-1} \zeta$ over $\lambda^{-1} \eta$ via ρ and $\lambda^{-1} \zeta$ is in the same measure class as ζ since (Z, ζ) is a Λ-space. Therefore, by Lemma 4.1.4, $\lambda^{-1} D_\rho(\lambda y)$ and $D_\rho(y)$ are in the same measure class for almost every y. Hence $D_{\rho_\lambda}(y)$ and $D_\rho(y)$ are in the same measure class for almost every y as needed. $\square$

4.15.1   The Original Contractive Factor Theorem

As a corollary we obtain a slightly improved form of the Contractive Factor Theorem:

Corollary 4.40 (Creutz-Shalom [CS12])**.** *Let $\Gamma < G$ be a lattice in a locally compact second countable group and let Λ contain and commensurate Γ and be dense in G. Let (X, ν) be a contractive (G, μ)-space (for some $\mu \in P(G)$ such that the support of μ generates G) and $\pi : (X, \nu) \to (Y, \eta)$ a Γ-map to a Λ-space. Then π is a Λ-map, (Y, η) is Λ-isomorphic to a G-space and over this isomorphism π is a G-map.*

Proof. As usual, take the relatively contractive map from (X, ν) to the one-point system and apply the Intermediate Contractive Factor Theorem. $\square$

4.15.2   The Piecewise Intermediate Contractive Factor Theorem

For our study of stabilizers, we will need a slightly stronger form of the Intermediate Contractive Factor Theorem:

Theorem 4.41. *Let Γ be a group and let Λ be a group that contains and commensurates Γ. Let (W, ρ) be a Λ-space such that the action restricted to Γ on (W, ρ) is contractive and let (X, ν) be a measure-preserving Λ-space. Set $(Y, \eta) = (W \times X, \rho \times \nu)$ to be the product space with the diagonal action. Let $p : (Y, \eta) \to (X, \nu)$ be the natural projection map. Let (Z, ζ) be a Γ-space and $\pi : (Y, \eta) \to (Z, \zeta)$ and $\varphi : (Z, \zeta) \to (X, \nu)$ be Γ-maps such that $\varphi \circ \pi = p$.*

Assume that Z is orbital over X: for any $\gamma \in \Gamma$ and $x \in X$ such that $\gamma x = x$, if $z \in Z$ such that $\varphi(z) = x$ then $\gamma z = z$.

Fix $\lambda \in \Lambda$, define the Borel set

$$E = E_\lambda = \{x \in X : \lambda x \in \Gamma x\},$$

and define the map $\theta_\lambda : \varphi^{-1}(E) \to Z$ as follows: for $z \in \varphi^{-1}(E)$ choose $\gamma \in \Gamma$ such that $\lambda \varphi(z) = \gamma \varphi(z)$ and define $\theta_\lambda(z) = \gamma z$ (this is well-defined since Z is orbital).

Then $\pi(\lambda y) = \theta_\lambda(\pi(y))$ for almost every $y \in p^{-1}(E)$. In particular, for almost every y such that $\lambda p(y) = p(y)$ we have that $\pi(\lambda y) = \pi(y)$.

The proof of the theorem will proceed as a series of Propositions. Retain the notation above throughout:

Proposition 4.15.1. $\theta_\lambda(\varphi^{-1}(E)) = \varphi^{-1}(\lambda E)$.

Proof. Let $z \in \theta_\lambda(\varphi^{-1}(E))$. Then $z = \theta_\lambda(w)$ for some $w \in \varphi^{-1}(E)$ so $\lambda\varphi(w) = \gamma\varphi(w)$ for some $\gamma \in \Gamma$ hence $z = \gamma w$ by the definition of θ_λ. Then $\varphi(z) = \varphi(\gamma w) = \gamma\varphi(w) = \lambda\varphi(w) \in \lambda E$. Therefore $\theta_\lambda(\varphi^{-1}(E)) \subseteq \varphi^{-1}(\lambda E)$.

Conversely, let $z \in \varphi^{-1}(\lambda E)$. Then $\varphi(z) = \lambda x$ for some $x \in E$ so there exists $\gamma \in \Gamma$ such that $\varphi(z) = \lambda x = \gamma x$. Now $\varphi(\gamma^{-1}z) = \gamma^{-1}\varphi(z) = x \in E$ and $\lambda x = \gamma x$ so $\theta_\lambda(\gamma^{-1}z) = \gamma(\gamma^{-1}z) = z$. Therefore $z = \theta_\lambda(\gamma^{-1}z) \in \theta_\lambda(\varphi^{-1}(E))$ so $\varphi^{-1}(\lambda E) \subseteq \theta_\lambda(\varphi^{-1}(E))$. $\qquad\square$

Proposition 4.15.2. θ_λ *is invertible: there exists* $\theta_\lambda^{-1} : \theta_\lambda(\varphi^{-1}(E)) \to E$ *such that* $\theta_\lambda^{-1}\theta_\lambda$ *is the identity on* $\varphi^{-1}(E)$ *and* $\theta_\lambda\theta_\lambda^{-1}$ *is the identity on* $\theta_\lambda(\varphi^{-1}(E))$.

Proof. Let $w \in \theta_\lambda(\varphi^{-1}(E))$. Then $w = \theta_\lambda(z)$ for some $z \in \varphi^{-1}(E)$ so $w = \gamma z$ for some $\gamma \in \Gamma$ such that $\lambda\varphi(z) = \gamma\varphi(z)$. Note that if $\gamma, \gamma' \in \Gamma$ are both such that $\lambda\varphi(z) = \gamma\varphi(z) = \gamma'\varphi(z)$ then $\gamma^{-1}\gamma'\varphi(z) = \varphi(z)$ so as Z is orbital then $\gamma^{-1}\gamma'z = z$. Define $\theta_\lambda^{-1}(w) = \gamma^{-1}w$. This is then well-defined since $(\gamma')^{-1}w = (\gamma')^{-1}\gamma\gamma^{-1}w = (\gamma')^{-1}\gamma z = z = \gamma^{-1}w$ because $\gamma^{-1}\gamma'z = z$. Then $\theta_\lambda^{-1}(\theta_\lambda(z)) = \theta_\lambda^{-1}(w) = z$ and $\theta_\lambda(\theta_\lambda^{-1}(w)) = \theta_\lambda(z) = w$ hence the proof is complete (since θ_λ maps onto its image). $\qquad\square$

Proposition 4.15.3. $\Gamma_0 = \Gamma \cap \lambda^{-1}\Gamma\lambda$ *is a lattice in* Γ *and* E *is* Γ_0-*invariant*.

Proof. Γ_0 has finite index in Γ since Λ commensurates Γ hence is a lattice. Observe that for $\gamma_0 \in \Gamma_0$ and $x \in E$, writing $\gamma_0 = \lambda^{-1}\gamma\lambda$ for some $\gamma \in \Gamma$ we have that

$$\lambda\gamma_0 x = \lambda\lambda^{-1}\gamma\lambda x = \gamma\lambda x \in \gamma\Gamma x = \Gamma x$$

and therefore the set E is Γ_0-invariant, that is $\lambda\gamma_0 x \in \Gamma x$ whenever $\lambda x \in \Gamma x$. $\qquad\square$

Proposition 4.15.4. *Define the map* $\pi_\lambda : Y \to Z$ *as follows: for* $y \in p^{-1}(E)$ *set* $\pi_\lambda(y) = \theta_\lambda^{-1}(\pi(\lambda y))$ *and for* $y \notin p^{-1}(E)$ *set* $\pi_\lambda(y) = \pi(y)$. *Likewise define the map* $\varphi_\lambda : Z \to X$ *by* $\varphi_\lambda(z) = \lambda^{-1}\varphi(\theta_\lambda(z))$ *for* z *such that* $\varphi(z) \in E$ *and* $\varphi_\lambda(z) = \varphi(z)$ *for* z *such that* $\varphi(z) \notin E$.
Then $\varphi_\lambda \circ \pi_\lambda = \varphi \circ \pi = p$ *and both* π_λ *and* φ_λ *are* Γ_0-*equivariant*.

Proof. Note that in fact $\varphi_\lambda = \varphi$ since for $z \in \varphi^{-1}(E)$ and $\gamma \in \Gamma$ such that $\lambda\varphi(z) = \gamma\varphi(z)$ we have that $\lambda^{-1}\varphi(\gamma z) = \lambda^{-1}\gamma\varphi(z) = \varphi(z)$ but we will find it helpful to distinguish these maps since the measures $\pi_*\eta$ and $(\pi_\lambda)_*\eta$ may be distinct and we will be treating φ_λ as a map $(Z, (\pi_\lambda)_*\eta) \to (X, \nu)$ and φ as a map $(Z, \pi_*\eta) \to (X, \nu)$.

Now for y such that $p(y) \in E$, observe that

$$\varphi_\lambda(\pi_\lambda(y)) = \lambda^{-1}\varphi(\theta_\lambda\theta_\lambda^{-1}\pi(\lambda y)) = \lambda^{-1}\varphi(\pi(\lambda y)) = \lambda^{-1}p(\lambda y) = p(y)$$

since p is Λ-equivariant. Clearly for y such that $p(y) \notin E$ we have $\varphi_\lambda(\pi_\lambda(y)) = \varphi_\lambda(\pi(y)) = \varphi(\pi(y)) = p(y)$. Hence $\varphi_\lambda \circ \pi_\lambda = p$.

Observe that for $\gamma_0 \in \Gamma_0$, writing $\gamma_0 = \lambda^{-1}\gamma\lambda$ for some $\gamma \in \Gamma$, we have that for y such that $p(y) \in E$, also $p(\gamma_0 y) = \gamma_0 p(y) \in E$ since E is Γ_0-invariant, so

$$\pi_\lambda(\gamma_0 y) = \theta_\lambda^{-1}\pi(\lambda\gamma_0 y) = \theta_\lambda^{-1}\pi(\gamma\lambda y) = \theta_\lambda^{-1}\gamma\pi(\lambda y) = \theta_\lambda^{-1}\gamma\theta_\lambda\theta_\lambda^{-1}\pi(\lambda y) = \theta_\lambda^{-1}\gamma\theta_\lambda\pi_\lambda(y).$$

Now observe that for z such that $\varphi(z) \in E$ (which includes $\pi_\lambda(y)$ for $p(y) \in E$), write $\gamma' \in \Gamma$ such that $\theta_\lambda z = \gamma' z$ and observe that then $\lambda \varphi(z) = \gamma' \varphi(z)$ and so

$$\gamma \gamma' \gamma_0^{-1} \varphi(\gamma_0 z) = \gamma \gamma' \varphi(z) = \gamma \lambda \varphi(z) = \lambda \gamma_0 \varphi(z) = \lambda \varphi(\gamma_0 z)$$

which in turn means that

$$\theta_\lambda(\gamma_0 z) = \gamma \gamma' \gamma_0^{-1}(\gamma_0 z) = \gamma \gamma' z = \gamma \theta_\lambda(z)$$

and therefore

$$\pi_\lambda(\gamma_0 y) = \theta_\lambda^{-1} \gamma \theta_\lambda \pi_\lambda(y) = \theta_\lambda^{-1} \theta_\lambda \gamma_0 \pi_\lambda(y) = \gamma_0 \pi_\lambda(y).$$

Of course, for y such that $p(y) \notin E$ we have that $\gamma_0 y \notin E$ and so

$$\pi_\lambda(\gamma_0 y) = \pi(\gamma_0 y) = \gamma_0 \pi(y) = \gamma_0 \pi_\lambda(y)$$

and we conclude that π_λ is Γ_0-equivariant. Note that $\varphi_\lambda = \varphi$ so φ_λ is likewise Γ_0-equivariant. $\square$

Proposition 4.15.5. *The maps $\pi, \varphi, \pi_\lambda, \varphi_\lambda$ are all relatively contractive Γ_0-maps.*

Proof. p is a relatively contractive Γ-map hence is a relatively contractive Γ_0-map since Γ_0 has finite index in Γ (Theorem 4.36). Since $\varphi_\lambda \circ \pi_\lambda = \varphi \circ \pi = p$ then the maps are all relatively contractive (Theorems 4.21 and 4.15). $\square$

Proposition 4.15.6. *ζ_λ is in the same measure class as ζ.*

Proof. Let $B \subseteq Z$ be measurable such that $B \cap \varphi^{-1}(E) = \emptyset$. Then $\pi^{-1}(B) \cap p^{-1}(E) = \pi^{-1}(B \cap \varphi^{-1}(E)) = \emptyset$ and $\pi_\lambda^{-1}(B) \cap p^{-1}(E) = \pi_\lambda^{-1}(B \cap \varphi_\lambda^{-1}(E)) = \emptyset$ since $\varphi = \varphi_\lambda$ pointwise. So $\pi_\lambda(y) = \pi(y)$ for $y \in \pi^{-1}(B)$ and for $y \in \pi_\lambda^{-1}(B)$. Then

$$\zeta_\lambda(B) = \eta(\pi_\lambda^{-1}(B)) \leq \eta(\pi_\lambda^{-1}(\pi(\pi^{-1}(B)))) = \eta(\pi^{-1}(B)) = \zeta(B)$$

and likewise

$$\zeta(B) = \eta(\pi^{-1}(B)) \leq \eta(\pi^{-1}(\pi_\lambda(\pi_\lambda^{-1}(B)))) = \eta(\pi_\lambda^{-1}(B)) = \zeta_\lambda(B)$$

hence $\zeta(B) = \zeta_\lambda(B)$ for $B \subseteq \varphi^{-1}(E^C)$.

Now let $B \subseteq Z$ be measurable such that $B \subseteq \varphi^{-1}(E)$. For $x \in E$, measurably choose $\gamma_x \in \Gamma$ such that $\lambda x = \gamma_x x$. Write $F_\gamma = \{x \in E : \gamma_x = \gamma\}$. Define the disjoint sets

$$B_\gamma = B \cap \varphi^{-1}(B_\gamma).$$

Then $\theta_\lambda(B_\gamma) = \gamma B_\gamma$ by the definition of θ_λ.

Suppose first that $\zeta(B) = 0$ but that $\zeta_\lambda(B) > 0$. Then

$$0 < \zeta_\lambda(B) = \eta(\lambda^{-1} \pi^{-1}(\theta_\lambda(B))) = \lambda \eta(\pi^{-1}(\theta_\lambda(B)))$$

so, since η is in the same measure class as $\lambda \eta$,

$$0 < \eta(\pi^{-1}(\theta_\lambda(B))) = \zeta(\theta_\lambda(B)).$$

Now

$$\zeta(\theta_\lambda(B)) = \zeta(\theta_\lambda(\bigsqcup_\gamma B_\gamma)) = \sum_\gamma \zeta(\gamma B_\gamma) = \sum_\gamma \gamma^{-1}\zeta(B_\gamma)$$

and therefore there exists $\gamma \in \Gamma$ such that $\gamma^{-1}\zeta(B_\gamma) > 0$. Since ζ is Γ-quasi-invariant then $\zeta(B_\gamma) > 0$ for some $\gamma \in \Gamma$. But then $\zeta(B) \geq \zeta(B_\gamma) > 0$ contradicting that $\zeta(B) = 0$.

Suppose now that $\zeta(B) > 0$ but that $\zeta_\lambda(B) = 0$. Observe that

$$\zeta_\lambda(B) = (\pi_\lambda)_*\eta(B) = \eta(\lambda^{-1}\pi^{-1}(\theta_\lambda(B)))$$
$$= \lambda\eta(\pi^{-1}(\theta_\lambda(\bigsqcup_\gamma B_\gamma))) = \sum_\gamma \gamma^{-1}\lambda\eta(\pi^{-1}(B_\gamma))$$

and therefore $\gamma^{-1}\lambda\eta(\pi^{-1}(B_\gamma)) = 0$ for all $\gamma \in \Gamma$. By the Λ-quasi-invariance of η, then $\eta(\pi^{-1}(B_\gamma)) = 0$ for all $\gamma \in \Gamma$. But then

$$\zeta(B) = \eta(\pi^{-1}(B)) = \eta(\bigsqcup_\gamma \pi^{-1}(B_\gamma)) = 0$$

contradicting that $\zeta(B) > 0$. $\square$

Proof of Theorem 4.41. We are now in the situation of having $\pi : (Y, \eta) \to (Z, \zeta)$, $\varphi : (Z, \zeta) \to (X, \nu)$, $\pi_\lambda : (Y, \eta) \to (Z, \zeta_\lambda)$ and $\varphi_\lambda : (Z, \zeta_\lambda) \to (X, \nu)$ all Γ_0-maps of Γ_0-spaces such that $\varphi \circ \pi = \varphi_\lambda \circ \pi_\lambda = p$ is a relatively contractive Γ_0-map and such that the disintegration measures $D_\varphi(x)$ and $D_{\varphi_\lambda}(x)$ are in the same measure class for almost every x (which follows from the previous proposition and Lemma 4.1.4). By Theorem 4.27, as (Y, η) is a product of a contractive space and a measure-preserving space, then $\pi = \pi_\lambda$ almost surely and $\zeta_\lambda = \zeta$.

Therefore for almost every y such that $p(y) \in E$ we have that

$$\pi(\lambda y) = \theta_\lambda \theta_\lambda^{-1}\pi(\lambda y) = \theta_\lambda \pi_\lambda(y) = \theta_\lambda \pi(y).$$

$\square$

5   Weak Amenability of Actions of Lattices

A key fact in our study of stabilizers of commensurators and lattices is that if an action of the commensurator has infinite stabilizers then the restriction of the action to the lattice is weakly amenable (the equivalence relation corresponding to the action of the lattice is amenable):

Theorem 5.1. *Let $\Gamma < G$ be a lattice in a locally compact second countable group and let Λ be a countable dense subgroup of G such that $\Gamma <_c \Lambda$.*

Assume that for every ergodic measure-preserving action of G either the restriction of the action to Λ has finite stabilizers or the restriction of the action to Γ has finite orbits.

Let $\Lambda \curvearrowright (X, \nu)$ be an ergodic measure-preserving action. Then either Λ has finite stabilizers or the restriction of the action to Γ is weakly amenable.

Proof. Let (B, ρ) be any Poisson boundary of G with respect to a measure whose support generates G. Then $G \curvearrowright (B, \rho)$ is contractive and amenable (Zimmer [Zim84]; reproduced as Proposition 2.4.2). Then $\Gamma \curvearrowright (B, \rho)$ amenably since Γ is closed in G (Zimmer [Zim84]; reproduced as Proposition 2.4.3). Let A be an affine orbital Γ-space over (X, ν). Then there exists Γ-maps $\pi : B \times X \to A$ and $p : A \to X$ such that $p \circ \pi$ is the natural projection to X (Zimmer [Zim84]; reproduced as Proposition 2.4.5). Since (B, ρ) is a stationary space, being a Poisson boundary, by Theorem 4.37, the restriction of the action on (B, ρ) to Γ makes it a Γ-contractive space.

By the Intermediate Contractive Factor Theorem (the piecewise version–Theorem 4.41), for almost every $x \in X$ and $b \in B$ and for any $\lambda \in \mathrm{stab}_\Lambda(x)$ we have $\pi(\lambda(b, x)) = \pi(b, x)$.

By standard arguments (see Appendix B in [Zim84]), there exist Borel models for the spaces B, X and A, and the maps π and φ. Moreover, there is a Borel section $X \to (B \to A)$ for π: for almost every x there is a Borel map $\pi_x : B \times \{x\} \to A_x$ where $A_x = \varphi^{-1}(\{x\})$. The conclusion of the Intermediate Contractive Factor Theorem is that $\pi_x \circ \lambda = \pi_x$ for almost every x and all $\lambda \in \mathrm{stab}_\Lambda(x)$. Since composition is weakly continuous on the space of Borel maps, treating the Λ action as a Borel map $B \to B$, then $\pi_x \circ g = \pi_x$ for almost every x and all $g \in \overline{\mathrm{stab}_\Lambda(x)}$; that is, $\pi(gb, x) = \pi(b, x)$ for almost every x and b and all $g \in \overline{\mathrm{stab}_\Lambda(x)}$.

Define the map $s : X \to S(G)$, where $S(G)$ is the Borel space of closed subgroups of G equipped with the conjugation action by G, by $s(x) = \overline{\mathrm{stab}_\Lambda(x)}$. Observe that $s(\lambda x) = \lambda \overline{\mathrm{stab}_\Lambda(x)} \lambda^{-1} = \lambda \cdot s(x)$ so s is a Λ-map. Let $\eta \in P(S(G))$ be $\eta = s_* \nu$.

Let $(\widetilde{X}, \widetilde{\nu})$ be an action of G giving rise to the invariant random subgroup η. Such an action exists by Theorem 3.3. Then $(S(G), \eta)$ is a G-factor of $(\widetilde{X}, \widetilde{\nu})$ and $\eta = \widetilde{s}_* \widetilde{\nu}$ where $\widetilde{s}(\widetilde{x}) = \mathrm{stab}_G(\widetilde{x})$. Then anything true of the stabilizer $\mathrm{stab}_G(\widetilde{x})$ of almost every $\widetilde{x} \in \widetilde{X}$ is also true of the closure of the stabilizer $\overline{\mathrm{stab}_\Lambda(x)}$ of almost every $x \in X$.

Since Λ acts ergodically on (X, ν) and $(S(G), \eta)$ is a Λ-factor of (X, ν) then Λ acts ergodically on $(S(G), \eta)$. Since Λ is dense in G, G acts ergodically on $(S(G), \eta)$. Therefore we may assume G acts ergodically on $(\widetilde{X}, \widetilde{\nu})$ by Proposition 3.3.1.

By hypothesis, the G-action on $\widetilde{X}$ either has finite orbits when restricted to Γ or the restriction to Λ of the action has finite stabilizers. Suppose first that the action is such that $\Lambda \cap \mathrm{stab}_G(\widetilde{x})$ is finite for almost every $\widetilde{x}$ (for some affine orbital Γ-space over (X, ν)). Then $\overline{\mathrm{stab}_\Lambda(x)} \cap \Lambda$ is finite for almost every x and therefore $\mathrm{stab}_\Lambda(x)$ is finite for almost every x meaning the Λ-action on (X, ν) has finite stabilizers, in which case the proof is complete.

So assume instead that $G \curvearrowright (\widetilde{X}, \widetilde{\nu})$ has finite orbits when restricted to Γ (for every affine orbital Γ-space over (X, ν)). Then $\Gamma \cap \mathrm{stab}_G(\widetilde{x})$ has finite index in Γ for $\widetilde{\nu}$-almost every $\widetilde{x}$ (since the Γ-orbits are finite almost surely). Therefore $\Gamma \cap \overline{\mathrm{stab}_\Lambda(x)}$ has finite index in Γ for ν-almost every x. Let $\Gamma_x = \Gamma \cap \overline{\mathrm{stab}_\Lambda(x)}$ be this lattice. Note that $\pi(\gamma b, x) = \pi(b, x)$ for every $\gamma \in \Gamma_x$ and almost every $b \in B$.

For each such x, let A_x be the fiber over x in A and define the map $\pi_x : B \to A_x$ by $\pi_x(b) = \pi(b, x)$. Now (B, β) is a contractive G-space hence is a contractive Γ_x-space (by Theorem 4.37 since Γ_x is a lattice in G) and we will now treat $(A_x, (\pi_x)_* \beta)$ as a Γ_x-space with the trivial action. Observe that for any $\gamma \in \Gamma_x$ and almost every $b \in B$,

$$\pi_x(\gamma b) = \pi(\gamma b, x) = \pi(b, x) = \pi_x(b) = \gamma \pi_x(b)$$

and therefore π_x is a Γ_x-map meaning that $(A_x, (\pi_x)_*\beta)$ is a contractive Γ_x-space. Since the Γ_x-action on it is trivial, $(\pi_x)_*\beta$ must be a point mass. Let $c_x \in A_x$ be the point $(\pi_x)_*\beta$ is supported on. Then $\pi(b, x) = c_x$ for almost every b so the mapping $x \mapsto c_x$ inverts φ. Moreover, this map provides an invariant section for A since for any $\gamma \in \Gamma$ we have that $c_{\gamma x} = \pi(b, \gamma x)$ for almost every $b \in B$ and so $c_{\gamma x} = \pi(\gamma b, \gamma x) = \gamma \pi(b, x) = \gamma c_x$ for almost every $b \in B$ and $x \in X$ so $x \mapsto c_x$ is Γ-equivariant.

As this holds for all affine orbital Γ-spaces over (X, ν) the action of Γ on (X, ν) is weakly amenable. $\qquad\square$

Corollary 5.2. *Let $\Gamma < G$ be a lattice in a locally compact second countable group with property (T) and let Λ be a countable dense subgroup of G such that $\Gamma <_c \Lambda$.*

Assume that for every ergodic measure-preserving action of G either the restriction of the action to Λ has finite stabilizers or the restriction of the action to Γ has finite orbits.

Then any ergodic measure-preserving action $\Lambda \curvearrowright (X, \nu)$ either has finite stabilizers or the restriction of the action to Γ has finite orbits.

Proof. By Theorem 5.1, if the action of Λ does not have finite stabilizers then the restriction of the action to Γ is weakly amenable. By Proposition 2.4.1 then almost every Γ-orbit is finite since Γ inherits property (T) from G. $\qquad\square$

6   The One-One Correspondence

We obtain a correspondence between invariant random subgroups of Λ and of the relative profinite completion (see Section 2.1.2) using the previous corollary.

6.1   Invariant Random Subgroups of Commensurators

We can restate our previous corollary in terms of invariant random subgroups:

Corollary 6.1. *Let $\Gamma < G$ be a lattice in a locally compact second countable group with property (T) and let Λ be a countable dense subgroup of G such that $\Gamma <_c \Lambda$.*

Assume that for every ergodic measure-preserving action of G either the restriction of the action to Λ has finite stabilizers or the restriction of the action to Γ has finite orbits.

Then any ergodic invariant random subgroup $\eta \in P(S(\Lambda))$ of Λ is either finite (η-almost every $H \in S(\Lambda)$ is finite) or η contains Γ up to finite index: for η-almost every $H \in S(\Lambda)$, we have $[\Gamma : H \cap \Gamma]$ is finite.

Proof. An ergodic invariant random subgroup can always be realized as the stabilizer subgroups of a measure-preserving Λ-action (Theorem 3.3). By Corollary 5.2 this action either has finite stabilizers, in which case the invariant random subgroup is finite, or has finite Γ-orbits which means that a finite index subgroup of Γ fixes each point. $\qquad\square$

6.2   The One-One Correspondence of Invariant Random Subgroups

Theorem 6.2. *Let $\Gamma < G$ be a lattice in a locally compact second countable group with property (T) and let Λ be a countable dense subgroup of G such that $\Gamma <_c \Lambda$.*

Assume that for every ergodic measure-preserving action of G either the restriction of the action to Λ has finite stabilizers or the restriction of the action to Γ has finite orbits.

Then there is a one-one, onto correspondence between commensurability classes of infinite ergodic invariant random subgroups of Λ and commensurability classes of open ergodic invariant random subgroups of $\Lambda /\!\!/ \Gamma$.

We prove some preliminary facts before proving the theorem.

Proposition 6.2.1. *Let $\Gamma <_c \Lambda$ such that every infinite ergodic invariant random subgroup of Λ contains Γ up to finite index. Let $\tau : \Lambda \to \mathrm{Symm}(\Lambda /\!\!/ \Gamma)$ be the map defining the relative profinite completion (see Section 2.1.2).*

The map $c : S(\Lambda) \to S(\Lambda /\!\!/ \Gamma)$ given by $c(L) = \overline{\tau(L)}$ is a Λ-equivariant map taking infinite ergodic invariant random subgroups of Λ to open ergodic invariant random subgroups of $\Lambda /\!\!/ \Gamma$.

Proof. For notational purposes, write

$$H = \Lambda /\!\!/ \Gamma = \overline{\tau(\Lambda)} \qquad \text{and} \qquad K = \overline{\tau(\Gamma)}$$

and note that K is a compact open subgroup of H.

Let $\nu \in P(S(\Lambda))$ be an infinite ergodic invariant random subgroup of Λ. By hypothesis, ν contains Γ up to finite index almost surely. For $L \in S(\Lambda)$, let

$$K_L = \overline{\tau(L \cap \Gamma)}.$$

Since $L \cap \Gamma$ has finite index in Γ almost surely, we have that K_L has finite index in K almost surely: $[\tau(\Gamma) : \tau(L \cap \Gamma)] \leq [\Gamma : L \cap \Gamma] < \infty$ so $[\overline{\tau(\Gamma)} : \overline{\tau(L \cap \Gamma)}] < \infty$ since finite index passes to closures. Therefore K_L is a compact open subgroup (since K is a compact open subgroup of the locally compact totally disconnected group H). In particular, $c(L)$ contains K_L and therefore $c(L)$ is an open subgroup of H almost surely.

Therefore c maps $S(\Lambda)$ to open subgroups of H. Recall that $H \curvearrowright S(H)$ by conjugation and therefore $\Lambda \curvearrowright S(H)$ by $\lambda \cdot L = \tau(\lambda) L \tau(\lambda)^{-1}$. For $\lambda \in \Lambda$ and $L \in S(\Lambda)$

$$c(\lambda \cdot L) = \overline{\tau(\lambda L \lambda^{-1})} = \tau(\lambda) \overline{\tau(L)} \tau(\lambda)^{-1} = \lambda \cdot c(L)$$

and therefore this mapping is Λ-equivariant. Let $\eta \in P(S(H))$ be the pushforward of ν under this map. Then η is $\tau(\Lambda)$-invariant hence H-invariant since $\tau(\Lambda)$ is dense in H and H acts continuously on $S(H)$. Since ν is ergodic, so is η. $\qquad\square$

Proposition 6.2.2. *The map $d : S(\Lambda /\!\!/ \Gamma) \to S(\Lambda)$ by $d(M) = \tau^{-1}(M \cap \tau(\Lambda))$ has the following properties:*

    (i) $c(d(M)) = M$ *for all open $M \in S(\Lambda /\!\!/ \Gamma)$;*

    (ii) $d(M \cap Q) = d(M) \cap d(Q)$ *for all $M, Q \in S(\Lambda /\!\!/ \Gamma)$;*

    (iii) $L < d(c(L))$ *for all $L \in S(\Lambda)$;*

    (iv) $[d(c(L)) : L] < \infty$ *for all $L \in S(\Lambda)$ such that $[\Gamma : \Gamma \cap L] < \infty$; and*

    (v)   *for open $M, Q \in S(\Lambda /\!\!/ \Gamma)$ with $Q < M$, if $[M : Q] < \infty$ then $[d(M) : d(Q)] < \infty$.*

Proof. Let M be an open subgroup of $\Lambda/\!\!/\Gamma$. Then

$$c(d(M)) = \overline{\tau(\tau^{-1}(M \cap \tau(\Lambda)))} = \overline{M \cap \tau(\Lambda)} = M$$

since M is open (hence also closed) and $\tau(\Lambda)$ is dense in $\Lambda/\!\!/\Gamma$, proving the first statement.

Now let $M, Q \in S(\Lambda/\!\!/\Gamma)$. Then

$$d(M) \cap d(Q) = \tau^{-1}(M \cap \tau(\Lambda)) \cap \tau^{-1}(Q \cap \tau(\Lambda)) = \tau^{-1}(M \cap Q \cap \tau(\Lambda)) = d(M \cap Q)$$

proving the second statement.

Let $L \in S(\Lambda)$. Then $d(c(L)) = \tau^{-1}(\overline{\tau(L)} \cap \tau(\Lambda))$ and $\tau(L) \subseteq \tau(\Lambda)$ so L is a subgroup of $d(c(L))$, proving the third statement. Now let L be an infinite subgroup of Λ. Define the group

$$Q = c(L) \cap \tau(\Lambda).$$

Then $\tau(L)$ is dense in Q and $K = \overline{\tau(\Gamma)}$ is open in $H = \overline{\tau(\Lambda)}$ so $Q \subseteq K\tau(L)$. Let $h \in Q$, then $h = kn$ for some $k \in K$ and $n \in \tau(L)$. Therefore $hn^{-1} \in K$ and also $hn^{-1} \in \tau(\Lambda)$. By Proposition 2.1.3, $K \cap \tau(\Lambda) = \tau(\Gamma)$ so we have that $hn^{-1} \in \tau(\Gamma)$. Hence

$$Q \subseteq \tau(\Gamma)\tau(L) = \tau(\Gamma L).$$

We will use the notation $[A : B]$ when A and B are merely subsets (and not necessarily subgroups) to refer to the smallest number of elements of A such that the left translates of B by those elements cover A. Observe that, since L contains Γ up to finite index,

$$[Q : \tau(L)] \le [\tau(\Gamma L) : \tau(L)] \le [\Gamma L : L] = [\Gamma : \Gamma \cap L] < \infty$$

so Q is a finite index extension of $\tau(L)$.

Now write $R = \tau^{-1}(Q) = \tau^{-1}(\overline{\tau(L)} \cap \tau(\Lambda))$. Then $\tau(R) = Q$. Write $R_0 = R \cap \ker(\tau)$ and $L_0 = L \cap \ker(\tau)$. Since $R_0 \subseteq \ker(\tau)$ and $\ker(\tau) \subseteq \Gamma$, by the isomorphism theorems we have that

$$[R_0 : L_0] \le [\ker(\tau) : L \cap \ker(\tau)] = [L\ker(\tau) : L] \le [L\Gamma : L] = [\Gamma : \Gamma \cap L] < \infty.$$

By Lemma 6.2.3 below,

$$[R : L] \le [\tau(R) : \tau(L)][R_0 : L_0] = [Q : \tau(L)][R_0 : L_0] < \infty$$

since Q is a finite index extension of $\tau(L)$. Therefore L has finite index in $\tau^{-1}(\overline{\tau(L)} \cap \tau(\Lambda)) = d(c(L))$ proving the fourth statement.

Now let M, Q be open subgroups of $\Lambda/\!\!/\Gamma$ such that $[M : Q] < \infty$. Observe that $Q \cap \overline{\tau(\Gamma)}$ is then open so $\overline{\tau(\Gamma)}/Q \cap \overline{\tau(\Gamma)}$ is both compact and discrete, hence finite. Since $\overline{\tau(d(Q))} = Q$ then we have $[\overline{\tau(\Gamma)} : \overline{\tau(d(Q))} \cap \overline{\tau(\Gamma)}] < \infty$. Therefore $[\tau(\Gamma) : \tau(d(Q)) \cap \tau(\Gamma)] < \infty$.

Since $\ker(\tau) = \tau^{-1}(\{e\}) = d(\{e\})$,

$$[\ker(\tau) : d(Q) \cap \ker(\tau)] = [d(\{e\}) : d(Q) \cap d(\{e\})] = [d(\{e\}) : d(Q \cap \{e\})] = 1$$

hence by Lemma 6.2.3,

$$[\Gamma : \Gamma \cap d(Q)] \le [\tau(\Gamma) : \tau(\Gamma \cap d(Q))][\ker(\tau) : d(Q) \cap \ker(\tau)] = [\tau(\Gamma) : \tau(\Gamma \cap d(Q))] < \infty.$$

Similarly, $[d(M) \cap \ker(\tau) : d(Q) \cap \ker(\tau)] = 1$, so by Lemma 6.2.3,

$$[d(M) : d(Q)] \leq [\tau(d(M)) : \tau(d(Q))][d(M) \cap \ker(\tau) : d(Q) \cap \ker(\tau)] < \infty$$

since $[\overline{\tau(d(M))} : \overline{\tau(d(Q))}] = [c(d(M)) : c(d(Q))] = [M : Q] < \infty$, proving the final statement. $\square$

Lemma 6.2.3. *Let $\phi : C \to D$ be a group homomorphism and $A \subseteq C$ and $B \subseteq A$ be subsets. Then*

$$[A : B] \leq [\phi(A) : \phi(B)][\ker(\phi) : B \cap \ker(\phi)].$$

Proof. Assume both indices on the right are finite, otherwise there is nothing to prove. Let X be a finite system of representatives for $\phi(A)/\phi(B)$ (that is, $\phi(A) \subseteq \cup_{x \in X} x\phi(B)$). Let Y be a finite system of representatives for $\ker(\phi)/B \cap \ker(\phi)$. Let $\widetilde{X}$ contain one element $\widetilde{x}$ for each $x \in X$ such that $\phi(\widetilde{x}) = x$ so $|\widetilde{X}| = |X|$.

Let $a \in A$. Then $\phi(a) = x\phi(b)$ for some $x \in X$ and $b \in B$. So $\phi(\widetilde{x}^{-1}ab^{-1}) = e$ hence $\widetilde{x}^{-1}ab^{-1} \in \ker(\phi)$ and therefore $\widetilde{x}^{-1}ab^{-1} = yk$ for some $y \in Y$ and some $k \in B \cap \ker(\phi)$. Then $a = \widetilde{x}ykb$. Now $kb \in B$ and there are at most $|\widetilde{X}||Y| = |X||Y|$ choices for $\widetilde{x}y$ so the claim follows. $\square$

Proof of Theorem 6.2. Let c and d denote the maps in the previous propositions. The correspondence will be given by the map c on commensurability classes. By Corollary 6.1, any infinite ergodic invariant random subgroup ν of Λ contains Γ up to finite index almost surely. By Proposition 6.2.1, $c_*\nu$ is then an open ergodic invariant random subgroup of $\Lambda /\!/ \Gamma$.

Let ν_1 and ν_2 be infinite ergodic invariant random subgroups of Λ such that ν_1 and ν_2 are commensurate invariant random subgroups. Let $\alpha \in P(S(\Lambda) \times S(\Lambda))$ be a joining of η_1 and η_2 witnessing the commensuration. Define $\beta \in P(S(\Lambda /\!/ \Gamma) \times S(\Lambda /\!/ \Gamma))$ by $\beta = (c \times c)_*\alpha$. Then β is a joining of $c_*\nu_1$ and $c_*\nu_2$ that is clearly measure-preserving. Since, in general $\overline{X \cap Y} \subseteq \overline{X} \cap \overline{Y}$, for any $H, L \in S(\Lambda)$,

$$[c(H) : c(H) \cap c(L)] = [\overline{\tau(H)} : \overline{\tau(H)} \cap \overline{\tau(L)}] \leq [\overline{\tau(H)} : \overline{\tau(H) \cap \tau(L)}].$$

For α-almost every H, L, we have that $[H : H \cap L] < \infty$ and since τ is a homomorphism then $[\tau(H) : \tau(H) \cap \tau(L)] < \infty$. Therefore $[c(H) : c(H) \cap c(L)] < \infty$ since finite index passes to closures. Likewise, $[c(L) : c(H) \cap c(L)] < \infty$.

Hence for β-almost every M, Q, the subgroup $M \cap Q$ has finite index in both M and Q. Therefore β makes $c_*\eta_1$ and $c_*\eta_2$ commensurate invariant random subgroups. Hence c defines a correspondence from commensurability classes of infinite ergodic invariant random subgroups of Λ to commensurability classes of open ergodic invariant random subgroups of $\Lambda /\!/ \Gamma$.

Now let ν_1 and ν_2 be infinite ergodic invariant random subgroups of Λ such that $c_*\nu_1$ and $c_*\nu_2$ are commensurate open ergodic invariant random subgroups of $\Lambda /\!/ \Gamma$. Let $\beta \in P(S(\Lambda /\!/ \Gamma) \times S(\Lambda /\!/ \Gamma))$ be a joining of $c_*\nu_1$ and $c_*\nu_2$ such that for β-almost every M, Q, the subgroup $M \cap Q$ has finite index in M and Q. Define $\nu_3 = d_*c_*\nu_1$. Then by Proposition 6.2.2 (iv), $[d(c(L)) : L] < \infty$ for ν_1-almost every $L \in S(\Lambda)$. Define $\rho \in P(S(\Lambda) \times S(\Lambda))$ by

$$\rho = \int_L \delta_L \times \delta_{d(c(L))} \ d\nu_1(L).$$

Then ρ is a joining of ν_1 and ν_3 and clearly $L \cap d(c(L)) = L$ has finite index in both L and $d(c(L))$ almost surely so ρ makes ν_1 and ν_3 commensurate invariant random subgroups. Likewise ν_2 and $\nu_4 = d_*c_*\nu_2$ are commensurate invariant random subgroups. Since commensurability is an equivalence relation (Proposition 3.8), it is enough to show that ν_3 and ν_4 are commensurate.

Define $\alpha \in P(S(\Lambda) \times S(\Lambda))$ by $\alpha = (d \times d)_*\beta$. Then α is a joining of $d_*c_*\nu_1 = \nu_3$ and $d_*c_*\nu_2 = \nu_4$. By Proposition 6.2.2 (ii), for open $M, Q \in S(\Lambda/\!\!/\Gamma)$, $d(M) \cap d(Q) = d(M \cap Q)$. Observe that ν_3 and ν_4 are infinite ergodic invariant random subgroup of Λ hence $d(M)$ and $d(Q)$ both contain Γ up to finite index almost surely. Then $d(M \cap Q)$ contains Γ up to finite index almost surely. For β-almost every M, Q we also know that $[M : M \cap Q] < \infty$. So by Proposition 6.2.2 (v),

$$[d(M) : d(M) \cap d(Q)] = [d(M) : d(M \cap Q)] < \infty$$

almost surely. Hence for α-almost every H, L the subgroup $H \cap L$ has finite index in both so ν_3 and ν_4 are commensurate invariant random subgroups. Therefore the correspondence is one-one.

Let $\eta \in P(S(\Lambda/\!\!/\Gamma))$ be an open ergodic invariant random subgroup of $\Lambda/\!\!/\Gamma$. For M an open subgroup of $\Lambda/\!\!/\Gamma$ we have that $d(M) = \tau^{-1}(M \cap \tau(\Lambda))$ is infinite since otherwise $M \cap \tau(\Lambda)$ is finite but $\tau(\Lambda)$ is dense. Therefore $d_*\eta$ is an infinite invariant random subgroup of Λ and must be ergodic since $c_*d_*\eta = \eta$ by Proposition 6.2.2 (i). Therefore the correspondence is onto. $\qquad\square$

6.3   The Dichotomy for Actions of Commensurators

We now are ready to state the conclusion of our study of stabilizer subgroups that will be the main ingredient in the various consequences we prove in the rest of the paper:

Corollary 6.3. *Let $\Gamma < G$ be a lattice in a locally compact second countable group with property (T) and let Λ be a countable dense subgroup of G such that $\Gamma <_c \Lambda$.*

Assume that for every ergodic measure-preserving action of G either the restriction of the action to Λ has finite stabilizers or the restriction of the action to Γ has finite orbits.

Assume that every ergodic measure-preserving action of $\Lambda/\!\!/\Gamma$ with open stabilizer subgroups is necessarily on the trivial space.

Then any ergodic measure-preserving action of Λ on a probability space either has finite orbits or has finite stabilizers.

Proof. Let $\Lambda \curvearrowright (X, \nu)$ be an ergodic measure-preserving action that does not have finite stabilizers. By the one-one correspondence theorem, the invariant random subgroup of stabilizer subgroups corresponds to an ergodic open invariant random subgroup η of $H = \Lambda/\!\!/\Gamma$. This invariant random subgroup corresponds to an ergodic action of H with open stabilizer groups and so by hypothesis then $\eta = \delta_H$ meaning $\overline{\tau(\mathrm{stab}_\Lambda(x))} = H$ for almost every x.

By the one-one correspondence construction we then have that

$$[\Lambda : \mathrm{stab}_\Lambda(x)] = [\tau^{-1}(\overline{\mathrm{stab}_\Lambda(x)} \cap \tau(\Lambda)) : \mathrm{stab}_\Lambda(x)] = [d(c(\mathrm{stab}_\Lambda(x))) : \mathrm{stab}_\Lambda(x)] < \infty$$

for almost every x. This means that almost every Λ-orbit is finite so by ergodicity (X, ν) consists of exactly one such orbit. $\qquad\square$

7 Howe-Moore Groups

We now discuss the properties one can impose on the ambient group G to ensure that for every nontrivial ergodic measure-preserving action of G the restriction of the action to Λ has finite stabilizers. The main property we impose on the ambient group will be the Howe-Moore property.

7.1 Actions of Subgroups of Simple Lie Groups

The next fact is a consequence of the Stuck-Zimmer Theorem [SZ94] and also follows from earlier work by Zimmer, [Zim87] Lemma 6, and of Iozzi, [Ioz94] Proposition 2.1, showing that the stabilizers of any nontrivial irreducible action of a semisimple real Lie group are discrete. However, we opt to include the following elementary argument proving what we need directly.

Theorem 7.1. *Let G be a connected (real) Lie group with trivial center and let $\Lambda < G$ be any countable subgroup. Let $G \curvearrowright (X, \nu)$ be a faithful weakly mixing measure-preserving action. Then the restriction of the action to Λ is essentially free.*

Proof. For $x \in X$ let $C(x)$ be the connected component of the identity in the stabilizer subgroup $\mathrm{stab}_G(x)$. Let $n(x)$ be the dimension of $C(x)$. Then $n(gx)$ is the dimension of $C(gx) = gC(x)g^{-1}$ hence $n(x)$ is G-invariant. By ergodicity then $n(x) = n$ is constant almost surely.

Since the action of G is weakly mixing, the diagonal action $G \curvearrowright (X^2, \nu^2)$ is ergodic. Let $n_1(x, y)$ be the dimension of the connected component of the identity $C(x, y)$ in $\mathrm{stab}_G(x, y)$ and then $n_1(x, y) = n_1$ is constant almost surely by ergodicity.

When $n = 0$, the stabilizer subgroup $\mathrm{stab}_G(x)$ is discrete for almost every x (since the stabilizer subgroup is closed). Assume now that $n \neq 0$. Since $\mathrm{stab}_G(x, y) = \mathrm{stab}_G(x) \cap \mathrm{stab}_G(y)$ we have that $C(x, y) = C(x) \cap C(y)$. Suppose that $n = n_1$. Then for almost every x and y we have that $C(x, y) = C(x) \cap C(y)$ has the same dimension as $C(x)$ and $C(y)$.

In general, if $H < G$ are real Lie groups with the same dimension then H has finite index in G. If, in addition, G and H are connected then $H = G$.

Therefore, if $C(x, y) = C(x) \cap C(y)$ has the same dimension as $C(x)$ and $C(y)$ then in fact $C(x) = C(y)$. So if $n > 0$ this then means there is a positive dimension subgroup in the kernel of the action contradicting that the action is faithful. So if $n = n_1$ then $n = n_1 = 0$.

So instead we have that $n_1 < n$. Proceeding by induction, since G acts ergodically on (X^m, ν^m) for any $m \in \mathbb{N}$, we conclude that for almost every $\tilde{x} \in X^{n+1}$ the stabilizer subgroup $\mathrm{stab}_G(\tilde{x})$ is discrete.

The conclusion now follows from the following proposition. $\qquad\square$

Proposition 7.1.1. *Let G be a nondiscrete locally compact second countable group and $\Lambda < G$ a countable subgroup such that Λ does not intersect the center of G. Let $G \curvearrowright (X, \nu)$ be a measure-preserving action such that almost every stabilizer subgroup is discrete. Then the restriction of the action to Λ is essentially free.*

Proof. Fix a compact model of X where G acts continuously. Suppose the Λ-action is not essentially free. Then there exists $\lambda \in \Lambda$, $\lambda \neq e$, such that $E = \{x \in X : \lambda x = x\}$ has positive measure. Let $g_n \to e$ such that $g_n^{-1}\lambda g_n \neq \lambda$ for all n (possible as G is nondiscrete and λ is not in the center of G). Then $\nu(g_n E \triangle E) \to 0$ since G acts continuously. Take a subsequence along which $\nu(g_n E \triangle E) < 2^{-n-1}\nu(E)$. Then

$$\nu(E \cap \bigcap_n g_n E) = \nu(E) - \nu(E \triangle \bigcap_n g_n E) \geq \nu(E) - \sum_{n=1}^{\infty} \nu(E \triangle g_n E) > \frac{1}{2}\nu(E) > 0$$

For $x \in E \cap (\cap_n g_n E)$ we have that $\lambda x = x$ and $g_n^{-1}\lambda g_n x = x$ for all n, hence $g_n^{-1}\lambda g_n \in \mathrm{stab}_G(x)$ and $\lambda \in \mathrm{stab}_G(x)$. But $g_n^{-1}\lambda g_n \neq \lambda$ and $g_n^{-1}\lambda g_n \to \lambda$ contradicting that $\mathrm{stab}_G(x)$ is discrete. □

Theorem 7.2. *Let G be a product of noncompact connected simple locally compact second countable groups with the Howe-Moore property. Let $\Lambda < G$ be a countable subgroup of G such that the Λ intersection with any proper subproduct of G is finite. Then the restriction of any nontrivial ergodic measure-preserving action of G to Λ has finite stabilizers.*

Proof. Let $G \curvearrowright (X, \nu)$ be a nontrivial ergodic action. Then the kernel of the action is some subproduct G' of the G-factors (as all are normal). Let $G_0 = G/G'$. By a result of Rothman [Rot80] reproduced as Theorem 2.36, each simple factor of G_0 being a Howe-Moore group that is simple and connected is necessarily a simple Lie group. Hence G_0 is a minimally almost periodic group, being a semisimple Lie group without compact factors, so any ergodic action of G_0 is weakly mixing. Since each factor is simple, G_0 has trivial center. Therefore $G_0 \curvearrowright (X, \nu)$ is a faithful weakly mixing action so Theorem 7.1 implies that $\mathrm{proj}_{G_0} \Lambda$ acts essentially freely. Therefore $\mathrm{proj}_{G_0} \mathrm{stab}_\Lambda(x) = \{e\}$ for almost every x hence $\mathrm{stab}_\Lambda(x) \subseteq G'$ and $|\Lambda \cap G'| < \infty$ by hypothesis. □

7.2 Actions of Lattices in Howe-Moore Groups

In the case when G is not connected we have a similar result:

Definition 7.3. A countable discrete group Γ is **locally finite** when every finitely generated subgroup is finite.

Theorem 7.4. *Let G be a locally compact second countable group and $\Gamma < G$ be a lattice in G. Let $G \curvearrowright (X, \nu)$ be an ergodic measure-preserving action of G such that the restriction of the action to Γ is mixing. Then either the kernel of the G-action is noncompact or $\mathrm{stab}_\Gamma(x) = \mathrm{stab}_G(x) \cap \Gamma$ is locally finite almost surely.*

Proof. Let $E = \{x \in X : \mathrm{stab}_\Gamma(x)$ is locally finite$\}$. Then E is Γ-invariant. Assume that $\nu(E) < 1$. By ergodicity then $\nu(E) = 0$. Then $\mathrm{stab}_\Gamma(x)$ contains a finitely generated infinite subgroup for almost every x. Since there are countably many finitely generated infinite subgroups of Γ, there exists an infinite finitely generated subgroup $\Gamma_0 < \Gamma$ and a positive measure set $F \subseteq X$ such that $\Gamma_0 < \mathrm{stab}_\Gamma(x)$ for each $x \in F$.

Since the action of Γ on (X, ν) is mixing, we have that $\nu(F) = 1$ (as Γ_0 is infinite so is unbounded in Γ and therefore must also be mixing but Γ_0 acts trivially on F). Therefore Γ_0

is contained in the kernel of the G-action which is therefore noncompact (as Γ is a lattice so Γ_0 is unbounded in G). □

Corollary 7.5. *Let G be a noncompact locally compact second countable group with the Howe-Moore property and $\Gamma < G$ be a lattice. Let $G \curvearrowright (X, \nu)$ be a nontrivial ergodic measure-preserving action. Then $\mathrm{stab}_\Gamma(x)$ is locally finite for almost every x.*

Proof. The Howe-Moore property applied to the Koopman representation for $G \curvearrowright (X, \nu)$ implies that $G \curvearrowright (X, \nu)$ is mixing (a result of Schmidt [Sch84] reproduced as Theorem 2.35). Since Γ is a lattice in G then $\Gamma \curvearrowright (X, \nu)$ by restriction is also mixing. If $\mathrm{stab}_\Gamma(x)$ is not locally finite almost surely then by Theorem 7.4 the kernel N of the G-action is a noncompact closed normal subgroup. Since G has Howe-Moore any proper normal subgroup is compact (Proposition 2.5.1) so the kernel is all of G. □

The following argument is due to R. Tucker-Drob [TD12] and we are grateful to him for allowing us to present it here:

Corollary 7.6. *Let G be a noncompact locally compact second countable group with the Howe-Moore property and $\Gamma < G$ be a lattice. Let $G \curvearrowright (X, \nu)$ be a nontrivial ergodic measure-preserving action. Then $\mathrm{stab}_\Gamma(x)$ is finite for almost every x.*

Proof. (Tucker-Drob [TD12]) By the previous corollary, the stabilizer subgroups are locally finite almost surely. Hall and Kulatilaka [HK64] showed that any infinite locally finite group contains an infinite abelian subgroup.

Since G has Howe-Moore, the action is mixing and has compact kernel K. Let $G' = G/K$ and $\Gamma' = \Gamma/\Gamma \cap K$. Then $G' \curvearrowright (X, \nu)$ is a faithful mixing action and Γ' is a lattice in G' (and is a finite index subgroup of Γ).

Let $\gamma \in \Gamma'$, $\gamma \neq e$, such that there exists an infinite abelian subgroup $A < \Gamma'$ with $\gamma \in A$. Let $E_\gamma = \{x \in X : \gamma x = x\}$. Then for $a \in A$, $\gamma a x = a \gamma x = a x$ so E_γ is an A-invariant set. Since the action is mixing and faithful, and since A is infinite and discrete, $\nu(E_\gamma) = 0$.

Let $F = \{x \in X : \mathrm{stab}_{\Gamma'}(x) \text{ contains an infinite abelian subgroup}\}$ and suppose $\nu(F) > 0$. Since Γ' is countable there then exists some $\gamma \neq e$ such that $\nu\{x \in F : \gamma x = x\} > 0$. But this contradicts the above since γ is then contained in an infinite abelian subgroup. □

7.3   A Normal Subgroup of the Commensurator

Proposition 7.3.1. *Let Γ be a finitely generated countable group that is not virtually abelian and let Λ be a countable group such that $\Gamma <_c \Lambda$. Let $\Lambda \curvearrowright (X, \nu)$ be a measure-preserving action such that $\mathrm{stab}_\Lambda(x)$ is infinite almost surely. If $\mathrm{stab}_\Gamma(x)$ are finite on a positive measure set then Λ contains an infinite normal subgroup $N \lhd \Lambda$ such that $[\Gamma : \Gamma \cap N] = \infty$.*

Proof. Since there are only countably many finite subgroups of Γ, let us assume there exists some finite subgroup $\Sigma < \Gamma$ such that $\mathrm{stab}_\Gamma(x) = \Sigma$ for all $x \in E$ where $\nu(E) > 0$.

For $\lambda \in \Lambda$ define the set

$$E_\lambda = \{x \in X : \lambda x = x\} \cap E$$

and denote by $\Gamma_\lambda = \Gamma \cap \lambda \Gamma \lambda^{-1}$ the subgroup with finite index in Γ and $\lambda \Gamma \lambda^{-1}$.

By hypothesis, $\nu(E_\lambda) > 0$ for infinitely many $\lambda \in \Lambda$ (since otherwise $\mathrm{stab}_\Lambda(x)$ is finite for all $x \in E$ which has positive measure, see [Ver12]). For such λ, define $\overline{E_\lambda} \subset X$ to be $\cup_{\gamma \in \Gamma_\lambda} \gamma E_\lambda$. For any $\epsilon > 0$ there exists a finite set $F \subseteq \Gamma_\lambda$ such that $\nu(\overline{E_\lambda}) - \nu(\cup_{f \in F} f E_\lambda) < \epsilon$. Take $\epsilon = \nu(E_\lambda)$. Then there is a finite set $F \subseteq \Gamma_\lambda$ such that $\nu(\cup_{f \in F} f E_\lambda) > \nu(\overline{E_\lambda}) - \nu(E_\lambda)$.

Then for each $\gamma \in \Gamma_\lambda$ there exists $f \in F$ such that $\nu(\gamma E_\lambda \cap f E_\lambda) > 0$ because $\gamma E_\lambda \subseteq \overline{E_\lambda}$ and $\nu(\gamma E_\lambda) = \nu(E_\lambda)$. For $x \in f^{-1} \gamma E_\lambda \cap E_\lambda$ we have that $x \in E$, $\lambda x = x$ and $\lambda \gamma^{-1} f x = \gamma^{-1} f x$, therefore

$$\lambda^{-1}(f^{-1}\gamma)\lambda(f^{-1}\gamma)^{-1}x = \lambda^{-1}(f^{-1}\gamma)(f^{-1}\gamma)^{-1}x = \lambda^{-1}x = x$$

and so $\lambda^{-1}(f^{-1}\gamma)\lambda(f^{-1}\gamma)^{-1} \in \Sigma$. This in turn means that $\gamma\lambda\gamma^{-1} \in f\lambda\Sigma f^{-1} \subseteq F\lambda\Sigma F^{-1}$. Since F and Σ are finite then the centralizer

$$C_{\Gamma_\lambda}(\lambda) = \{\gamma \in \Gamma_\lambda : \gamma\lambda\gamma^{-1} = \lambda\}$$

has finite index in Γ_λ. Therefore $[\Gamma : C_\Gamma(\lambda)] < \infty$ since Γ_λ has finite index in Γ.

Consider the subgroup

$$N = \{\lambda \in \Lambda : [\Gamma : C_\Gamma(\lambda)] < \infty\}$$

which is infinite by the above (it is a subgroup since $C_\Gamma(\lambda_1) \cap C_\Gamma(\lambda_2) \subseteq C_\Gamma(\lambda_1\lambda_2)$). Since $\Gamma <_c \Lambda$, for $\lambda_0 \in \Lambda$,

$$\begin{aligned}
\lambda_0 N \lambda_0^{-1} &= \{\lambda \in \Lambda : [\Gamma : C_\Gamma(\lambda_0^{-1}\lambda\lambda_0)] < \infty\} \\
&= \{\lambda \in \Lambda : [\Gamma : \lambda_0^{-1}C_\Gamma(\lambda)\lambda_0] < \infty\} \\
&= \{\lambda \in \Lambda : [\lambda_0\Gamma\lambda_0^{-1} : C_\Gamma(\lambda)] < \infty\} \\
&= \{\lambda \in \Lambda : [\Gamma : C_\Gamma(\lambda)] < \infty\} = N
\end{aligned}$$

where the last line follows since $\Gamma \cap \lambda_0\Gamma\lambda_0^{-1}$ has finite index in Γ by commensuration.

Therefore N is an infinite normal subgroup of Λ. If $[\Gamma : \Gamma \cap N] < \infty$ then there exists $\Gamma_0 = \Gamma \cap N$ of finite index in Γ such that for every $\gamma \in \Gamma_0$ we have that $[\Gamma_0 : C_{\Gamma_0}(\gamma)] < \infty$. Hence for any finite set $F \subseteq \Gamma_0$ we have that $[\Gamma_0 : C_{\Gamma_0}(F)] < \infty$. As Γ is finitely generated so is Γ_0 so let S be a finite generating set of Γ_0 and then $C_{\Gamma_0}(S)$ has finite index in Γ_0. But $C_{\Gamma_0}(S)$ commutes with Γ_0 so Γ_0 is virtually abelian hence so is Γ. $\qquad\square$

7.4   Ensuring Actions of the Ambient Group "Behave"

Theorem 7.7. *Let G be a noncompact compactly generated locally compact second countable group with the Howe-Moore property. Let $\Gamma < G$ be a lattice and let $\Lambda < G$ be a dense subgroup such that $\Gamma <_c \Lambda$. Assume that for every compact normal subgroup $M \triangleleft G$ we have that $|M \cap \Lambda| < \infty$.*

Then for any nontrivial ergodic measure-preserving action of G the restriction of the action to Λ has finite stabilizers.

Proof. By Corollary 7.6, since G has the Howe-Moore property almost every Γ-stabilizer is finite, hence the restriction of the action to Γ has finite stabilizers. Also, Γ is finitely generated since G is compactly generated. Since G is not a compact extension of an abelian group (being Howe-Moore), Γ cannot be virtually abelian. Suppose the restriction of the

action to Λ does not have finite stabilizers. Then by Proposition 7.3.1 there exists an infinite normal subgroup $N \lhd \Lambda$ such that $[\Gamma : \Gamma \cap N] < \infty$. But $\Gamma < \Lambda < G$ satisfy the hypotheses of the Normal Subgroup Theorem for Commensurators (Theorem 2.45), as any proper closed normal subgroup is compact by Howe-Moore, so for any normal subgroup $N \lhd \Lambda$ we have that $[\Gamma : \Gamma \cap N] < \infty$ or $|N| < \infty$. This contradiction means the Λ-action has finite stabilizers. $\qquad \square$

7.5   Ensuring Actions of the Relative Profinite Completion "Behave"

To handle invariant random subgroups coming from the relative profinite completion we also need:

Theorem 7.8. *Let H be a simple nondiscrete locally compact second countable totally disconnected group with the Howe-Moore property. If $H \curvearrowright (X, \nu)$ is an ergodic measure-preserving action with open stabilizer subgroups then (X, ν) is trivial.*

Proof. Suppose (X, ν) is nontrivial so that $\mathrm{stab}_H(x) \neq H$ almost surely. For almost every $x \in X$, since $\mathrm{stab}_H(x)$ is open in H and H has Howe-Moore then $\mathrm{stab}_H(x)$ is compact almost surely. There are only countably many compact open subgroups of H (as H is second countable) so there exists $E \subseteq X$ with $\nu(E) > 0$ and K_0 a compact open subgroup such that $\mathrm{stab}_H(x) = K_0$ for all $x \in E$. Now for $h \in H \setminus N_H(K_0)$ we have that $hE \cap E = \emptyset$ (since $\mathrm{stab}_H(hx) = hK_0h^{-1}$ for $x \in E$). As $\nu(E) > 0$ and ν is preserved by H there exists a finite collection $h_1, h_2, \ldots, h_n \in H$ such that $X = \sqcup_{j=1}^n h_j E$. Then $K = \cap_{j=1}^n h_j K_0 h_j^{-1}$ is a compact open subgroup and $K < \mathrm{stab}_H(x)$ for almost every x hence K is in the kernel of the H-action. As H is simple and K is nontrivial (since H is nondiscrete) then the kernel is all of H so X is trivial. $\qquad \square$

Proposition 7.5.1. *Let $H = H_1 \times \cdots \times H_m$ be a product of locally compact second countable groups where each H_j has the property that any ergodic measure-preserving action of H_j with open stabilizer subgroups is necessarily on the trivial space. Then any ergodic measure-preserving action of H with open stabilizer subgroups is necessarily on the trivial space.*

Proof. Let $H \curvearrowright (X, \nu)$ be an ergodic measure-preserving action. Let $(S(H), \eta)$ be the ergodic open invariant random subgroup corresponding to the (X, ν) stabilizers. Fix j and consider the map $p_j : S(H) \to S(H_j)$ by $p_j(L) = L \cap H_j$ (meaning that $p_1(L) = L \cap H_1 \times \{e\} \times \cdots \times \{e\}$). Treat $S(H_j)$ as an H-space where H_i acts trivially on $S(H_j)$ for $i \neq j$. Then p_j is an H-map from $(S(H), \eta)$ to $(S(H_j), \eta_j)$ where $\eta_j = (p_j)_* \eta$.

   Since η-almost every $L \in S(H)$ is open so is η_j-almost every $L_j < H_j$. Since η is H-ergodic, η_j is H_j-ergodic hence corresponds to an ergodic action of H_j with open stabilizer subgroups (Theorem 3.3). By hypothesis then $\eta_j = \delta_{H_j}$. As this holds for each j, for η-almost every $L < H$ we have that $L \cap H_j = H_j$ hence $\langle H_1, \cdots, H_m \rangle \subseteq L$ and therefore $\eta = \delta_H$. So $\mathrm{stab}_\Lambda(x) = H$ for almost every x and therefore (X, ν) is the trivial space. $\qquad \square$

Proposition 7.5.2. *Let H be a restricted infinite product $\prod' H_j$ of locally compact second countable groups where each H_j has the property that any ergodic measure-preserving action of H_j with open stabilizer subgroups is necessarily on the trivial space. Then any ergodic measure-preserving action of H with open stabilizer subgroups is necessarily on the trivial space.*

Proof. This follows exactly as in the proof of Proposition 7.5.1.   □

8   Actions of Commensurators in Howe-Moore (T) Groups

Here we apply the results of the previous sections to derive concrete consequences about actions of commensurators in groups with Howe-Moore and property (T):

Corollary 8.1. *Let G be a noncompact locally compact second countable group with the Howe-Moore property and property (T). Let $\Gamma < G$ be a lattice and $\Lambda < G$ be a countable dense subgroup such that $\Gamma <_c \Lambda$ and such that Λ has finite intersection with every compact normal subgroup of G.*

Then any ergodic measure-preserving action of Λ either has finite stabilizers or the restriction of the action to Γ has finite orbits.

Proof. This follows from Corollary 5.2 and Theorem 7.7.   □

Corollary 8.2. *Let G be a noncompact locally compact second countable group with the Howe-Moore property and property (T). Let $\Gamma < G$ be a lattice and $\Lambda < G$ be a countable dense subgroup such that $\Gamma <_c \Lambda$ and such that Λ has finite intersection with every compact normal subgroup of G.*

The commensurability classes of infinite ergodic invariant random subgroups of Λ are in one-one, onto correspondence with the commensurability classes of open ergodic invariant random subgroups of $\Lambda /\!\!/ \Gamma$.

Proof. This follows from Theorem 6.2 and Theorem 7.7.   □

Corollary 8.3. *Let G be a noncompact locally compact second countable group with the Howe-Moore property and property (T). Let $\Gamma < G$ be a lattice and $\Lambda < G$ be a countable dense subgroup such that $\Gamma <_c \Lambda$ and such that Λ has finite intersection with every compact normal subgroup of G.*

Assume that $\Lambda /\!\!/ \Gamma$ is isomorphic to a finite (or restricted infinite) product $\prod H_j$ such that each H_j is a simple nondiscrete locally compact second countable group with the Howe-Moore property.

Then any ergodic measure-preserving action of Λ either has finite orbits or has finite stabilizers.

Proof. By Theorem 7.7, any nontrivial ergodic action of G has finite stabilizers when restricted to Λ. Theorem 7.8 applied to each H_j says that open ergodic invariant random subgroups of H_j correspond to the trivial space. Theorem 6.3 combined with Propositions 7.5.1 and 7.5.2 then gives the conclusion.   □

Corollary 8.4. *Let G be a product of connected noncompact locally compact second countable groups with the Howe-Moore property and property (T). Let $\Gamma < G$ be a lattice and $\Lambda < G$ be a countable dense subgroup such that $\Gamma <_c \Lambda$ and such that Λ has finite intersection with every proper closed normal subgroup of G.*

Assume that $\Lambda /\!\!/ \Gamma$ is isomorphic to a finite (or restricted infinite) product $\prod H_j$ such that each H_j is a simple nondiscrete locally compact second countable group with the Howe-Moore property.

Then any ergodic measure-preserving action of Λ either has finite orbits or has finite stabilizers.

Proof. By Theorem 7.2, any nontrivial ergodic action of G has finite stabilizers when restricted to Λ. Theorem 7.8 applied to the H_j shows that any ergodic action of H_j with open stabilizers is on the trivial space. Theorem 6.3 combined with Propositions 7.5.1 and 7.5.2 then gives the conclusion. $\qquad\square$

9 Actions of Lattices in Products of Howe-Moore (T) Groups

A consequence of the previous results is a generalization of the Bader-Shalom Normal Subgroup Theorem for Lattices in Product Groups to measure-preserving actions for certain product groups:

Theorem 9.1. *Let G be a product of at least two simple nondiscrete noncompact locally compact second countable groups with the Howe-Moore property, at least one of which has property (T), at least one of which is totally disconnected and such that every connected simple factor has property (T). Let $\Gamma < G$ be an irreducible lattice.*

Then any ergodic measure-preserving action of Γ either has finite orbits or has finite stabilizers.

Proof. Write G_0 to be the product of all the connected simple factors of G. In the case when there are no connected simple factors instead take G_0 to be a simple factor with property (T). Write H to be the product of all the simple factors not in G_0. So H is totally disconnected and nondiscrete.

Write $G = G_0 \times H$ and let K be a compact open subgroup of H. Let $L = \Gamma \cap (G_0 \times K)$. Then $\text{proj}_K L$ is dense in K since Γ is irreducible. L is a lattice in $G_0 \times K$ since K is open.

Set $\Gamma_0 = \text{proj}_{G_0} L$. Since K is compact, Γ_0 has finite covolume in G_0 since L does in $G \times K$. Moreover, Γ_0 is discrete since L is discrete. Therefore Γ_0 is a lattice in G_0.

Set $\Lambda_0 = \text{proj}_{G_0} \Gamma$. Then Λ_0 is dense in G_0 since Γ is irreducible and $\Gamma_0 <_c \Lambda_0$ since $K <_c H$.

By Propositions 2.1.5 and 2.1.4, $\Gamma /\!\!/ L$ is isomorphic to $H/\ker(\tau_{H,K})$ since $\text{proj} : \Gamma \to H$ is a homomorphism with dense image and pullback of K equal to L. Since $\ker(\tau_{H,K})$ is contained in K and H is semisimple then the kernel is trivial so $\Gamma /\!\!/ L$ is isomorphic to H.

Set $N = \Gamma \cap \{e\} \times H$ and write M for the subgroup of H such that $N = \{e\} \times M$. Then $N \triangleleft \Gamma$ since $\{e\} \times H \triangleleft G \times H$ and M is discrete in H so $M = \overline{\text{proj}_H N} \triangleleft \overline{\text{proj}_H \Gamma} = H$ by the irreducibility of Γ. Since H is simple, M is trivial so $\Gamma \cap \{e\} \times H$ is trivial. This means that $\text{proj}_G : \Gamma \to \Lambda_0$ is an isomorphism and so $\Lambda_0 /\!\!/ \Gamma_0 \simeq H$.

By Corollary 8.3 or Corollary 8.4 (depending on whether G_0 is a single factor or a product of connected factors) then any ergodic measure-preserving action of Λ_0 either has finite orbits or has finite stabilizers. The same then holds for $\Gamma \simeq \Lambda_0$. $\qquad\square$

We remark that the above construction, writing an irreducible lattice in a product of nondiscrete groups, at least one of which is totally disconnected, as the commensurator of a lattice in one of the groups can also be reversed:

Theorem 9.2. *Let Γ be a lattice in a locally compact second countable group G and let Λ be a subgroup of G such that $\Gamma <_c \Lambda$. Then Λ sits diagonally as a lattice in $G \times (\Lambda /\!\!/ \Gamma)$.*

Proof. Let $\tau : \Lambda \to \Lambda /\!\!/ \Gamma$ be the map defining the relative profinite completion and let

$$\Lambda_0 = \{(\lambda, \tau(\lambda)) : \lambda \in \Lambda\} < G \times (\Lambda /\!\!/ \Gamma)$$

be the diagonal embedding of Λ.

Let F be a fundamental domain for G/Γ: F is of finite volume, $F \cap \Gamma = \{e\}$ and $\Gamma \cdot F = G$. Let $K = \overline{\tau(\Gamma)}$ be the canonical compact open subgroup. Let $\lambda_0 \in \Lambda_0 \cap F \times K$. Then $\lambda_0 = (\lambda, \tau(\lambda))$ for some $\lambda \in \Lambda \cap F$ such that $\tau(\lambda) \in K$. Now $K = \overline{\tau(\Gamma)}$ and by Proposition 2.1.3, $K \cap \tau(\Lambda) = \tau(\Gamma)$ so $\tau(\lambda) \in \tau(\Gamma)$ meaning that $\lambda \in \Gamma$ (as the kernel of τ is contained in Γ). But $\lambda \in \Lambda \cap F$ so $\lambda \in \Gamma \cap F = \{e\}$. Therefore $F \times K$ is a subset of $G \times (\Lambda /\!\!/ \Gamma)$ of finite volume such that $\Lambda_0 \cap F \times K = \{e\}$ and, in particular, Λ_0 is discrete in $G \times (\Lambda /\!\!/ \Gamma)$.

Let $(g, h) \in G \times \Lambda /\!\!/ \Gamma$ be arbitrary. Write $h = \tau(\lambda')k'$ for some $\lambda' \in \Lambda$ and $k' \in K$. Write $(\lambda')^{-1}g = \gamma f$ for some $\gamma \in \Gamma$ and $f \in F$. Set $\lambda = \lambda'\gamma$. Then $\tau(\lambda)\tau(\gamma^{-1})k' = \tau(\lambda')k' = h$ and $k = \tau(\gamma^{-1})k' \in K$. Also $g = \lambda'\gamma f = \lambda f$. Therefore $(g, h) = (\lambda, \tau(\lambda))(f, k) \in \Lambda_0 \cdot (F \times K)$.

Therefore $F \times K$ is a fundamental domain for Λ_0 hence Λ_0 is a lattice as claimed. $\quad\square$

We remark that if both G and $\Lambda /\!\!/ \Gamma$ are semisimple with finite center then Λ sits as an irreducible lattice if and only if Γ is irreducible and Λ is dense.

A consequence of this reverse construction is a special case of the Normal Subgroup Theorem for Commensurators [CS12] following immediately from the Bader-Shalom Normal Subgroup Theorem for lattices in products:

Corollary 9.3. *Let Γ be an irreducible integrable lattice in a just noncompact locally compact second countable group G and let Λ be a dense subgroup of G such that $\Gamma <_c \Lambda$. Assume that $\Lambda /\!\!/ \Gamma$ is just noncompact. Then Λ is just infinite.*

Proof. Write Λ as an irreducible lattice in the product $G \times \Lambda /\!\!/ \Gamma$. Observe that Λ will be integrable (as a lattice) since Γ is. As the relative profinite completion is totally disconnected, it is not isomorphic to $\mathbb{R}$ and therefore the Bader-Shalom Normal Subgroup Theorem implies that Λ has no nontrivial normal subgroups of infinite index. $\quad\square$

10 Commensurators and Lattices in Lie Groups

The primary example of a class of groups our results apply to is commensurators of lattices and lattices in higher-rank Lie groups.

10.1   Actions of Commensurators in Semisimple Higher-Rank Lie Groups

Theorem 10.1. *Let G be a semisimple Lie group (real or p-adic or both) with finite center where each simple factor has rank at least two. Let $\Gamma < G$ be an irreducible lattice. Let $\Lambda < G$ be a countable dense subgroup such that $\Gamma <_c \Lambda$ and that Λ has finite intersection with every proper subfactor of G.*

Then any ergodic measure-preserving action of Λ either has finite stabilizers or the restriction of the action to Γ has finite orbits.

Moreover, the commensurability classes of infinite ergodic invariant random subgroups of Λ are in one-one, onto correspondence with the commensurability classes of open ergodic invariant random subgroups of $\Lambda /\!\!/ \Gamma$.

Proof. First note that if we show that the commensurability classes are in one-one, onto correspondence then any ergodic measure-preserving action of Λ that does not have finite stabilizers must have finite Γ-orbits since any infinite ergodic invariant random subgroup of Λ then contains Γ up to finite index. So we need only prove the one-one correspondence.

The case when G is a real Lie group follows from Corollary 8.2 combined with Theorem 7.2 and the case when G is simple follows from Corollary 8.2 directly since every factor in G has Howe-Moore and property (T).

So we may assume that G has at least two factors, at least one of which is totally disconnected. By Theorem 9.1, any measure-preserving ergodic action of Γ either has finite stabilizers or has finite orbits. In particular, if $G \curvearrowright (X, \nu)$ is a measure-preserving ergodic action such that the restriction to Γ does not have finite orbits then there must exist a positive measure Γ-invariant subset where the Γ-stabilizers are finite. Since $\Lambda \curvearrowright (X, \nu)$ is ergodic we may then apply Proposition 7.3.1 and the Normal Subgroup Theorem for Commensurators to conclude that the restriction of the action to Λ has finite stabilizers. So $\Gamma <_c \Lambda < G$ satisfy the hypotheses of Theorem 6.2 and the result follows. $\square$

10.2   Relative Profinite Completions of Arithmetic Lattices

Theorem 10.2. *Let K be a global field, let $\mathcal{O}$ be the ring of integers, let V be the set of places (inequivalent valuations) on K, let V_∞ be the infinite places (archimedean valuations in the case of a number field) and let K_v be the completion of K over $v \in V$.*

Let $V_\infty \subseteq S \subseteq V$ be any collection of valuations and let $\mathcal{O}_S$ be the ring of S-integers: $\mathcal{O}_S = \{k \in K : v(k) \geq 0 \text{ for all } v \notin S\}$. Let $\mathcal{O}_v = \mathcal{O}_{V_\infty \cup \{v\}}$ be the ring of v-integers and let $\overline{\mathcal{O}_v}$ be the closure of the v-integers in K_v.

Let $\mathbf{G}$ be a simple algebraic group defined over K. Then for $V_\infty \subseteq S' \subseteq S$, the relative profinite completion $\mathbf{G}(\mathcal{O}_S) /\!\!/ \mathbf{G}(\mathcal{O}_{S'})$ is isomorphic to the restricted product

$$\prod_{v \in S \setminus S'}' \mathbf{G}(K_v) = \{(g_v)_{v \in S \setminus S'} : g_v \notin \mathbf{G}(\overline{\mathcal{O}_v}) \text{ for only finitely many } v \in S \setminus S'\}.$$

Proof. That $\mathbf{G}(\mathcal{O}_{S'})$ is commensurated by $\mathbf{G}(\mathcal{O}_S)$ follows from the fact that any fixed element in $\mathcal{O}_S$ has a negative valuation on only finitely many valuations in $S \setminus S'$. Let $\varphi : \mathbf{G}(\mathcal{O}_S) \to \prod_{v \in S \setminus S'}' \mathbf{G}(K_v)$ be the natural diagonal embedding. Then $\varphi(\mathbf{G}(\mathcal{O}_S))$ is

dense (since S' contains V_∞) and $\varphi^{-1}(\prod'_{v \in S \setminus S'} \mathbf{G}(\overline{\mathcal{O}_v})) = \mathbf{G}(\mathcal{O}_{S'})$. By Proposition 2.1.5, $\mathbf{G}(\mathcal{O}_S) /\!\!/ \mathbf{G}(\mathcal{O}_{S'})$ is isomorphic to $\prod'_{v \in S \setminus S'} \mathbf{G}(K_v) /\!\!/ \prod'_{v \in S \setminus S'} \mathbf{G}(\overline{\mathcal{O}_v})$. By Proposition 2.1.4, it is isomorphic to $\prod'_{v \in S \setminus S'} \mathbf{G}(K_v)/M$ where M is the largest closed normal subgroup contained in $\prod'_{v \in S \setminus S'} \mathbf{G}(\overline{\mathcal{O}_v})$. Since $\mathbf{G}$ is simple, the only normal subgroups are of the form $\prod'_{v \in S''} \mathbf{G}(K_v)$ for $S'' \subsetneq S \setminus S'$. But M is contained in the v-integers for each $v \in S \setminus S'$ so M must be trivial. $\qquad\square$

Corollary 10.3. *Let $\mathbf{G}$ be a simple algebraic group over $\mathbb{Q}$ and let $S' \subseteq S$ be sets of primes containing ∞. Then the relative profinite completion $\mathbf{G}(\mathbb{Z}_S) /\!\!/ \mathbf{G}(\mathbb{Z}_{\mathbb{S}'})$ is isomorphic to the restricted product $\prod'_{p \in S \setminus S'} \mathbf{G}(\mathbb{Q}_p)$. In particular $\mathbf{G}(\mathbb{Q}) /\!\!/ \mathbf{G}(\mathbb{Z})$ is isomorphic to $\prod'_{p \in \mathbb{P}} \mathbf{G}(\mathbb{Q}_p)$ where $\mathbb{P}$ is the set of all primes.*

10.3 Actions of Lattices in Semisimple Higher-Rank Lie Groups

Corollary 10.4. *Let G be a semisimple Lie group (real or p-adic or both) with no compact factors, finite center, at least one factor with rank at least two and such that each real simple factor has rank at least two. Let $\Gamma < G$ be an irreducible lattice. Then any ergodic measure-preserving action of Γ either has finite orbits or has finite stabilizers.*

Proof. By Margulis' S-Arithmeticity Theorem, and using that compact kernels and commensuration do not affect the conclusion, we may assume $\Gamma = \mathbf{G}(\mathbb{Z}_S)$ and $G = \prod_{p \in S \cup \{\infty\}} \mathbf{G}(\mathbb{Q}_p)$ where $\mathbf{G}$ is a semisimple algebraic group over $\mathbb{Q}$ and S is a finite set of primes containing ∞.

First observe that since Γ is irreducible, the intersection of Γ with any proper subfactor of G is finite (that is, $\mathbf{G}(\mathbb{Z}_S) \cap \prod_{p \in S \setminus Q} \mathbf{G}(\mathbb{Q}_p)$ is finite for any nonempty $Q \subseteq S$ since the lattice is embedded diagonally). Also, Γ has finite intersection with any compact factor because the original choice of G has no compact factors.

The case when every simple factor of G is real reduces to the Stuck-Zimmer Theorem (Theorem 2.44). So instead we may assume there is some p-adic factor in G. When G is a simple p-adic group, the methods of Stuck-Zimmer combined with the more general Nevo-Zimmer Intermediate Factor Theorem [NZ99a] (for local fields of characteristic zero) give the conclusion since in this case there is no issue with action of G being nonirreducible (however, in the case when there are two factors, such an issue does arise and their work does not apply).

Therefore we may assume that there are at least two noncompact simple factors, one of which is totally disconnected, so combined with the fact that each noncompact simple factor of $\mathbf{G}(\mathbb{Q}_p)$ is Howe-Moore, Theorem 9.1 then implies the conclusion. $\qquad\square$

Corollary 10.5. *Let G be a semisimple Lie group (real or p-adic or both) with no compact factors, trivial center, at least one factor with rank at least two and such that each real simple factor has rank at least two. Let $\Gamma < G$ be an irreducible lattice. Then any ergodic measure-preserving action of Γ on a nonatomic probability space is essentially free.*

Proof. Let $\Gamma \curvearrowright (X, \nu)$ be an ergodic measure-preserving action on a nonatomic probability space. Corollary 10.4 implies that the action either has finite stabilizers or has finite orbits.

The case of finite orbits is ruled out by the space being nonatomic so the action has finite stabilizers. For a finite subgroup $F < \Gamma$ let $E_F = \{x \in X : \mathrm{stab}(x) = F\}$. Since there are only countably many finite groups there is some F with $\nu(E_F) > 0$. Since $gE_F = E_{gFg^{-1}}$, $\nu(E_{gFg^{-1}}) = \nu(E_F)$ for all g so there are at most a finite number of finite groups appearing as stabilizers. Therefore there is a subgroup $\Gamma_0 < \Gamma$ of finite index such that Γ_0 normalizes the finite subgroup F. Then $F \lhd \Gamma_0$ and Γ_0 is a lattice in G hence by Margulis' Normal Subgroup Theorem [Mar91], F is contained in the center of G. Therefore F is trivial so all the stabilizer groups are trivial. $\qquad\square$

10.4   Actions of Rational Groups in Simple Higher-Rank Lie Groups

Corollary 10.6. *Let $\mathbf{G}$ be a simple algebraic group defined over $\mathbb{Q}$ with p-rank at least two for some prime p, possibly ∞, such that $\mathbf{G}(\mathbb{R})$ is either compact or has rank at least two. Let S be any (finite or infinite) set of primes containing ∞ and p. Then every ergodic measure-preserving action of $\mathbf{G}(\mathbb{Z}_S)$ either has finite orbits or has finite stabilizers.*

Proof. The case when S contains only one prime q, possibly ∞, such that $\mathbf{G}(\mathbb{Q}_q)$ is non-compact is a consequence of Corollary 10.4. So assume S contains more than one such prime. Let $S' = \{p, \infty\}$. By Theorem 10.2, the relative profinite completion $\mathbf{G}(\mathbb{Z}_S) /\!\!/ \mathbf{G}(\mathbb{Z}_{\mathbb{S}'})$ is isomorphic to $\prod'_{p \in S \setminus S'} \mathbf{G}(\mathbb{Q}_p)$. The above facts about Lie groups imply that each factor of the relative profinite completion has Howe-Moore. Therefore Corollary 8.3 applied to $\mathbf{G}(\mathbb{Z}_{\mathbb{S}'}) <_c \mathbf{G}(\mathbb{Z}_S) < \prod_{p \in S'} \mathbf{G}(\mathbb{Q}_p)$ (recall $\mathbb{Q}_\infty = \mathbb{R}$) implies the result. $\qquad\square$

Corollary 10.7. *Let $\mathbf{G}$ be a simple algebraic group defined over $\mathbb{Q}$ with p-rank at least two for some prime p, possibly ∞, such that $\mathbf{G}(\mathbb{R})$ is either compact or has rank at least two. Then every nontrivial ergodic measure-preserving action of $\mathbf{G}(\mathbb{Q})$ is essentially free.*

Proof. Since $\mathbf{G}$ is simple as an algebraic group over $\mathbb{Q}$ the group $\mathbf{G}(\mathbb{Q})$ has no finite normal subgroups and therefore the previous corollary implies the conclusion. $\qquad\square$

10.5   Actions of Rational Groups in Simple Higher-Rank Groups

Theorem 10.8. *Let K be a global field, let $\mathcal{O}$ be the ring of integers, let V be the set of places (inequivalent valuations) on K, let V_∞ be the infinite places (archimedean valuations in the case of a number field), let K_v be the completion of K over $v \in V$ and let $\mathcal{O}_v$ be the ring of v-integers. Let $V_\infty \subseteq S \subseteq V$ and let $\mathcal{O}_S$ be the ring of S-integers.*

Let $\mathbf{G}$ be a simple algebraic group defined over K such that $\mathbf{G}$ has v_0-rank at least two for some $v_0 \in S$ (possibly in V_∞), $\mathbf{G}(K_v)$ is noncompact for some $v \in S$, $v \neq v_0$, and $\mathbf{G}(K_{v_\infty})$ is compact or of higher-rank for all $v_\infty \in V_\infty$. Then every ergodic measure-preserving action of $\mathbf{G}(\mathcal{O}_S)$ either has finite orbits or has finite stabilizers.

Proof. Let $S' = V_\infty \cup \{v_0\}$. Let $\Gamma = \mathbf{G}(\mathcal{O}_{S'})$, let $\Lambda = \mathbf{G}(\mathcal{O}_S)$ and let $G = \prod_{v \in S'} \mathbf{G}(K_v)$. Then $\Gamma <_c \Lambda$ and Λ is dense in G (since S contains some valuation $v \neq v_0$ where $\mathbf{G}(K_v)$ is noncompact). Γ is an irreducible lattice in G and each simple factor of G has property (T) and the Howe-Moore property.

By Theorem 10.2, $\Lambda /\!\!/ \Gamma$ is isomorphic to $\prod'_{v \in S \setminus S'} \mathbf{G}(K_v)$ which is a product of simple locally compact groups with the Howe-Moore property. Corollary 8.3 applied to $\Gamma <_c \Lambda < G$ then implies the result. $\qquad\square$

Corollary 10.9. *Let* $\mathbf{G}$ *be a simple algebraic group defined over a global field K with v-rank at least two for some place v such that the v_∞-rank is at least two for every infinite place v_∞. Then every nontrivial ergodic measure-preserving action of* $\mathbf{G}(K)$ *is essentially free.*

References

[ABB$^+$11] Miklós Abert, Nicolas Bergeron, Ian Biringer, Tsachik Gelander, Nikolay Nikolov, Jean Raimbault, and Iddo Samet, *On the growth of Betti numbers of locally symmetric spaces*, Comptes Rendus Mathématique. Académie des Sciences. Paris **349** (2011), no. 15–16, 831–835.

[AEG94] S. Adams, G. Elliott, and T. Giordano, *Amenable actions of groups*, Transactions of the American Mathematical Society **344** (1994), no. 2, 803–822.

[AGV12] Miklós Abert, Yair Glasner, and Bálint Virág, *Kesten's theorem for invariant random subgroups*, Preprint (2012), arXiv:1201.3399.

[AM66] L. Auslander and C. C. Moore, *Unitary representations of solvable Lie groups*, Memoirs of the American Mathematical Society (1966), 66–77.

[AS93] S. Adams and G. Stuck, *Splitting of nonnegatively curved leaves in minimal sets of foliations*, Duke Mathematical Journal **71** (1993), no. 1, 71–92.

[Bek07] Bachir Bekka, *Operator-algebraic supperrigidity for* $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$, Inventiones Mathematica **169** (2007), no. 2, 401–425.

[Ben04] Yves Benoist, *Five lectures on lattices in semisimple groups*, Lecture Notes available online, 2004.

[BG04] N. Bergeron and D. Gaboriau, *Asymptotique des nombres de Betti, invariants ℓ^2 et laminations [Asymptotics of Betti numbers and ℓ^2-invariants and laminations]*, Commentarii Mathematici Helvetici **79** (2004), no. 2, 362–395.

[Bow12] Lewis Bowen, *Invariant random subgroups of the free group*, Preprint (2012), arXiv:1204.5939.

[BS06] Uri Bader and Yehuda Shalom, *Factor and normal subgroup theorems for lattices in products of groups*, Inventiones Mathematicae **163** (2006), no. 2, 415–454.

[CFW81] A. Connes, J. Feldman, and B. Weiss, *An amenable equivalence relation is generated by a single transformation*, Ergodic Theory and Dynamical Systems **1** (1981), no. 4, 431–450.

[CM09] Pierre-Emmanuel Caprace and Nicolas Monod, *Isometry groups of non-positively curved spaces: Discrete subgroups*, Journal of Topology **2** (2009), no. 4, 701–746.

[Cre11] Darren Creutz, *Commensurated subgroups and the dynamics of group actions on quasi-invariant measure spaces*, Ph.D. thesis, University of California: Los Angeles, 2011.

[CS12] Darren Creutz and Yehuda Shalom, *A normal subgroup theorem for commensurators of lattices*, Preprint, 2012.

[CW85] A. Connes and E. J. Woods, *Approximately transitive flows and IPTFI factors*, Ergodic Theory and Dynamical Systems **5** (1985), 203–236.

[Dye59]   H. A. Dye, *On groups of measure-preserving transformations I*, American Journal of Mathematics **81** (1959), 119–159.

[FG10]   Hillel Furstenberg and Eli Glasner, *Stationary dynamical systems*, Dynamical Numbers—Interplay Between Dynamical Systems and Number Theory, Contemporary Mathematics, vol. 532, American Mathematical Society, 2010, pp. 1–28.

[Fur63]   Hillel Furstenberg, *A Poisson formula for semi-simple Lie groups*, The Annals of Mathematics **77** (1963), no. 2, 335–386.

[Fur00]   Alex Furman, *Random walks on groups and random transformations*, Handbook of Dynamical Systems, Elsevier, 2000.

[Gla03]   Eli Glasner, *Ergodic theory via joinings*, Mathematical Surveys and Monographs, vol. 101, American Mathematical Society, 2003.

[Gri11]   R.I. Grigorchuk, *Some topics in the dynamics of group actions on rooted trees*, Proceedings of the Steklov Institute of Mathematics **273** (2011), 64–175.

[GS12]   Rostislav Grigorchuk and Dmytro Savchuk, *Self-similar groups acting essentially freely on the boundary of the binary rooted tree*, Preprint (2012), arXiv:1212.0605.

[HK64]   P. Hall and C. R. Kulatilaka, *A property of locally finite groups*, Journal of the London Mathematical Society **39** (1964), 235–239.

[HM79]   R.E. Howe and C.C. Moore, *Asymptotic properties of unitary representations*, Journal of Functional Analysis **32** (1979), 72–96.

[Ioz94]   Alessandra Iozzi, *Equivariant maps and purely atomic spectrum*, Journal of Functional Analysis **124** (1994), 211–227.

[Jaw94]   Wojciech Jaworski, *Strongly approximately transitive group actions, the Choquet-Deny theorem, and polynomial growth*, Pacific Journal of Mathematics **165** (1994), no. 1, 115–129.

[Jaw95]   _____, *Strong approximate transitivity, polynomial growth, and spread out random walks on locally compact groups*, Pacific Journal of Mathematics **170** (1995), no. 2, 517–533.

[JM12]   Kate Juschenko and Nicolas Monod, *Cantor systems, piecewise translations and simple amenable groups*, Preprint (2012), arXiv:1204.2132.

[Jol05]   Paul Jolissaint, *The Haagerup property for measure-preserving standard equivalence relations*, Ergodic Theory and Dynamical Systems **25** (2005), no. 1, 161–174.

[Kai02]   Vadim Kaimanovich, *SAT actions and ergodic properties of the horosphere foliation*, pp. 261–282, Springer, 2002.

[Kak51]   Shizuo Kakutani, *Random ergodic theorems and Markov processes with a stable distribution*, Proceedings of the Second Berkeley Symposium on Mathematics, Statistics and Probability, University of California Press, 1951, pp. 247–261.

[Kaz67]   D. Kazhdan, *Connection of the dual space of a group with the structure of its closed subgroups*, Functional Analysis and Applications **1** (1967), 63–65.

[Kif86]   Y. Kifer, *Ergodic theory of random transformations*, Progress in Probability and Statistics, vol. 10, Birkh'auser, 1986.

[LM92]   Alexander Lubotzky and Shahar Mozes, *Asymptotic properties of unitary representations of tree automorphisms*, Harmonic Analysis and Discrete Potential Theory, Plenum Press, 1992.

[Mac62]   George Mackey, *Point realizations of transformation groups*, Illinois Journal of Math **6** (1962), 327–335.

[Mac66]   _____, *Ergodic theory and virtual groups*, Annals of Mathematics **166** (1966), 187–207.

[Mar79]   Gregory Margulis, *Finiteness of quotient groups of discrete subgroups*, Funktsionalńyi Analiz i Ego Prilozheniya **13** (1979), 28–39.

[Mar91]   _____, *Discrete subgroups of semisimple Lie groups*, Springer-Verlag, 1991.

[Mat06]   Hiroki Matsui, *Some remarks on topological full groups of Cantor minimal systems*, International Journal of Mathematics **17** (2006), no. 2, 231–251.

[NZ99a]   Amos Nevo and Robert Zimmer, *A generalization of the intermediate factor theorem*, Preprint (1999).

[NZ99b]   _____, *Homogenous projective factors for actions of semi-simple lie groups*, Inventiones Mathematicae (1999), 229–252.

[Oza07]   Narutaka Ozawa, *Boundaries of reduced free group C^*-algebras*, Bulletin of the London Mathematical Society **39** (2007), no. 1, 35–38.

[Pop06]   Sorin Popa, *On a class of type II_1 factors with Betti numbers invariants*, Annals of Mathematics. Second Series **163** (2006), no. 3, 809–899.

[Ram71]   A. Ramsay, *Virtual groups and group actions*, Advances in Mathematics **6** (1971), 253–322.

[Rob93]   A. Guyan Robertson, *Property (T) for II_1 factors and unitary representations of Kazhdan groups*, Mathematische Annalen **256** (1993), 547–555.

[Rot80]   Sheldon Rothman, *The von Neumann kernel and minimally almost periodic groups*, Transactions of the American Mathematical Society (1980), 401–421.

[Sch80]   G. Schlichting, *Operationen mit periodischen Stabilisatoren*, Archiv der Math. **34** (1980), 97–99.

[Sch84]   Klaus Schmidt, *Asymptotic properties of unitary representations and mixing*, Proc. London Math. Soc. (3) **48** (1984), no. 3, 445–460.

[Sch96]   _____, *From infinitely divisible representations to cohomological rigidity, analysis, geometry and probability*, vol. 10, Hindustan Book Agency, Delhi, 1996.

[SW09]   Yehuda Shalom and George Willis, *Commensurated subgroups of arithmetic groups, totally disconnected groups and adelic rigidity*, Preprint (2009), arXiv:0911.1966.

[SZ94]   Garrett Stuck and Robert Zimmer, *Stabilizers for ergodic actions of higher rank semisimple groups*, The Annals of Mathematics **139** (1994), no. 3, 723–747.

[TD12]   Robin Tucker-Drob, *Mixing actions of countable groups are almost free*, Preprint (2012).

[Tza00]   Kroum Tzanev, *C^*-algébres de Hecke at K-theorie*, Ph.D. thesis, Université Paris 7 – Denis Diderot, 2000.

[Tza03]   _____, *Hecke C^*-algébres and amenability*, Journal of Operator Theory **50** (2003), 169–178.

[Var63]   V.S. Varadarajan, *Groups of automorphisms of Borel spaces*, Transactions of the American Mathematical Society **109** (1963), no. 2, 191–220.

[Ver11]   A. M. Vershik, *Nonfree actions of countable groups and their characters*, Journal of Mathematical Science **174** (2011), no. 1, 1–6.

[Ver12]   ———, *Totally nonfree actions and the infinite symmetric group*, Moscow Mathematical Journal **12** (2012), 193–212.

[Zim77]   Robert Zimmer, *Hyperfinite factors and amenable group actions*, Inventiones Mathematicae **41** (1977), no. 1, 23–31.

[Zim82]   ———, *Ergodic theory, semi-simple lie groups, and foliations by manifolds of negative curvature*, Publications Mathématique de l'IHÉS (1982), 37–62.

[Zim84]   ———, *Ergodic theory and semisimple groups*, Birkhauser, 1984.

[Zim87]   ———, *Split rank and semisimple automorphism group of a G-structure*, Journal of Differential Geometry **26** (1987), 169–173.