Profinite Methods in Abstract Groups

(*On the sixtieth birthday of Stuart Margolis*)

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International Conference on Geometric, Combinatorial and Dynamics Aspects of Semigroup and Group Theory
Bar Ilan, June 2013

Most results in this presentation are joint work with Pavel Zalesskii

Notation: $\mathcal{C}$ denotes a nonempty class of finite groups satisfying the following conditions:

- If $K \leq H$ and $H \in \mathcal{C}$, then $K \in \mathcal{C}$,
- If $K \lhd H$ and $H \in \mathcal{C}$, then $H/K \in \mathcal{C}$,
- If

$$1 \to K \to G \to H \to 1$$

is an exact sequence of finite groups and $H \in \mathcal{C}$, then $G \in \mathcal{C}$.

For example, $\mathcal{C}$ could be the class of

(i) all finite groups, or

(ii) all finite solvable groups, or

(iii) all finite p-groups, where p is a fixed prime number.

Definition 1 *One says that an abstract group R is*

"conjugacy $\mathcal{C}$-separable"

if for every pair of elements $x, y \in R$, these elements are conjugate in R if and only if their images in every quotient of R which is in $\mathcal{C}$ are conjugate.

This is a *"profinite property"* in the following sense:

Assume that R is residually $\mathcal{C}$, that is,

$$\bigcap_{U \triangleleft R, R/U \in \mathcal{C}} U = 1.$$

Then R is naturally embedded in its *pro-$\mathcal{C}$ completion* $R_{\hat{\mathcal{C}}}$

$$R \hookrightarrow R_{\hat{\mathcal{C}}} = \varprojlim_{U \triangleleft R, R/U \in \mathcal{C}} R/U$$

Then R being conjugacy $\mathcal{C}$-separable means that whenever two elements $x, y \in R$ are conjugate in $R_{\hat{\mathcal{C}}}$, then they are conjugate in R.

Definition 2 *One says that a subgroup H of an abstract group R is*

"conjugacy $\mathcal{C}$-distinguished"

if whenever $a \in R$, then H contains a conjugate of a in R if and only if the images of a and H in any quotient of R that is in $\mathcal{C}$ satisfy analogous properties; or equivalently, if a^R denotes the conjugacy class of a in R, then $a^R \cap H = \emptyset$ if and only if there exists an open (in the pro-$\mathcal{C}$ topology of R) normal subgroup N of R such that $a^R \cap HN = \emptyset$.

Being conjugacy $\mathcal{C}$-distinguished is a 'profinite property' as well.

In this talk we are interested in finitely generated groups

$$R$$

that are extensions of free groups by groups in $\mathcal{C}$ (*free-by-$\mathcal{C}$ groups*), i.e., R is a group which contains a normal subgroup Φ that is free of finite rank and

$$R/\Phi \in \mathcal{C}.$$

Example. It is easy to give examples of these types of groups: let $A, B \in \mathcal{C}$ and consider the exact sequence

$$1 \longrightarrow \Phi \longrightarrow A * B \xrightarrow{\varphi} A \times B \longrightarrow 1,$$

where φ is the homomorphism that sends A identically to A, and B identically to B, and where $\Phi = \mathrm{Ker}(\varphi)$. It is not difficult to see that Φ (the *cartesian subgroup of the free product $A * B$*) is a free group with basis

$$\{[a, b] \mid 1 \neq a \in A, 1 \neq b \in B\}.$$

In fact this example is rather representative. In general we have

Proposition *If R is a finitely generated group which is an extension of a free group by a group in $\mathcal{C}$, then there exists a finite connected graph Δ and a graph of groups that are in $\mathcal{C}$,*

$$(\mathcal{G}, \Delta),$$

over Δ so that R is the fundamental group $\Pi^{abs} = \Pi_1^{abs}(\mathcal{G}, \Delta)$ of this graph of groups:

$$R = \Pi_1^{abs}(\mathcal{G}, \Delta).$$

[This follows immediately from a characterization of finite extensions of free groups due to *Serre* and *Karrass-Pietrovski-Solitar*.]

An important consequence is that R acts in a natural way on a tree with finite stabilizers (en $\mathcal{C}$): specifically R operates on the universal covering graph

$$S^{abs} = S^{abs}(\mathcal{G}, \Delta)$$

of the graph of groups $(\mathcal{G}, \Delta)$.

Since the groups of $(\mathcal{G}, \Delta)$ are in $\mathcal{C}$, the graph of groups $(\mathcal{G}, \Delta)$ can be viewed as a graph of pro-$\mathcal{C}$ groups. Therefore one has a corresponding fundamental pro-$\mathcal{C}$ group

$$\Pi = \Pi_1(\mathcal{G}, \Delta)$$

and a corresponding universal pro-$\mathcal{C}$ cover

$$S = S(\mathcal{G}, \Delta)$$

of the graph $(\mathcal{G}, \Delta)$ of pro-$\mathcal{C}$ groups.

In this case (because $\Pi_1^{abs}(\mathcal{G}, \Delta)$ is a group residually $\mathcal{C}$) one has that $\Pi_1(\mathcal{G}, \Delta)$ is the pro-$\mathcal{C}$ completion of the group $\Pi_1^{abs}(\mathcal{G}, \Delta)$; in particular

$$\Pi_1^{abs}(\mathcal{G}, \Delta) \leq_{dense} \Pi_1(\mathcal{G}, \Delta).$$

On the other hand (because the groups $\mathcal{G}(m)$ ($m \in \Delta$) are finite, and hence closed in the pro-$\mathcal{C}$ topology of $\Pi_1^{abs}(\mathcal{G}, \Delta)$), we have

$$S^{abs}(\mathcal{G}, \Delta) \subseteq_{dense} S(\mathcal{G}, \Delta)$$

(as graphs).

In this talk I will concentrate on the following sample result

Theorem 3 *Let R be a finitely generated free-by-$\mathcal{C}$ abstract group and let H be a finitely generated subgroup of R which is closed in its pro-$\mathcal{C}$ topology. Then H is conjugacy $\mathcal{C}$-distinguished.*

This generalizes a result of Wilson-Zalesskii, who proved this in the case when $\mathcal{C}$ is the class of all finite groups. Observe that in this case a finitely generated subgroup H of a free-by-finite group R is automatically closed in the profinite topology of R.

The following result is central in the work that I am presenting here:

Theorem 1 *Let R be a group which is an extension of a free group of finite rank by a finite group in $\mathcal{C}$. Let H be a subgroup of R finitely generated and closed (in the pro-$\mathcal{C}$ topology of R). Then*

$$\overline{\mathcal{N}_R(H)} = \mathcal{N}_{R_{\hat{\mathcal{C}}}}(\bar{H}),$$

where if X is a subset of R, we denote by $\bar{X}$ its topological closure in

$$R_{\hat{\mathcal{C}}} = \varprojlim_{U \lhd R, R/U \in \mathcal{C}} R/U,$$

the pro-$\mathcal{C}$ completion of R.

For the proof we distinguish two cases.

Case 1 : H is infinite. In this case we need to study the H-invariant pro-$\mathcal{C}$ subtrees of S^{abs}:

Proposition *We continue with the same hypotheses and the same notation. Say*

$$H = \langle h_1, \ldots, h_r \rangle$$

is an infinite finitely generated subgroup of

$$R = \Pi_1^{abs}(\mathcal{G}, \Delta),$$

and assume H is closed in the pro-$\mathcal{C}$ topology of $\Pi_1^{abs}(\mathcal{G}, \Delta)$. Denote by $\bar{H}$ topological closure of H in $R_{\hat{\mathcal{C}}}$. Then S^{abs} has a unique minimal H-invariant subtree D^{abs}, and its topological closure $\overline{D^{abs}} = D$ in S is the unique minimal $\bar{H}$-invariant pro-$\mathcal{C}$ subtree of S.

In addition

(a) $S^{abs} \cap D = D^{abs}$, *and*

(b) $H \backslash D^{abs} = \bar{H} \backslash D$ *is finite.*

Case 2 : H is finite. In this case the following result plays a fundamental role.

Proposition *Let R be a group which is an extension of a free group of finite rank by a finite group in $\mathcal{C}$. Let*

$$H_1 \quad \textit{and} \quad H_2$$

be finitely generated and closed subgroups of R (closed in the pro-$\mathcal{C}$ topology of R). Then

$$\overline{H_1 \cap H_2} = \overline{H_1} \cap \overline{H_2}$$

in

$$R_{\hat{\mathcal{C}}} = \varprojlim_{U \triangleleft R,\, R/U \in \mathcal{C}} R/U,$$

the pro-$\mathcal{C}$ completion of R.

Applications:

Recall that R is conjugacy $\mathcal{C}$-separable if whenever two elements $x, y \in R$ are conjugate in $R_{\hat{\mathcal{C}}}$, then they are conjugate in R.

Theorem 2 *Let R be a group which is an extension of a free group of finite rank by a finite group in $\mathcal{C}$. Then R is conjugacy $\mathcal{C}$-separable.*

This generalizes results of

– *J. Dyer* (who proved it when $\mathcal{C}$ is the class of all finite groups), and

– *E. Toinet* (who has proved it when $\mathcal{C}$ is the class of all finite p-groups, where p is a fixed prime number).

Theorem 3 *Let R be a finitely generated free-by-$\mathcal{C}$ abstract group and let H be a finitely generated subgroup of R which is closed in its pro-$\mathcal{C}$ topology. Then H is conjugacy $\mathcal{C}$-distinguished.*

To prove this theorem we view R as the fundamental group of a graph of groups $(\mathcal{G}, \Delta)$, where the graph Δ is finite and $\mathcal{G}(m) \in \mathcal{C}$, for each $m \in \Delta$; furthermore, $R_{\hat{\mathcal{C}}}$ is the pro-$\mathcal{C}$ fundamental group of $(\mathcal{G}, \Delta)$ viewed as graph of groups in $\mathcal{C}$.

Case (i): a has finite order.

– Recall that $\bar{H} = H_{\hat{\mathcal{C}}}$.

– Since H is also a finitely generated free-by-$\mathcal{C}$ group, one has $H = \Pi^{abs}(\mathcal{G}', \Delta')$, over a finite graph Δ'; and $\bar{H} = H_{\hat{\mathcal{C}}}$ is the pro-$\mathcal{C}$ fundamental group of $(\mathcal{G}', \Delta')$, with $\mathcal{G}'(v) = \Pi^{abs}(v) \leq H$.

– Since $\gamma^{-1} a \gamma \in \bar{H}$ has finite order, it is conjugate in $\bar{H} = H_{\hat{\mathcal{C}}}$ to an element of some vertex group $\mathcal{G}'(w) = \Pi^{abs}(w) \leq H$.

– Therefore, since $H_{\hat{\mathcal{C}}} \leq R_{\hat{\mathcal{C}}}$, a is conjugate in $R_{\hat{\mathcal{C}}}$ to an element, say b, of H.

– Thus, by Theorem 2, there exists $c \in R$ with $c^{-1} a c = b \in H$.

Case (ii): a has infinite order.

– One may reduce to the following situation. Φ is an open free subgroup of R such that

$$\Phi = (\Phi \cap H) * L$$

and $\gamma \in \bar{\Phi} = \Phi_{\hat{\mathcal{C}}}$. Moreover, for some natural number n, $1 \neq a^n \in \Phi$.

– Using that a^n is conjugate in $\bar{\Phi} = \overline{(\Phi \cap H)} \amalg \bar{L}$, one proves that there exists $c \in \Phi$ with

$$c^{-1} a^n c \in \Phi \cap H,$$

and from this that $(\gamma^{-1} c) c^{-1} a^n c (c^{-1} \gamma) \in \overline{(\Phi \cap H)}$.

– Then
$$c^{-1} \gamma \in \overline{(\Phi \cap H)}.$$

This together with $\gamma^{-1} a \gamma \in \bar{H}$ implies that

– $(\gamma^{-1} c) c^{-1} a c (c^{-1} \gamma) \in \bar{H}$, and therefore $c^{-1} a c \in \bar{H} \cap R = H$.