Decidability of the Elementary Theory of a Torsion-Free Hyperbolic Group

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(based on joint results with A. Myasnikov)

Results

In this talk I will present the following results:

- Give an effective quantifier elimination for the elementary theory of a torsion-free hyperbolic group,
- Decidability of the elementary theory of such a group,
- Some algorithmic results for total relatively hyperbolic groups (with J. Macdonald, P. Weil).

Results

Let Γ be a non-elementary torsion-free hyperbolic group. We consider formulas in the language L_A that contains generators of Γ as constants.

Notice that in the language L_A every finite system of equations is equivalent to one equation and every finite disjunction of equations is equivalent to one equation.

It was proved by Sela (2009) that every first order formula in the theory of Γ is equivalent to a boolean combination of $\forall\exists$-formulas. Furthermore, a more precise result holds.

Theorem

(Kh, Miasn.) Every first order formula in Γ in the language L_A is equivalent to some boolean combination of formulas

$$\exists X \forall Y (U(P, X) = 1 \wedge V(P, X, Y) \neq 1), \tag{1}$$

where X, Y, P are tuples of variables.

Results

We will prove the following result.

Theorem

Let Γ be a torsion free hyperbolic group. There exists an algorithm given a first-order formula ϕ to find a boolean combination of formulas (1) that define the same set as ϕ over Γ.

Theorem

The $\forall\exists$-theory of a torsion-free hyperbolic group is decidable.

These results imply

Theorem

The elementary theory of a torsion-free hyperbolic group is decidable.

We proved a similar result for a free group in 2006 (solution of Tarski's problem).

Makanin (82, 85): Solution of equations in a free group F and decidability of the $\exists$-theory of F and of the positive theory of F.

Rips, Sela (95) : An algorithm to solve equations in torsion-free hyperbolic groups by reducing the problem to equations in free groups.

Dahmani, Guirardel (2009) Solution of equations in virtually free and hyperbolic groups.

Sela (2009), Dahmani (2009), Khar., Macdonald (2012): Decidability of the $\exists$-theory and of the positive theory of a torsion-free hyperbolic group .
Diekert, Gutierresz, Hagenah, The existential theory of equations with rational constraints in free groups is PSPACE-complete.

Γ-limit groups

Let Γ be a non-elementary torsion-free hyperbolic group. A group G is fully residually Γ if for any finite number of non-trivial elements in G there is a homomorphism $G \to \Gamma$ such that the images of these elements are non-trivial.
A finitely generated fully residually Γ group is called a Γ-limit group.
Warning: Nor all Γ-limit groups are finitely presented!

Our Γ-limit groups will be given as f.g. subgroups of NTQ groups.

Baumslag, Miasnikov, Remeslennikov, B. Plotkin , and Kh., Miasnikov introduced Algebraic Geometry for groups. The radical of $S = 1$ is

$$R(S) = \{W \in \Gamma[Z] | \forall B(S(B, A) = 1 \rightarrow W(B, A) = 1)\}.$$

Denote $\Gamma_{R(S)} = (F(Z) * \Gamma)/R(S)$, it is called the coordinate group.

NTQ groups

Let G be a group generated by A and let $S(X, A) = 1$ be a system of equations. Suppose S can be partitioned into subsystems

$$\begin{array}{rcl} S_1(X_1, X_2, \ldots, X_n, A) & = & 1, \\ S_2(X_2, \ldots, X_n, A) & = & 1, \\ & \ldots & \\ S_n(X_n, A) & = & 1 \end{array}$$

where $\{X_1, X_2, \ldots, X_n\}$ is a partition of X. Define groups G_i for $i = 1, \ldots, n+1$ by

$$\begin{array}{rcl} G_{n+1} & = & G \\ G_i & = & G_{R(S_i, \ldots, S_n)}. \end{array}$$

We interpret S_i as a subset of $G_{i-1} * F(X_i)$, i.e. letters from X_i are considered variables and letters from $X_{i+1} \cup \ldots \cup X_n \cup A$ are considered as constants from G_i.

NTQ groups

A system $S(X,A)=1$ is called *triangular quasi-quadratic* (TQ) if it can be partitioned as above such that for each i one of the following holds:

1. S_i is quadratic in variables X_i;
2. $S_i = \{[x,y]=1, [x,u]=1,\ x,y \in X_i, u \in U_i\}$ where U_i is a finite subset of G_{i+1} such that $\langle U_i \rangle = C_{G_{i+1}}(g)$ for some $g \in G_{i+1}$;
3. $S_i = \{[x,y]=1,\ x,y \in X_i\}$;
4. S_i is empty.

The system is called *non-degenerate triangular quasi-quadratic* (NTQ) if for every i the system $S_i(X_i,\dots,X_n,A)$ has a solution in the coordinate group $G_{R(S_{i+1},\dots,S_n)}$.

NTQ groups

NTQ groups are coordinate groups of NTQ systems. NTQ groups over torsion free hyperbolic groups are total relatively hyperbolic.

We will use the following definition of relative hyperbolicity. A group G with generating set A is relatively hyperbolic relative to a collection of finitely generated subgroups $\mathcal{P} = \{P_1, \ldots, P_k\}$ if the graph $Cay(G, A \cup B)$ (where B be the set of all elements in subgroups in $\mathcal{P}$) is a hyperbolic metric space, and the pair $\{G, \mathcal{P}\}$ has the Bounded Coset Penetration property.

Reduction to systems of equations over a free group

If $\Gamma =< A|R >$, consider the natural hom $\pi : F(A) \to \Gamma$. The problem of deciding whether or not a system of equations S over Γ has a solution was solved by Rips and Sela by constructing *canonical representatives* for certain elements of Γ in $F(A)$.

We use the reduction to find all solutions to $S(Z, A) = 1$ over Γ and to solve many other algorithmic problems.

We may assume that the system $S(Z, A) = 1$, in variables $z_1, \ldots, z_l$, consists of m constant equations and $q - m$ triangular equations, i.e.

$$S(Z, A) = \begin{cases} z_{\sigma(j,1)} z_{\sigma(j,2)} z_{\sigma(j,3)} = 1 & j = 1, \ldots, q - m \\ z_s = a_s & s = l - m + 1, \ldots, l \end{cases}$$

where $\sigma(j, k) \in \{1, \ldots, l\}$ and $a_i \in \Gamma$. One assigns to each element $g \in \Gamma$ a word $\theta_m(g) \in F$ satisfying

$$\theta_m(g) = g \text{ in } \Gamma$$

called its *canonical representative*.

Canonical representatives

Let $L = q \cdot 2^{5050(\delta+1)^6(2|A|)^{2\delta}}$. Suppose $\psi : F(Z, A) \to \Gamma$ is a solution of $S(Z, A) = 1$ and denote

$$\psi(z_{\sigma(j,k)}) = g_{\sigma(j,k)}.$$

Then there exist $h_k^{(j)}, c_k^{(j)} \in F(A)$ (for $j = 1, \dots, q - m$ and $k = 1, 2, 3$) such that

1. each $c_k^{(j)}$ has length less than L (as a word in F),
2. $c_1^{(j)} c_2^{(j)} c_3^{(j)} = 1$ in Γ,
3. there exists $m \leq L$ such that the canonical representatives satisfy the following equations in F:

$$\theta_m(g_{\sigma(j,1)}) = h_1^{(j)} c_1^{(j)} \left(h_2^{(j)}\right)^{-1} \quad (2)$$

$$\theta_m(g_{\sigma(j,2)}) = h_2^{(j)} c_2^{(j)} \left(h_3^{(j)}\right)^{-1} \quad (3)$$

$$\theta_m(g_{\sigma(j,3)}) = h_3^{(j)} c_3^{(j)} \left(h_1^{(j)}\right)^{-1}. \quad (4)$$

Let $^{-}$ denote the canonical epimorphism $F(Z, A) \to \Gamma_{R(S)}$. For a homomorphism $\phi : F(Z, A) \to K$ we define $\overline{\phi} : \Gamma_{R(S)} \to K$ by

$$\overline{\phi}(\overline{w}) = \phi(w),$$

Reduction to systems of equations over a free group

Lemma

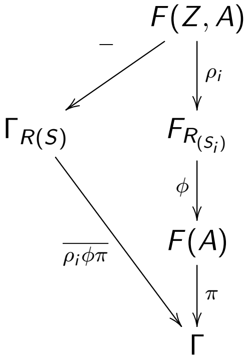
Let $\Gamma = \langle A \,|\, \mathcal{R} \rangle$ be a torsion-free δ-hyperbolic group and $\pi : F(A) \to \Gamma$ the canonical epimorphism. There is an algorithm that, given a system $S(Z, A) = 1$ of equations over Γ, produces finitely many systems of equations

$$S_1(X_1, A) = 1, \ldots, S_n(X_n, A) = 1 \qquad (5)$$

over F, constants $\lambda, \mu > 0$, and homomorphisms $\rho_i : F(Z, A) \to F_{R(S_i)}$ for $i = 1, \ldots, n$ such that

1. *for every F-homomorphism $\phi : F_{R(S_i)} \to F$, the map $\overline{\rho_i \phi \pi} : \Gamma_{R(S)} \to \Gamma$ is a Γ-homomorphism, and*
2. *for every Γ-homomorphism $\psi : \Gamma_{R(S)} \to \Gamma$ there is an integer i and an F-homomorphism $\phi : F_{R(S_i)} \to F(A)$ such that $\overline{\rho_i \phi \pi} = \psi$. Moreover, for any $z \in Z$, the word $z^{\rho_i \phi}$ labels a (λ, μ)-quasigeodesic path for z^{ψ}.*

Reduction to systems of equations over a free group



Effective construction of the solution set

Proposition

(K, Macdonald, Miasn)If Γ *is a torsion-free hyperbolic group, and* $S(X) = 1$ *a system of equations (having a solution in* Γ*), then there exists an algorithm to construct a finite number of strict fundamental sequences of solutions*

$$\sigma_1 \pi_1 \sigma_2 \ldots \pi_n \pi$$

from $\Gamma_{R(S)}$ *to* $\Gamma * F(Y)$ *that encode all solutions of* $S(X) = 1$ *in* Γ*.*

Key Technical Result

Proposition

Let H be a Γ-limit group given as the image of the group $\Gamma_{R(S)}$, where $S = S(Z, A)$, in the NTQ group N corresponding to a strict fundamental sequence in $\mathcal{T}(S, \Gamma)$. There is an algorithm to construct a presentation of H as a series of amalgamated products and HNN-extensions with abelian (or trivial) edge groups beginning with cyclic groups, Γ, and a finite number of subgroups of Γ given by finite generating sets.
Moreover, if $g_1, \ldots, g_k$ are generators of this presentation, and $h_1, \ldots, h_s$ are images of the generators of $\Gamma_{R(S)}$ in N (they are also generators of H), then there is an algorithm to express $g_1, \ldots, g_k$ in terms of $h_1, \ldots, h_s$ and vice versa.

Let

$$G =< H, t | t^{-1} u t = v >$$

For every hom $\psi : G \to \Gamma$ there are homs ψ_n, such that $\psi_n(h) = \psi(h),\ \psi_n(t) = u^n \psi(t)$.
For large n, $\psi_n(t)$ is much longer that $\psi_n(h)$, this can be seen when we construct canonical representatives.

Proposition

Let H be the image of the group $\Gamma_{R(S)}$, where $S = S(Z, A)$, in the NTQ group N corresponding to a strict fundamental sequence in the finite set from the previous proposition (denote it by $\mathcal{T}(S, \Gamma)$). Then there is an algorithm to find the generators (in N) of the vertex groups in the primary abelian JSJ decomposition of H. The algorithm also determines which vertex groups are QH and which are abelian.

Relative quasi-convexity

A subgroup R of G is called relatively quasiconvex if there exists a constant $K > 0$ such that for any element f of R and an arbitrary geodesic path p from 1 to f in $Cayley(G, A \cup B)$, for any vertex $v \in p$, there exists a vertex $w \in R$ such that

$$dist_A(v, w) \leq K.$$

Dahmani, Hruska, Martinez Pedroza, Osin, Wise etc.

One of the many technical results that we need to prove is the following theorem.

Theorem

(K.,Myasn., Weil) Let G be a toral relatively hyperbolic group (given by a finite presentation) and H and K finitely generated relatively quasi-convex subgroups of G (we are told they are) given by finite generating sets. Then one can effectively find a finite family $\mathcal{J}$ of non-trivial intersections $J = H^g \cap K \neq 1$ such that any non-trivial intersection $H^{g_1} \cap K$ has form J^k for some $k \in K$ and $J \in \mathcal{J}$. One can effectively find the generators of the subgroups from $\mathcal{J}$.

For limit groups this was proved by Kh.,Myasnikov, Serbin in 2005.