Simple-homotopy

Definition

Let K and L be finite CW complexes. There is an elementary expansion from K to L if $L = K \cup_f D^n$ where $f : D_-^{n-1} \to K$. We say that there is an elementary collapse from L to K. A homotopy equivalence is called simple if it is homotopic to a homotopy equivalence induced by a sequence of elementary expansions and collapses.

Theorem

If K is simply connected, then every homotopy equivalence $K \to L$ is simple.

Simple-homotopy

Example

There are finite complexes with $\pi_1 = Z_5$ that are homotopy equivalent but not simple-homotopy equivalent.

K is obtained by attaching D^2 to S^1 using a map of degree 5. L is obtained from K by wedging with S^2 and then attaching D^3 according to the prescription $1 - t + t^3$.[1]

In general, if K and L have fundamental group π and $f : K \to L$ is a homotopy equivalence, f is simple if and only if a torsion, $\tau(f)$ in the Whitehead group $Wh(\pi)$ of π is trivial. $Wh(\pi)$ measures whether an invertible matrix with entries in the integral group ring $Z\pi$ can be row and column reduced to a matrix with $\pm$ group elements along the diagonal.

[1] $(1 - t + t^2)(t + t^2 - t^4) = 1$

Hauptvermutung

Question: If K and L are homeomorphic simplicial complexes, must K and L be piecewise-linear(ly?) homeomorphic?

Originally, this was thought of as an approach to proving the topological invariance of simplicial homology. Of course, the introduction of the notion of homotopy equivalence gave a much easier proof of a much stronger theorem.

Theorem (Milnor)

There exist finite simplicial complexes K and L that are homeomorphic but that are not PL homeomorphic.

Topological invariance of torsion

Theorem (Chapman)

If $f : K \to L$ is a homeomorphism between simplicial complexes, then $\tau(f) = 0$.

Chapman's proof was modeled on Kirby-Siebenmann's work on the Hauptvermutung for PL manifolds, but in the setting of Hilbert cube manifolds.

Topological invariance of torsion

Definition

We call a homotopy equivalence $f : K \to L$ between simplicial complexes an ϵ-equivalence if there exist a homotopy equivalence $g : L \to K$ and homotopies $h_t : f \circ g \cong id$ and $k_t : g \circ f \cong id$ so that $\mathrm{diam}\{h_t(x) | 0 \leq t \leq 1\} < \epsilon$ for each $x \in L$ and $\mathrm{diam}\{f(k_t(y)) | 0 \leq t \leq 1\} < \epsilon$ for each $y \in K$.

Theorem (F.)

Given L, there is an $\epsilon > 0$ so that if $f : K \to L$ is an ϵ-equivalence, then $\tau(f) = 0$.

The first proof of this showed that $K \times Q$ and $L \times Q$ were homeomorphic, Q being the Hilbert cube, whence the result followed from Chapman. However, this point of view soon led to more direct proofs of Chapman's theorem.

Topological manifolds, $n \geq 5$

Theorem (Chapman-F.)

If M^n is a closed connected topological manifold, $n \geq 5$, then given $\epsilon > 0$, there is a $\delta > 0$ so that if $f : N \to M$ is an δ-equivalence, N closed, then f is ϵ-homotopic to a homeomorphism.

Due to the efforts of Freedman-Quinn, Perlman, and others, this result is now known in all dimensions.

Theorem (F)

If M^n is a closed connected topological manifold, $n \geq 5$, then there is an $\epsilon > 0$ so that if $f : M \to N$ is a map to a connected manifold of the same dimension such that $\mathrm{diam}\ f^{-1}(x) < \epsilon$ for each $x \in N$, then f is homotopic to a homeomorphism.

Grove-Petersen-Wu

Question: Do such homotopy equivalences occur naturally? (Yes, in geometric topology, but I'll give an application to differential geometry.)

Theorem (Grove-Petersen-Wu)

The collection of closed Riemannian n-manifolds, $n \geq 5$, with diameter $< D$, volume $> v$, and sectional curvature $> \kappa$ contains only finitely many homeomorphism (and therefore diffeomorphism) types.

As above, this result is now known for homeomorphisms in all dimensions. Some of Perlman's work generalizes Grove-Petersen-Wu. This example is included to give a general idea of what applications might look like.

Definition

A monotone function $\rho : [0, R) \to [0, \infty)$ is a contractibility function for a space X if $B_t(x)$ contracts to a point in $B_{\rho(t)}(x)$ for every $x \in X$ and $t \in [0, R]$, $\rho(0) = 0$ and $\rho(t) \geq t$.

Theorem (Grove-Petersen)

There is a function $\rho : [0, R) \to [0, \infty)$ *which is a contractibility function for every closed Riemannian n-manifold,* $n \geq 5$*, with diameter* $< D$*, volume* $> v$*, and sectional curvature* $> \kappa$*.*

This collection of Riemannian manifolds is precompact in Gromov-Hausdorff space. It is easy to see that manifolds with contractibility function ρ that are close enough together must be epsilon homotopy equivalent. Therefore, if limit points of the collection are manifolds, we're done. An argument involving crossing with a two-torus and peeling the factors off again removes this last hurdle.

Theorem (Dranishnikov-F. flexibility)

There exist Riemannian manifolds M_t *and* N_t, $0 \leq t < 1$ *and a function* $\rho : [0, R) \to [0, \infty)$ *which is a contractibility function for each* M_t *and* N_t, *so that* $\lim_{t\to 1} M_t = \lim_{t\to 1} N_t$ *with* M_t*'s homeomorphic to each other,* N_t*'s homeomorphic to each other, but* M_t*'s not homeomorphic to* N_t*'s. These manifolds do have the same simple-homotopy types and the same rational Pontrjagin classes.*

Theorem (Dranishnikov-F. rigidity)

If M_t *is two-connected and the homology of* M_t *contains no odd torsion, then the phenomenon above can't happen.*

The difference between this and the situation that Grove-Petersen encountered is that the common limit of the M_t's and N_t's can be infinite-dimensional and in this case the homeomorphism type can vary, but only by finitely many homeomorphism types.

Theorem (Dranishnikov-F.)

If $\mathcal{C}$ is a precompact collection of Riemannian n-manifolds, $n \neq 3$, such that there is a contractibility function $\rho : [0, R) \to [0, \infty)$ which is a contractibility function for each $M \in \mathcal{C}$, then $\mathcal{C}$ contains only finitely many homeomorphism types.

Actually, this theorem appears earlier in a paper of mine in the Duke Journal. The argument in the paper with Dranishnikov is different and more illuminating.