eigenvalue bounds and metric uniformization

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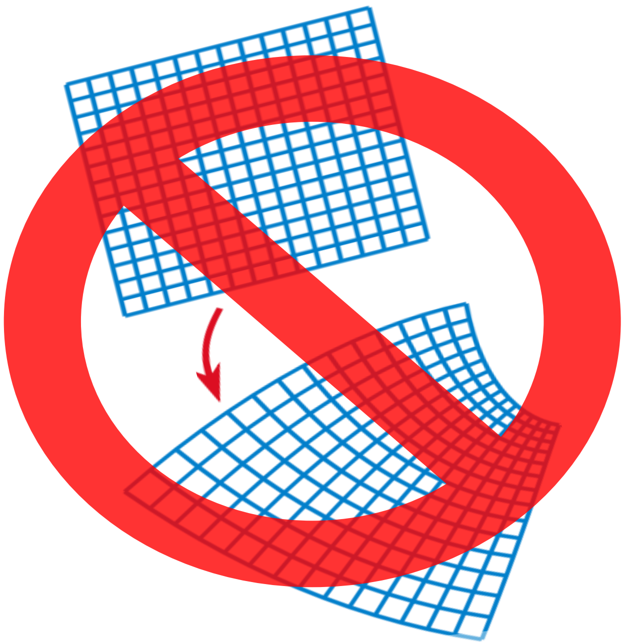
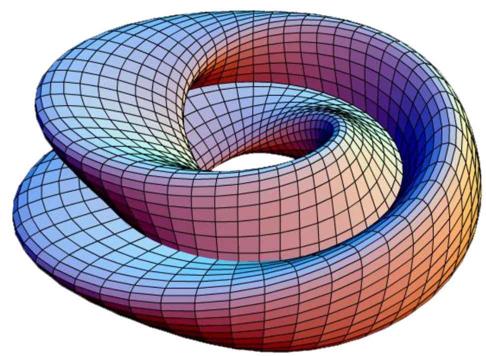
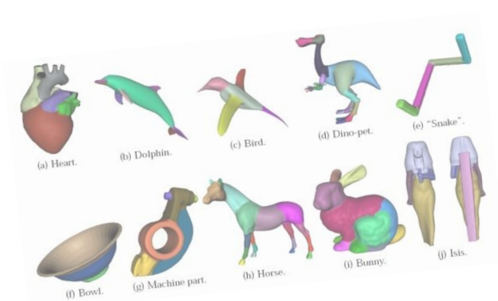
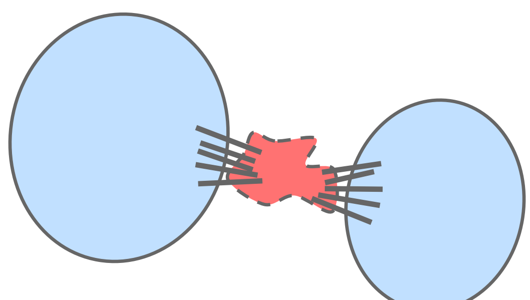
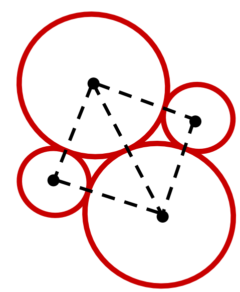




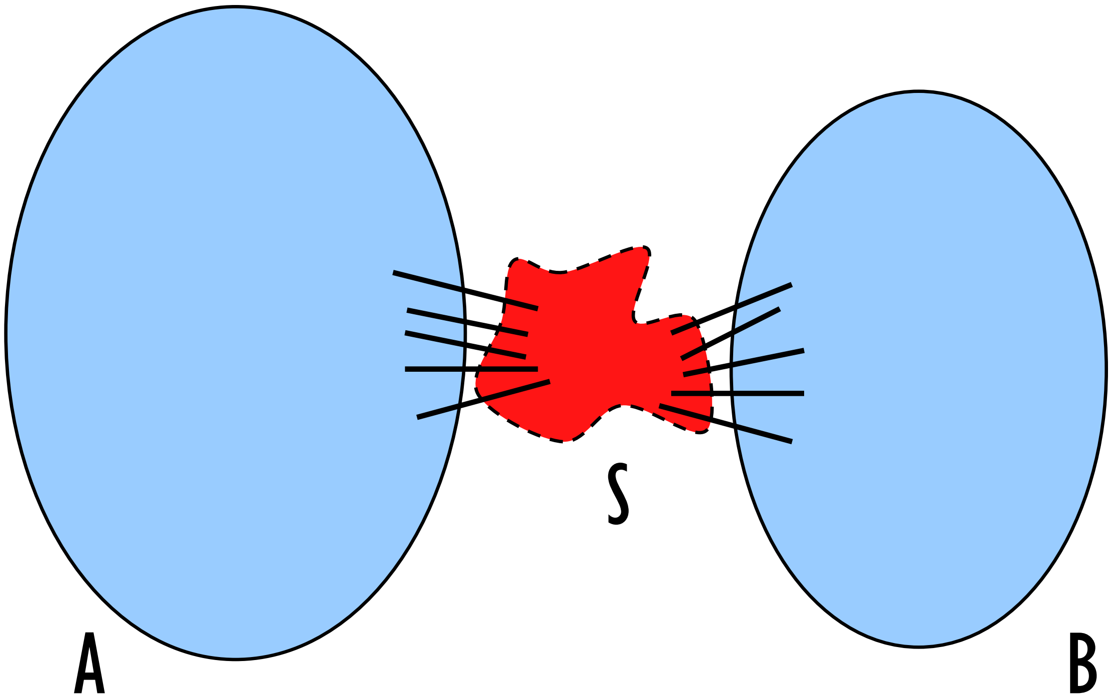






separators in planar graphs

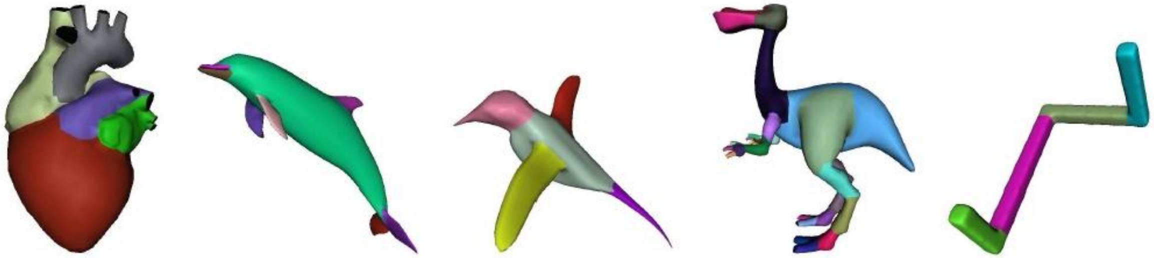
Lipton and Tarjan (1980) showed that every n-vertex **planar graph** has a set of $O(\sqrt{n})$ nodes that separates the graph into two roughly equal pieces.



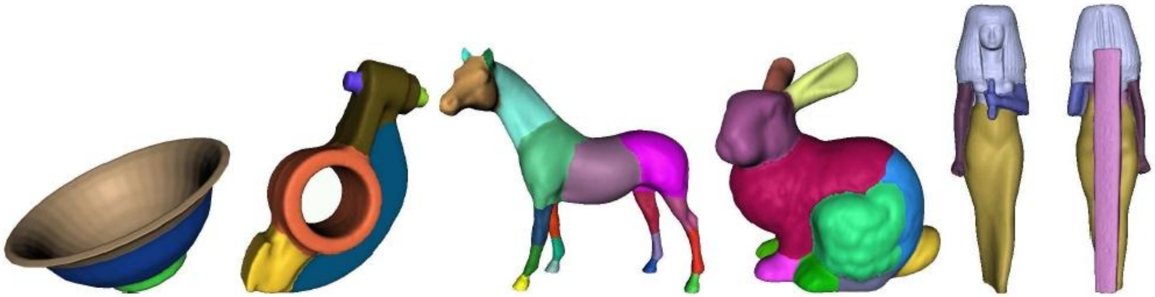
Useful for divide & conquer algorithms: E.g. there exist linear-time (1+ε)-approximations to the INDEPENDENT SET problem in planar graphs.

spectral partitioning

So we know good cuts exist. In practice, **spectral partitioning** does exceptionally well…



(a) Heart. (b) Dolphin. (c) Bird. (d) Dino-pet. (e) "Snake".



(f) Bowl. (g) Machine part. (h) Horse. (i) Bunny. (j) Isis.

spectral partitioning

So we know good cuts exist. In practice, **spectral partitioning** does exceptionally well…

Given a graph **G=(V,E)**, the **Laplacian** of **G** is

$$L_G = D - A$$

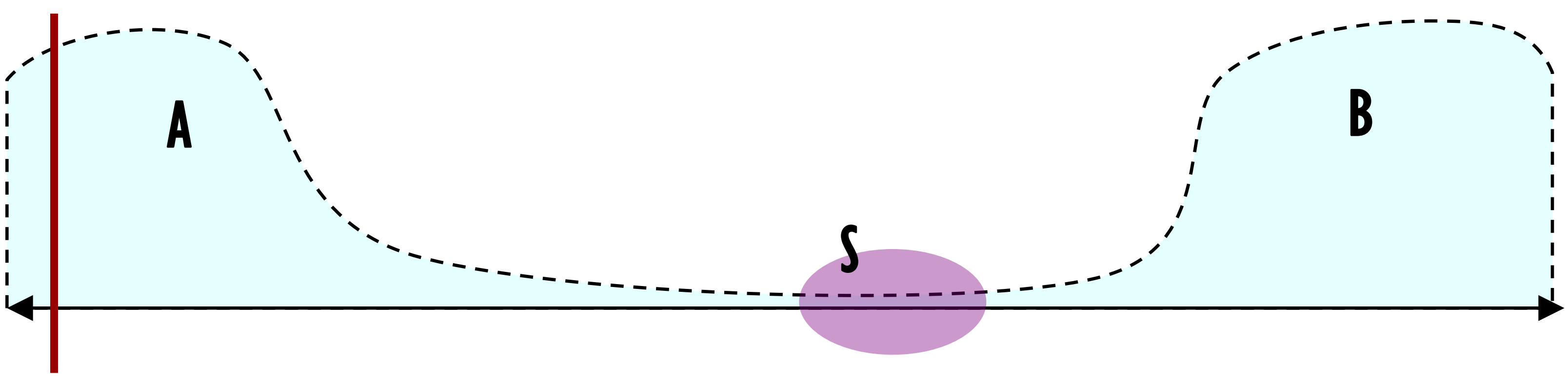
$$D = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & d_n \end{pmatrix}$$

$A = \text{ adjacency matrix of } G$

$$\lambda_1 = 0 \qquad v^{(1)} = (1, 1, \ldots, 1)$$

$$\lambda_2 = \min_{\substack{v \neq 0, \\ v \perp v^{(1)}}} \frac{\sum_{ij \in E} (v_i - v_j)^2}{\|v\|^2}$$

Arrange the vertices according to the 2nd eigenvector and sweep…



spectral partioning works

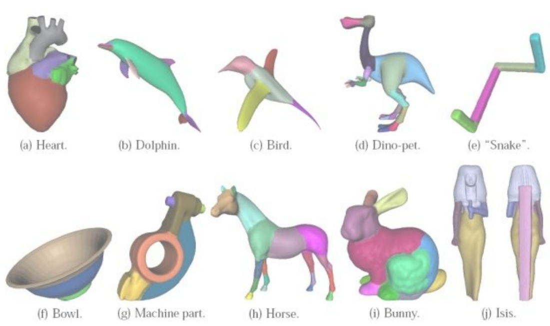
Spielman and Teng (1996) showed that spectral partioning will recover the Lipton-Tarjan $O(\sqrt{n})$ separator in bounded degree planar graphs.

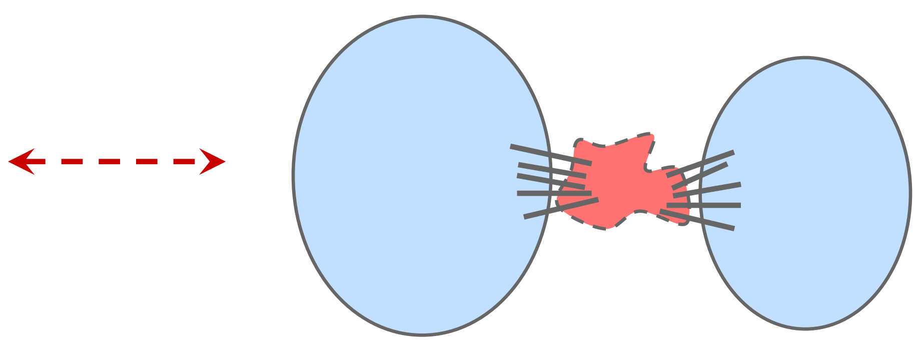
If **G** is an **n**-vertex planar graph with maximum degree **Δ**, then

$$\lambda_2(G) = O\left(\frac{\Delta}{n}\right)$$

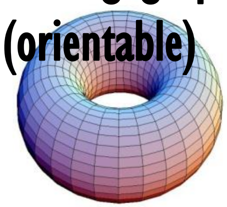
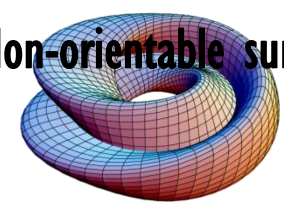
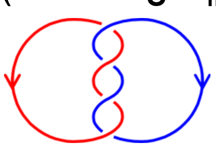
Cheeger's inequality implies that **G** has a cut with ratio $O\left(\sqrt{\frac{\Delta}{n}}\right)$,

so iteratively making spectral cuts yields a separator of size $O(\sqrt{\Delta n})$.

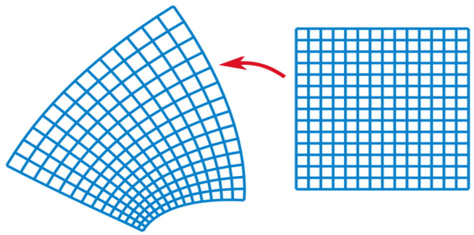




previous results

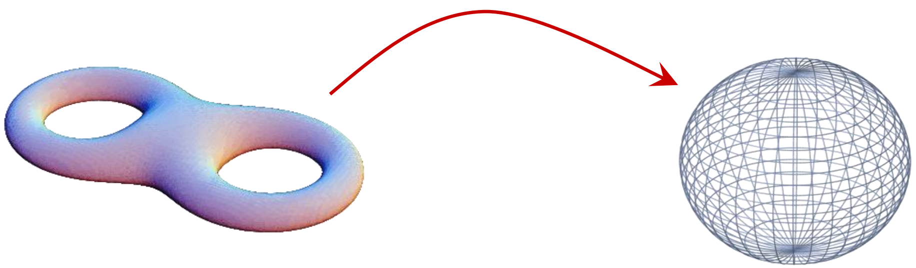
	separator size	eigenvalues (graphs)	eigenvalues (surfaces)
Planar graphs	$\sqrt{n}$ Lipton-Tarjan 1980	$\frac{\Delta}{n}$ Spielman-Teng 1996	$\frac{1}{\mathrm{vol}(M)}$ Hersch 1970
Genus g graphs (orientable) 	$\sqrt{gn}$ Gilbert-Hutchinson-Tarjan 1984	$\frac{g\,\mathrm{poly}(\Delta)}{n}$ Kelner 2004	$\frac{g}{\mathrm{vol}(M)}$ Yang-Yau 1980
Non-orientable surfaces 	GTH conjectured to be $\sqrt{gn}$	???	???
Excluded-minor graphs (excluding K_h) 	$h^{3/2}\sqrt{n}$ Alon-Seymour-Thomas 1990	ST conjectured to be $\frac{\Delta\,\mathrm{poly}(h)}{n}$	N/A

conformal mappings and circle packings

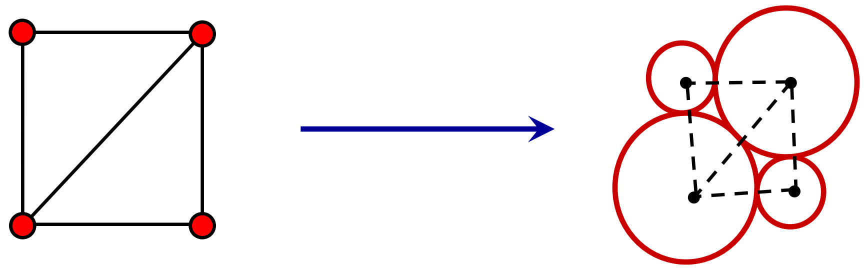


A conformal map preserves angles and their orientation.

Riemann-Roch: Every genus **g** surface admits a "nice" **O(g)**-to-**1** conformal mapping onto the Riemann sphere.



Koebe-Andreev-Thurston: (Discrete conformal uniformization)
Every planar graph can be realized as the adjacency graph of a circle packing on the sphere.

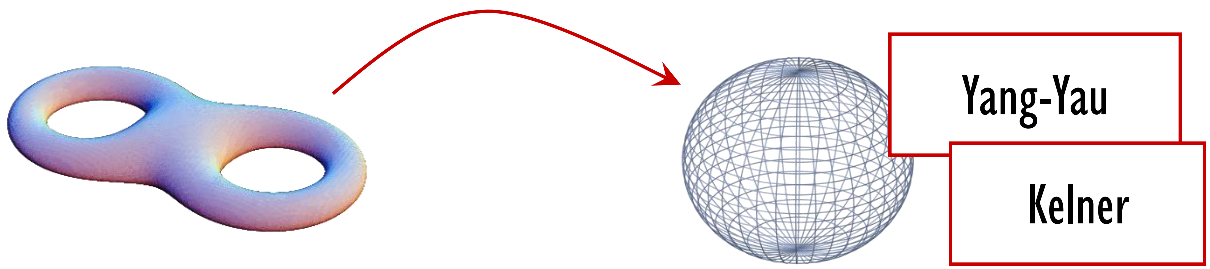


conformal mappings and circle packings

Main idea of previous bounds: These nice conformal representations can be used to produce a test vector for the Rayleight quotient, thus bounding the second eigenvalue.

It seems that we're out of luck without a conformal structure…

Riemann-Roch: Every genus **g** surface admits a "nice" **O(g)**-to-**1** conformal mapping onto the Riemann sphere.



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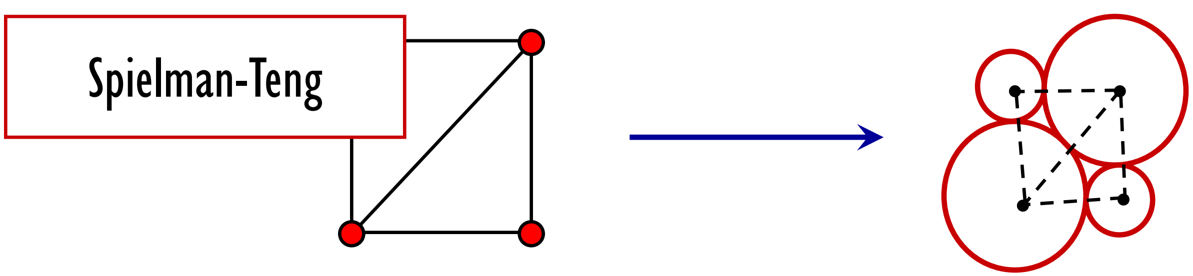
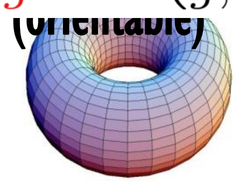
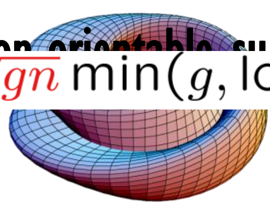
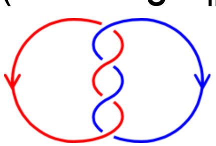
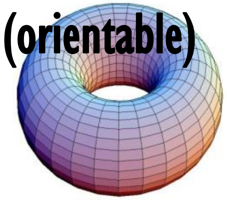
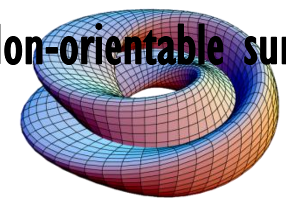
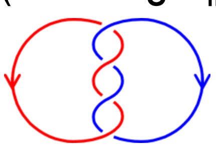


our results

	our results (unbounded degree)	separator size	eigenvalues (graphs)	(no conformal maps) our results
Planar graphs		$\sqrt{n}$ Lipton-Tarjan 1980	$\frac{\Delta}{n}$ Spielman-Teng 1996	$\frac{\Delta}{n}$
Genus g graphs (orientable) 	$\sqrt{gn}\,\min(g, \log n)$	$\sqrt{gn}$ Gilbert-Hutchinson-Tarjan 1984	$\frac{g\,\mathrm{poly}(\Delta)}{n}$ Kelner 2004	$\frac{g^3\Delta}{n}$
Non-orientable surfaces 	$\sqrt{gn}\,\min(g, \log n)$	GTH conjectured to be $\sqrt{gn}$	???	$\frac{g^3\Delta}{n}$
(excluding K_h) 	$h\sqrt{n}\,\min(h^2, \log n)$	$h^{3/2}\sqrt{n}$ Alon-Seymour-Thomas 1990	ST conjectured to be $\frac{\mathrm{poly}(h)\Delta}{n}$	$\frac{h^6\Delta}{n}$

higher spectra (Kelner-L-Price-Teng)

	separator size	eigenvalues (graphs)	kth eigenvalue
Planar graphs	$\sqrt{n}$ Lipton-Tarjan 1980	$\frac{\Delta}{n}$ Spielman-Teng 1996	$k\frac{\Delta}{n}$
Genus g graphs (orientable) 	$\sqrt{gn}$ Gilbert-Hutchinson-Tarjan 1984	$\frac{g\,\mathrm{poly}(\Delta)}{n}$ Kelner 2004	$k\frac{g^3\Delta}{n}$
Non-orientable surfaces 	GTH conjectured to be $\sqrt{gn}$	???	$k\frac{g^3\Delta}{n}$
Excluded-minor graphs (excluding K_h) 	$h^{3/2}\sqrt{n}$ Alon-Seymour-Thomas 1990	ST conjectured to be $\frac{\mathrm{poly}(h)\Delta}{n}$	$k\frac{h^6\Delta}{n}$

metric deformations

Let **G=(V,E)** be any graph with **n** vertices.

$$\frac{\lambda_2(G)}{2n} = \min_{f:V\to\mathbb{R}} \frac{\sum_{uv\in E} (f(u)-f(v))^2}{\sum_{u,v\in V} (f(u)-f(v))^2}$$

$$\min_{\text{metrics } d} \frac{\sum_{uv\in E} d(u,v)^2}{\sum_{u,v\in V} d(u,v)^2}$$

Bourgain's theorem [every n-point metric space embeds in a Hilbert space with O(log n) distortion] says these only differ by a factor of **$O(\log n)^2$**.

We'll consider the special class of **vertex weighted shortest-path metrics**:

Given $w : V \to \mathbb{R}_+$, let

$$\text{dist}_w(u,v) = \min \{ w(u_1)+w(u_2)+\cdots+w(u_k) : \langle u=u_1,u_2,\ldots,u_k=v\rangle \text{ is a u-v path in G} \}$$

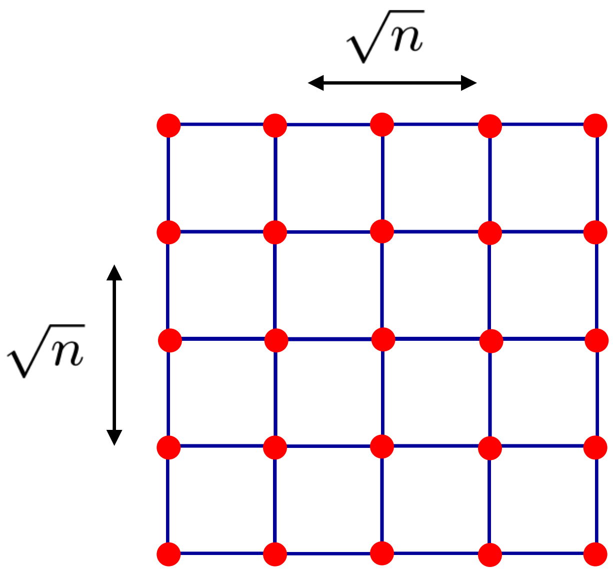
$$\frac{\sum_{uv\in E} \text{dist}_w(u,v)^2}{\sum_{u,v\in V} \text{dist}_w(u,v)^2} \leq 2\Delta \frac{\sum_{v\in V} w(v)^2}{\sum_{u,v\in V} \text{dist}_w(u,v)^2}$$

Goal: Show there exists a $w : V \to \mathbb{R}_+$ for which this is **$O(1/n^2)$**

metric deformations

Two examples:

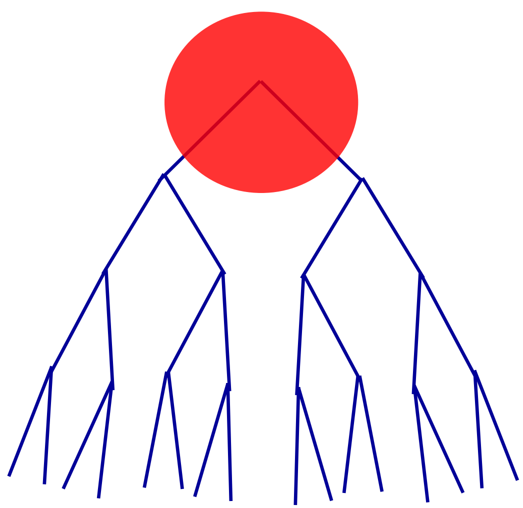
$$\frac{\sum_{v\in V} w(v)^2}{\sum_{u,v\in V} \mathrm{dist}_w(u,v)^2} \lesssim \frac{1}{n^2}$$



w(v)=1 $\forall$v $\in$ V

$$\sum_{v\in V} w(v)^2 = n$$

$$\sum_{u,v\in V} \mathrm{dist}_w(u,v)^2 \approx n^2 \cdot \left(\sqrt{n}\right)^2 = n^3$$



w(root)=1, w(v)=0 $\forall$v $\neq$ root

$$\sum_{v\in V} w(v)^2 = 1$$

$$\sum_{u,v\in V} \mathrm{dist}_w(u,v)^2 \approx n^2 \cdot 1 = n^2$$

metric deformations

$$\min_{w:V\to\mathbb{R}_+} \frac{\sum_{v\in V} w(v)^2}{\sum_{u,v\in V} \mathrm{dist}_w(u,v)^2}$$

feels like it should have a flow-ish dual…
but our objective function is **not convex**.

Instead, consider:

$$\Lambda_G(w) = \frac{\sqrt{\sum_{v\in V} w(v)^2}}{\sum_{u,v\in V} \mathrm{dist}_w(u,v)}$$

Notation:

For **u,v ∈ V**, let $\mathcal{P}_{uv}$ be the set of **u-v** paths in **G**.
Let $\mathcal{P} = \bigcup_{u,v\in V} \mathcal{P}_{uv}$ be the set of all paths in **G**.

$$\min_{w:V\to\mathbb{R}_+} \Lambda_G(w) \begin{cases} \min & \sqrt{\sum_{v\in V} w_v^2} \\ \text{s.t.} & \sum_{u,v\in V} d_{uv} = 1 \\ & d_{uv} \le \sum_{v\in p} w_v \quad \forall p \in \mathcal{P}_{uv} \end{cases}$$

By Cauchy-Schwarz, we have:

$$\frac{\sum_{v\in V} w(v)^2}{\sum_{u,v\in V} \mathrm{dist}_w(u,v)^2} \le n^2 \Lambda_G(w)^2$$

So our goal is now:

$$\min_{w:V\to\mathbb{R}_+} \Lambda_G(w) = O\left(\frac{1}{n^2}\right)$$

Notation:

For u,v $\in$ V, let $\mathcal{P}_{uv}$ be the set of u-v paths in G.
Let $\mathcal{P} = \bigcup_{u,v \in V} \mathcal{P}_{uv}$ be the set of all paths in G.

A **flow** is an assignment $F : \mathcal{P} \to \mathbb{R}_+$

For v $\in$ V, the **congestion of v under F** is

$$C_F(v) = \sum_{p \in \mathcal{P} : v \in p} F(p)$$

The **2-congestion of F** is

$$\text{con}_2(F) = \sqrt{\sum_{v \in V} C_F(v)^2}$$

F is a **complete flow** every u,v$\in$V satisfy

$$\sum_{p \in \mathcal{P}_{uv}} F(p) \geq 1$$

$$\Lambda_G(w) = \frac{\sqrt{\sum_{v \in V} w(v)^2}}{\sum_{u,v \in V} \text{dist}_w(u,v)}$$

DUALITY

$$\min_{w \to \mathbb{R}_+} \Lambda_G(w) = \left(\min_{F : \mathcal{P} \to \mathbb{R}_+} \text{con}_2(F) \right)^{-1}$$

where the minimum is over all complete flows

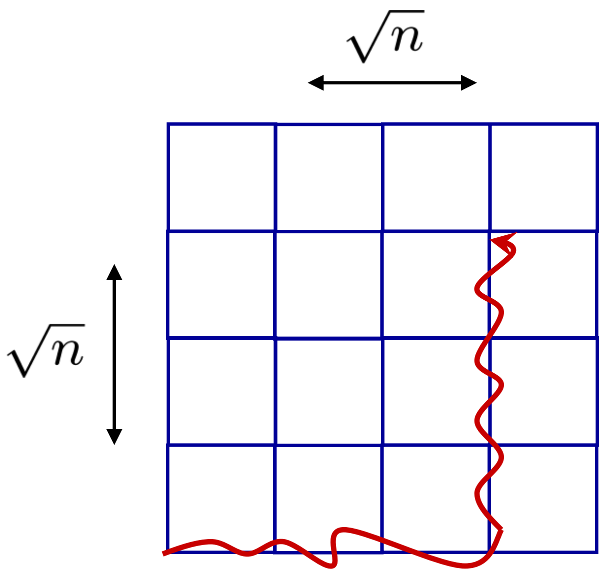
So now our goal is to show that:

For any complete flow F in G,
we must have $\text{con}_2(F) = \Omega(n^2)$.

congestion lower bounds

Two examples:

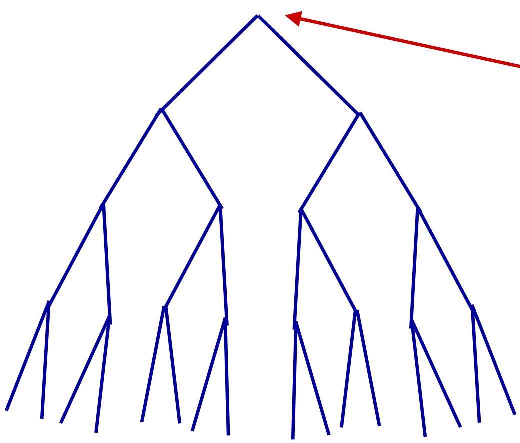
$$\mathrm{con}_2(F) = \sqrt{\sum_{v \in V} C_F(v)^2}$$



A complete flow has "total length" about $n^2\sqrt{n}$

To minimize **$\mathrm{con}_2(F)$**, we would spread this out evenly, and we get:

$$\mathrm{con}_2(F) \approx \sqrt{n \cdot \left(\frac{n^2\sqrt{n}}{n}\right)^2} = n^2$$



In any complete flow, this guy suffers **$\Omega(n^2)$** congestion.

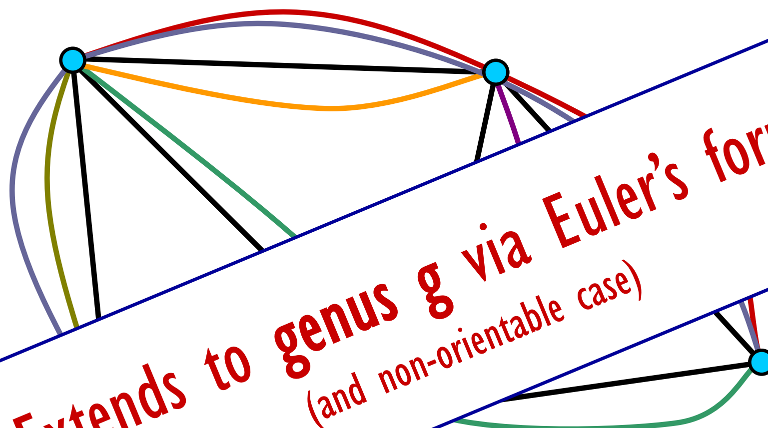
$$\mathrm{con}_2(F) \geq C_F(\mathrm{root}) = \Omega(n^2)$$

congestion lower bounds

THEOREM: If $G=(V,E)$ is an n-vertex planar graph, then for any complete flow F in G, we have $[\mathrm{con}_2(F)]^2 = \sum_{v\in V} C_F(v)^2 = \Omega(n^4)$.

PROOF: By randomized rounding, we may assume that F is an **integral** flow.

Let's imagine a drawing of G in the plane…



Cl sical results of Leighton (1984) and Chvatal-Newborn-Szemeredi (1982) say e crossing number of of the complete at least $\Omega(n^4)$.

Extends to genus g via Euler's formula
(and non-orientable case)

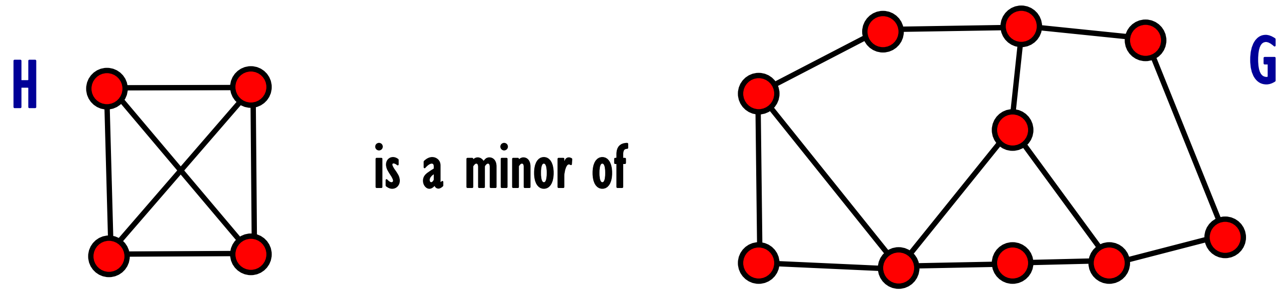
F ces a drawing of the complete graph K_n in the plane… where edges of K_n cross only at vertices of G.

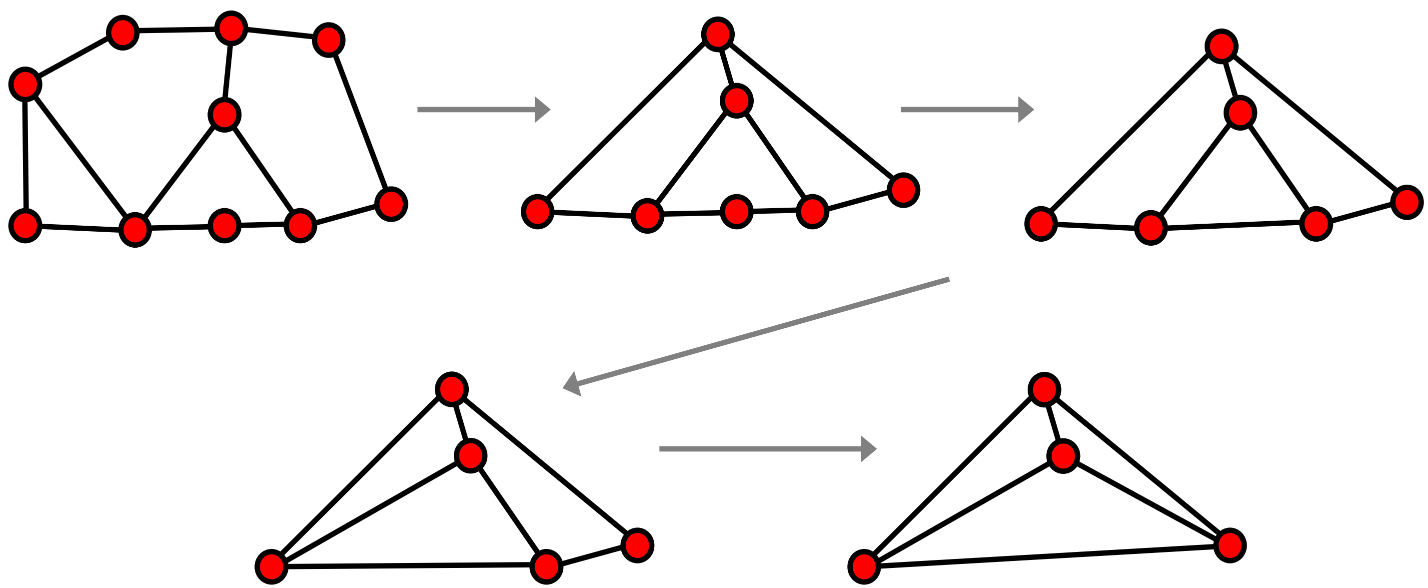
How many edge crossings of K_n at $v\in V$? At most $C_F(v)^2$.

So $\sum_{v\in V} C_F(v)^2 \geq$ (# edge crossings of K_n) $\geq \Omega(n^4)$

H-minor free graphs

A graph **H** is a **minor** of **G** if **H** can be obtained from **G** by **contracting** edges and **deleting** edges and isolated nodes.





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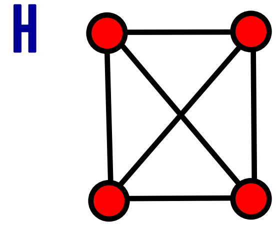
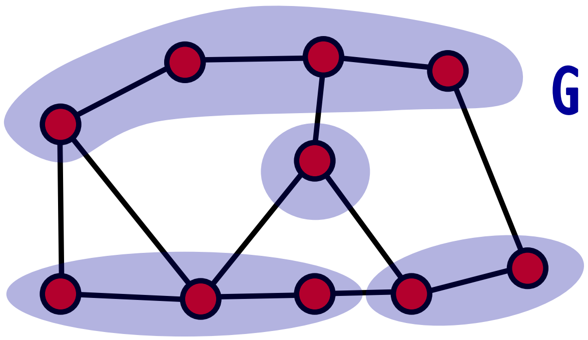


is a minor of



vertices of H $\longrightarrow$ disjoint connected subgraphs of G

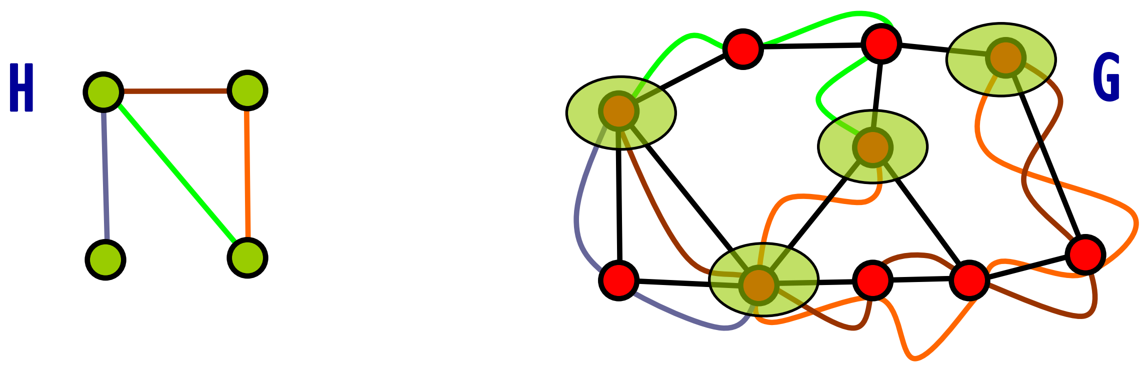
edges of H $\longrightarrow$ subgraphs that touch

A graph G is H-minor-free if it does **not** contain H as a minor

(e.g. planar graphs = graphs which are K_5 and $K_{3,3}$-minor-free)

H-flows

Def: An **H-flow** in **G** is an **integral** flow in **G** whose "demand graph" is isomorphic to **H**.



If φ is an **H**-flow, let φ_{ij} be the **i-j** path in **G**, for $(i,j) \in E(H)$, and define

$$\text{inter}(\varphi) = \#\{\ (i,j), (i',j') \in E(H) : |\{i,j,i',j'\}|=4 \text{ and } \varphi_{ij} \cap \varphi_{i'j'} \neq \emptyset\ \}$$

Theorem: If **H** is **bipartite** and φ is an **H**-flow in **G** with $\text{inter}(\varphi)=0$, then **G** contains an **H** minor.

Corollary: If **G** is K_h-minor-free and φ is a K_{2h}-flow in **G**, then $\text{inter}(\varphi) > 0$.
[If φ is a K_{2h}-flow in **G** with $\text{inter}(\varphi)=0$, then it is also a $K_{h,h}$-flow in **G**, so **G** contains a $K_{h,h}$ minor, so **G** contains a K_h minor.]

congestion in minor-free graphs

THEOREM: If $G=(V,E)$ is an n-vertex K_h-minor-free graph, then for any complete flow (i.e. any K_n-flow) F in G, we have $[\mathrm{con}_2(F)]^2 = \sum_{v\in V} C_F(v)^2 = \Omega(n^4/h^3)$.

PROOF: It suffices to prove that $\mathrm{inter}(F) = \Omega(n^4/h^3)$, because for any integral flow φ,

$$\mathrm{inter}(\varphi) \leq \sum_{v\in V} \left(\sum_{(i,j),(i',j')\in E(H)} \mathbf{1}_{v\in\varphi_{ij}} \cdot \mathbf{1}_{v\in\varphi_{i'j'}} \right) = \sum_{v\in V} C_\varphi(v)^2$$

Since $\mathrm{inter}(\varphi) > 0$ for any K_{2h}-flow φ, we have $\mathrm{inter}(\varphi) \geq r-2h+1$ for any K_r-flow φ.

Let $S_p \subseteq V$ be a random subset where each vertex occurs independently with probability p.

Let $n_p = |S_p|$. We can consider the K_{n_p}-flow F_p induced by restricting to the terminals in S_p.

Now, we have $p^4\ \mathrm{inter}(F) = \mathbb{E}[\mathrm{inter}(F_p)] \geq \mathbb{E}[n_p-2h+1] = pn - 2h + 1$.

Setting $p \approx 4h/n$ yields $\mathrm{inter}(F) = \Omega(n^4/h^3)$.