Gromov's Monster Group

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based on *Example of random groups* by G. Arzhantseva and T. Delzant

Geometry and analysis of large networks.

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Introduction

Gromov's monster group M: a "quite simple two-dimensional creature" with surprising properties

- M does not coarsely embeds into a Hilbert space.
- M does not satisfy the Baum-Connes conjecture with coefficients.

Particularity: M coarsely contains an infinite expander.

Definition

A map $f : X \to Y$ between two metric spaces is a *coarse embedding* if for every $(x_n), (x'_n) \in X^{\mathbf{N}}$

$$\lim_{n \to +\infty} |x_n - x'_n| = +\infty \text{ iff } \lim_{n \to +\infty} |f(x_n) - f(x'_n)| = +\infty$$

Usual small cancellation theory

$\bar{G} = \langle S|R \rangle = \mathbf{F}(S)/ \ll R \gg$.
(elements of R are cyclically reduced).
R^* is the set of all cyclic conjugates of $R \cup R^{-1}$.

A *piece* is a common prefix of two distinct elements of R^*.

Small cancellation condition

$\bar{G} = \langle S|R \rangle$ satisfies $C'(\lambda)$ if

$$\frac{\text{length of the largest piece}}{\text{length of the smallest relation}} \leqslant \lambda.$$

Theorem

Let $\bar{G} = \langle S|R \rangle$ satisfying $C'(\lambda)$ for $\lambda < \frac{1}{6}$.

- $\bar{G}$ is non-elementary, word-hyperbolic.
- (Greendlinger Lemma) If w is a reduced word, trivial in $\bar{G}$, it contains "$1 - 3\lambda$" of a relation.

In particular the map $\mathbf{F}(S) \to \bar{G}$ induces an isometry from $B(1, r)$ onto its image where $r = \frac{1-3\lambda}{2}$.(length of the smallest relation).

Small cancellation theory for graphs

Let G be a torsion free, non-elementary δ-hyperbolic group generated by S.

Let θ be a graph labelled by $S \cup S^{-1}$, T its universal cover and $f : T \to \mathrm{Cay}(G)$ the map given by the labeling.

Δ and ρ are the small cancellation parameters associated to the pair $(f(T), \pi_1(\theta))$. We assume that $\pi_1(\theta) \subset G$ does not contain a proper power.

Theorem

Let $\alpha > 1$. There exists positive numbers ε, and K which only depend on α with the following property. Let $\beta > 0$. Assume that f is a $(\alpha, \beta, \frac{1}{2}\,\text{girth}(\theta))$-local quasi-isometry such that

$$\frac{\delta}{\rho}, \frac{\beta}{\rho}, \frac{\Delta}{\rho} \leqslant \varepsilon.$$

Then

- $\bar{G} = G / \ll \pi_1(\theta) \gg$ is torsion-free, non-elementary, word-hyperbolic. Its hyperbolicity constant only depends on δ, α, β, Δ and ρ.
- The map $G \to \bar{G}$ induces an isometry from $B(1, K\rho)$ onto its image

In particular, the map $\bar{f} : \theta \to \text{Cay}(\bar{G})$ given by the labeling satisfies

$$\left|\bar{f}(x) - \bar{f}(y)\right| \geqslant \frac{K\rho}{\operatorname{diam}\theta} \left(\frac{\alpha^{-1}}{2} |x - y| - \beta \right), \forall x, y \in \theta.$$

Random groups

Theorem

Fix a density d between 0 and 1. Choose a length l and pick at random a set R of $(2k)^{dl}$ uniformly chosen words of length l in the letters $a_1^{\pm 1}$,...,$a_k^{\pm 1}$.

- If $d < \frac{1}{2}$ then the probability that the group $\langle a_1, \dots, a_k | R \rangle$ is hyperbolic tends to 1 as l approaches $+\infty$.
- If $d > \frac{1}{2}$ then the probability that the group $\langle a_1, \dots, a_k | R \rangle$ is trivial or $\mathbf{Z}/2\mathbf{Z}$ tends to 1 as l approaches $+\infty$.

Let G be a group of rank k. We denote by P_t the probability that a random walk on G (with respect to the generating set) starting at 1 comes back to 1 after time t.

Definition

The *spectral radius* of the random walk on G is the number κ_G such that

$$\ln \kappa_G = \limsup_{t \to +\infty} \frac{1}{t} \ln P_t.$$

Critical density for a torsion-free hyperbolic group G with k generators: $-\frac{\ln \kappa_G}{\ln 2k}$ (Y. Ollivier 2004)

Let $\Theta = (\theta_n)$ be a sequence of graphs of girth ρ_n with $\lim_{n \to +\infty} \rho_n = +\infty$. We assume that the vertices of Θ are of uniformly bounded degree. Given $l > 0$, $b_n(l)$ denotes the number of distinct simple paths of length l in θ_n.

Definition

Let $b > 0$ and $\xi_0 \in (0, \frac{1}{2})$. The family Θ is (b, ξ_0)-thin if there exists $C > 0$ such that for all $\xi \in [\xi_0, \frac{1}{2})$ we have

$$b_n(\xi \rho_n) \leqslant C e^{b \xi \rho_n}$$

"Density condition" : $b \ll -\ln \kappa_G$.

Let G be a non-elementary torsion-free hyperbolic group of rank k. Let $\Theta = (\theta_n)$ be a (b, ξ_0)-thin sequence of graphs and T_n the universal cover of θ_n.

Assumption: there exists $\kappa > \kappa_G$ such that $b + \ln \kappa < 0$.

Put $\alpha = -\frac{2\ln(2k-1)}{b+\ln \kappa}$.
Fix $\ell : \mathbf{R}_+ \to \mathbf{R}_+$ a function such that $\lim_{x \to +\infty} \ell(x) = +\infty$.

Quasi-geodesic labeling

For each n pick a random labeling of θ_n. The probability that every simple path w of length $\|w\| \leqslant \frac{1}{2}\rho_n$ of θ_n satisfies

$$|m_n(w)|_G \geqslant \alpha^{-1}\|w\| - \ell(\rho_n) - \alpha^{-1}\xi_0\rho_n,$$

goes to 1 as n approaches $+\infty$.

Small cancellation condition

Let $\lambda > 0$. With asymptotic probability 1 as n approaches $+\infty$, a random labeling of θ_n induces an $(\alpha, \beta_n, \frac{1}{2}\,\mathrm{girth}(\theta_n))$-local quasi-isometric embedding $T_n \to \mathrm{Cay}(G)$, where

$$\beta_n = \ell(\rho_n) + \alpha^{-1}\xi_0\rho_n.$$

Moreover it satisfies $\Delta(\theta_n) \leqslant \lambda\,\mathrm{girth}(\theta_n)$

Uniform control on κ_G

Let G be group with Kazhdan's property (T) generated by a set S. Let κ be the Kazhdan constant of the pair (G, S). Then for every quotient $\bar{G}$ of G, $\kappa_{\bar{G}} \leqslant \kappa$.

Small cancellation (Recall)

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Then

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