ALGORITHMIC PROBLEMS IN VARIETIES

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December 9, 1994

Mankind always sets itself only such problems as it can solve.
Karl Marx, The Introduction to "A Critique
of Political Economy".

Variety's the very spice of life.
William Cowper

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*Research supported in part by grants from NSERC and FCAR

†Research supported in part by NSF grant DMS-9203981 and by the Center for Communication and Information Science, University of Nebraska-Lincoln

1 Prologue

1.1 Motivation

This survey is about the synthesis of two big parts of modern algebra: algorithmic problems and the theory of varieties. Each of these parts has a rich history and remarkable achievements reflected in numerous books and surveys. The algorithmic nature of varieties and the semantic nature of algorithmic problems make these two parts a perfect match. Their synthesis gave birth to new interesting problems and results which are the subject of this survey.

We will be mainly concerned with varieties of "classical" algebras — groups, semigroups, associative and Lie algebras. But we will try to present results in the most general form, so when possible, we will formulate statements for arbitrary universal algebras.

In addition to algorithmic problems themselves, we will be dealing with their good neighbors and relatives like residual finiteness, the Higman embedding property, the finite basis property, etc. We will also discuss the computational complexity of solvable algorithmic problems.

1.2 Overview

The survey has 7 sections. In the Introduction, we present the main algorithmic problems, their neighbors, and general connections between them. Sections 3-6 are "local surveys". There we present results about semigroups, associative algebras, Lie algebras, and groups respectively.

The seventh section, "Methods", is the biggest and the most non-standard part of the survey. It contains a description of the main ideas employed in the proofs of some important results reviewed in the survey. We not only extract the main ideas of the proofs, but also try to reconstruct the path that lead to these ideas. We show how to prove that the word problem is decidable in a variety and how to use Minsky machines and systems of differential equations in order to prove the undecidability of an algorithmic problem, or in order to prove that a decidable problem is computationally "hard". We present the essence of Mursky's and Kleiman's proofs of the undecidability of equational theories. We even try to explain why one method of proving undecidability is stronger than another. In general, we think that surveys must present not only results but also the intuition of the authors, and Section 7 is the place where we share our intuition with the reader.

We hope that the following thing will become apparent to the reader: the deeper one investigates algorithmic problems, the more different types of algebras seem alike. The formulations of results for different classes are getting more and more similar and the methods of proofs are getting more and more universal. Often one and the same method is used to obtain results in groups, semigroups, Lie and associative algebras. Of course, some specifics in the proofs remain. For example a result about groups usually is more difficult than a similar looking result about Lie algebras. But the main methods are basically the same and (what should be important for the reader) the proof of the more difficult result is easier to understand after reading the proofs of its simpler brothers for other types of algebras. But common formulations and methods do not exhaust all connections between different classes of algebras. Perhaps more important are common ideas, common parts of constructions and even a common mood arising there. This allows us to speak about the unified theory "Algorithmic problems in varieties."

We do not pretend that we have considered all possible algorithmic properties in this survey. We concentrated mainly on neighbors and relatives of the word problem. Besides we preferred properties which make sense for different types of algebras. If a property that we consider here has not been investigated in some natural class of algebras, then we think that it is obviously natural to investigate it in this class. We do not always explicitly pose the corresponding open problem. The list of references, though relatively big, does not pretend to be comprehensive.

Many results presented here are not in their original form. We wanted to show connections between different results, and sometimes this required reformulation. We also wanted to show the algorithmic nature of results that are presented. Original formulations often did not make this clear.

We tried to make the survey interesting not only for specialists in algorithmic problems and varieties, but also for the general mathematical public. We expect the reader to know the basic concepts from standard undergraduate algebra and logic courses. Some of the basic definitions are collected in Section 2.10. There are separate subsections with definitions in each of the "local surveys".

The sections titled "Semigroups" and "Associative algebras" and the related subsections in "Methods" as well as most of the text in the "Introduction" were written by the second author. The sections titled"Lie algebras" and "Groups" were written jointly, although most of the material there was collected by the first author.

1.3 Acknowledgements

We are very much obliged to our teacher L.N. Shevrin who has influenced our work over many years. He suggested that we write this survey, read the manuscript, and, as usual, gave us his criticism and advice. Many of the topics presented here were discussed during the last 20 years at the Algebraic Systems seminar in Sverdlovsk, headed by L.N. Shevrin.

We thank all those who helped us by discussing, explaining, correcting, and advising. In particular we mention S. I. Adian, J-C. Birget, Yu. A. Bahturin, L. A. Bokut', V. S. Drensky, D. Gildenhuys, V. Guba, N. Gupta, Yu. Gurevich, P. Jones, A. do Lago, S. Margolis, E. Mayr, G. McNulty, J. Meakin, M. Minsky, A. Razborov, V. N. Remeslennikov, J. Rhodes, P. Trotter, M. Volkov, and M. Zaitsev.

The referee found and fixed lots of misprints in the text, partially solved Problem 2.4, and gave us many useful comments. We are very grateful to him/her for that.

The survey was started in 1990 when both authors worked at the Ural State University (Sverdlovsk, U.S.S.R, now - Ekaterinburg, Russia). We continued working on the survey after the first author moved to McGill University and the second author moved to the California State University at Chico and then the University of Nebraska at Lincoln. In all these places we enjoyed stimulating intellectual environments.

2 Introduction

2.1 The World of Algorithmic Problems in Algebra

Some histories of mathematics say that the world of algorithmic problems in algebra has existed over an indefinitely long time, and even "prehistoric" mathematicians like Diophantus dealt with algorithmic problems. Some people claim that this world was created by Gauss, because he created the elimination method, from which all other methods descended. Nevertheless, we subscribe to the popular myth that the world of algorithmic problems in algebra rests on three papers by Thue [397], Tietze [399] and Dehn [80]. These authors studied the problem of deducibility of relations in associative calculuses (Thue systems), the homeomorphism problem for topological manifolds, and the problem of homotopy equivalence of a curve to a point or to another curve on a finite dimensional manifold. In spite of the fact that the first of these problems came from logic and the others came from topology, all of them turned out to be closely related to algebraic problems: the word problem in finitely presented semigroups and groups, the isomorphism and conjugacy problems for finitely presented groups (see Haken [135] for details).

Recall that the word problem is said to be decidable (we shall also use the word "solvable") in an algebra S presented by a set of generators X and a set of defining relations Σ, if there exists an algorithm which, given any pair of terms in the alphabet X, answers whether or not these terms represent the same element in S. The conjugacy problem for groups and the isomorphism problem may be introduced in a similar way.

As is well known the unsolvability of the word problem for semigroups was proved in 1947 by A.A.Markov [249], [250] and Post [303], and significantly more difficult results on the undecidability of the three group problems: the word problem, the conjugacy problem and the isomorphism problem — were proved by P.S.Novikov [285] at the beginning of the fifties. A few years later Boone ([48], [49],[50], [51], [52],[53]) gave another proof of the unsolvability of the word problem in groups. There are several other proofs of Novikov's result (see surveys [317], [6], [46]). One of the most beautiful proofs has been found by E. Rips and is yet unpublished. Actually this is perhaps the first proof which can be explained to a (smart) high school student. Rips uses the relations of Boone [54] and van Kampen diagrams [229]. The main idea is to show that certain labeled n-gons can be filled by certain labeled n_i-gons in only one, canonical, way. This is done by using some high school combinatorics on the plane. A trace of Rips' proof can be found in a book by Rotman [330] (see pictures on pages 372–378, and the two colored plates on the inside front covers of this book).

The results of Markov, Post, and Novikov stimulated and largely determined further progress in the study of algorithmic problems, both algebraic and topological. For example, soon after Novikov's results were published, S.I.Adian [3] proved the unsolvability of the problem of isomorphism to any given finitely presented group, and

Markov proved the unsolvability of the homeomorphism problem for n-dimensional manifolds ($n \geq 4$) [248]. See the survey by Haken [135] for an excellent discussion of these and other algorithmic results in algebra and topology.

The existence of a semigroup and a group with undecidable word problem allowed one to prove the undecidability of many properties of finitely presented semigroups and groups. Here are some examples of such properties: finiteness, triviality, commutativity and so on. All these properties and many others satisfy the following condition: They hold in some finitely presented algebra G_1 and do not hold in any algebra containing some finitely presented algebra G_2 (G_1 and G_2 may be different for different properties). Such properties have been called Markov properties after A.A.Markov proved the undecidability of each of them for finitely presented semigroups [249], [250]. In the class of groups, the undecidability of an arbitrary Markov property was proved by Adian [3] and Rabin [308]. Similar results were obtained for associative and Lie algebras by L.A.Bokut' [45], [43].

The investigation of algorithmic problems, arising from the needs of logic and topology, is now largely motivated by internal needs of algebra. Perhaps every algebraist has either proved the decidability (undecidability) of the word problem in some concrete algebra or used somebody else's results of this kind. Algorithmic problems often lie in the very kernel of difficult algebraic problems. It is enough to mention super-works on Burnside problems. The kernel of these works is the solution of the word and conjugacy problems in free Burnside groups [286], [4], [288], [153].

Algorithmic problems are important from a general, "philosophic", point of view too. Thus, the decidability of the word problem in a class of algebraic systems usually means that the study of structural properties of algebras in this class is not hopeless. And conversely, the undecidability of the word problem usually means big difficulties are to be expected in the investigation of this class "as a whole". See also the survey [46] where interesting general thoughts about algorithmic problems are presented.

Among all possible classes of algebras, the most natural and important ones are, of course, varieties. Indeed, first of all varieties are given by syntactically very simple formulas — identities. Varieties play the classifying role in algebra, a description of varieties of algebras in a given class may be considered as a rough description of algebras in this class. Furthermore varieties are precisely the classes of algebras closed under the three most popular algebraic operations: taking subalgebras, taking direct products, and taking homomorphic images. Therefore if we work with algebras in a variety and use just these three operations, we will never need any algebra outside the variety. Every variety has free objects, so the concepts of free algebras, finitely presented algebras, etc. are applicable for any variety. Therefore every algorithmic problem about finitely presented algebras makes sense for every variety of algebras. And, finally, many important classes of algebras are varieties. For example, the following classes are varieties: the class of all commutative semigroups (groups, Lie and associative algebras), the class of solvable groups (Lie algebras) of a given degree, the class of nilpotent groups (Lie or associative algebras) of a given degree, the class of Burnside groups (semigroups) of a given exponent (index and period). All these factors have created a big and robust interest in algorithmic problems in varieties during the last 15-20 years. This subject occupies large parts in most surveys devoted to algorithmic problems in algebra (see e.g.[317], [6],[252], [46], [150]). However the last five or six years have given us many important new results which have not been previously analyzed. In a sense, these results lead us to a new, higher, level of understanding: One can say that the period of examples has been followed by a period of descriptions. In this article we are trying to comprehend this new period.

2.2 Two Variants of Formulations

Let $\mathcal{V}$ be a variety. There are two types of formulations of algorithmic problems concerning finitely presented algebras in $\mathcal{V}$. The first type deals with algebras finitely presented inside $\mathcal{V}$, i.e. those given by a finite number of generators, a finite number of defining relations, and by all of the identities of $\mathcal{V}$ (the class of these algebras will be denoted by $FP(\mathcal{V})$. The second type of formulation deals with algebras which are finitely presented in some fixed "large" variety $\mathcal{U}$ (say, the variety of all groups) and considers algebras from the intersection $FP(\mathcal{U}) \cap \mathcal{V}$. It will usually be clear from the context what $\mathcal{U}$ is, and we will write $FP \cap \mathcal{V}$ instead of $FP(\mathcal{U}) \cap \mathcal{V}$.

If some algorithmic problem is decidable in every algebra from $FP(\mathcal{V})$ then we say that it is *decidable (solvable) in* $\mathcal{V}$ or *decidable (solvable) for* $\mathcal{V}$. If we consider the smaller class $FP(\mathcal{U}) \cap \mathcal{V}$ and a problem α is decidable there, then we say that α is *weakly decidable (solvable) in* $\mathcal{V}$ (relative to $\mathcal{U}$). By the same reasoning it is natural to call the undecidability of the problem α in the second variant of the formulation *strong undecidability* of α in $\mathcal{V}$ (with respect to $\mathcal{U}$).

Generally speaking, weak decidability depends on the choice of the large variety $\mathcal{U}$. For example if $\mathcal{U}$ is the variety of all groupoids and $\mathcal{V}$ is the variety of all semigroups then the word problem is weakly decidable in $\mathcal{V}$ with respect to $\mathcal{U}$ since it is solvable in every finitely presented groupoid [429], however it is undecidable in $\mathcal{V}$ with respect to $\mathcal{V}$ itself.

It is clear that the decidability of an algorithmic problem for a variety $\mathcal{V}$ implies the weak decidability of this problem relative to every bigger variety. An important fact is that weak decidability is always hereditary for subvarieties. At the same time — and the above example of the variety of semigroups and the variety of groupoids shows it — "ordinary" decidability sometimes fails to be hereditary for subvarieties. Thus general observations from the theory of varieties (see, for instance, the survey by Bakhturin and Ol'shansky [19]) hint that, for example, the study of varieties with solvable word problem should be a much more difficult thing than the study of varieties with weakly solvable word problem. It is amazing that these predictions are often false! The main reason for this is that it is usually much more difficult to construct an algebra with, say, an undecidable word problem, in $FP \cap \mathcal{V}$ than in $FP(\mathcal{V})$. Indeed, in the first case the identities of $\mathcal{V}$ should follow from the defining relations. This means that the defining relations must be strong. But at the same time they should be weak enough to ensure the undecidability of the word problem: it is intuitively clear that an algebra with undecidable word problem must contain "many" elements, so its defining relations cannot be too strong. Another cause is that it is relatively rare that a subvariety does not inherit the undecidability of an algorithmic problem, and even if it does not, this is compensated for by other properties (see Section 6 for details).

The arguments from the previous paragraph show that it is not easy to construct an algebra with an undecidable algorithmic problem and a non-trivial identity, or equivalently — to construct a variety where an algorithmic problem is strongly undecidable.

The first example of a proper variety of groups with strongly undecidable word problem was found in 1979 by the first author of this survey. She constructed a finitely presented group with undecidable word problem which is soluble of degree 3 [175]. Analogous examples in classes of semigroups, associative algebras and Lie algebras may be found in [268] or in [27] (see further sections of this survey).

2.3 The Special Role of the Word Problem

Algebras with undecidable word problem usually play central roles when one proves the undecidability of other algorithmic properties. We can refer, for example, to the proofs of results on the Markov properties cited above.

Another feature which makes the word problem special is that if the word problem is undecidable in a variety then many other algorithmic problems almost surely are undecidable there. This is not a "scientific fact", of course, just a part of the intuition. Every branch of mathematics has such non-scientific facts. They are sometimes much more helpful than strictly proved theorems.

Here we would like to present the following connection between the word problem and the isomorphism problem. Other connections will be mentioned later. Recall that an algebra is called Hopfian if it is not isomorphic to any of its proper homomorphic images.

Connection 2.1 *If the class $FP(\mathcal{V})$ contains a Hopfian algebra S with an undecidable word problem then the isomorphism problem is undecidable in $\mathcal{V}$. Moreover the problem of whether a given finitely presented algebra in $\mathcal{V}$ is isomorphic to S is undecidable. If S is finitely presented in a bigger variety $\mathcal{U}$ then the isomorphism problem for algebras from $FP(\mathcal{U}) \cap \mathcal{V}$ is undecidable.*

Proof. Indeed, let w and w' be two elements in S and let γ be the smallest congruence which glues w and w' together. Then, since S is Hopfian, S is isomorphic to S/γ if and only if $w = w'$. But S has an undecidable word problem and all algebras S/γ belong to our variety. Hence the isomorphism problem is undecidable there. The connection is established.

It is worth adding that every known variety with undecidable word problem contains a Hopfian algebra with an undecidable word problem. Thus we do not know any example of a variety of universal algebras with decidable word problem and undecidable isomorphism problem. It would be very interesting to construct such an example and even more interesting to prove that there exist no such examples. So we formulate the following problem.

Problem 2.1 *Is there a variety of universal algebras (groups, semigroups, associative and Lie algebras) with decidable word problem and undecidable isomorphism problem?*

There are some important algorithmic problems which are often weaker than the word problem. The finiteness problem is one of them. Recall that this problem asks for an algorithm to decide if a finitely presented algebra is finite. For example, in the variety of all solvable of class 5 groups the word problem is undecidable [315] and the finiteness problem is decidable [25].

2.4 Connections With the Decidability of Fragments of the Elementary Theory

Algorithmic problems in varieties have deep connections with fragments of the elementary theories of these varieties. Let us recall some necessary definitions. Suppose we are given a class $\mathcal{K}$ of universal algebras. The set $E\mathcal{K}$ of all first order formulas of the corresponding type which hold in this class is called the *elementary theory* of this class. The set $A\mathcal{K}$ of all universal formulas (i.e. sentences without existential quantifiers) from $E\mathcal{K}$ is called *the universal theory* of $\mathcal{K}$. The set $Q\mathcal{K}$ of all quasi-identities, i.e. formulas of the following form:

$$(\forall x_1, \ldots, \forall x_n)(s_1 = t_1 \& \ldots \& s_k = t_k \to s = t)$$

where s_i, t_i, s, t are terms in the signature of $\mathcal{K}$, over the alphabet $\{x_1, \ldots, x_n\}$, is called the *Q-theory* of $\mathcal{K}$. The set $I\mathcal{K}$ of all identities of $\mathcal{K}$ is called the *equational theory* of $\mathcal{K}$. Finally consider yet another set of formulas of the following form:

$$\& \Sigma \to u = v \tag{1}$$

where $\Sigma = \{s_1 = t_1, \ldots, s_n = t_n\}$ is a set of identities and $u = v$ is an identity. Formula (1) may be written in full as follows:

$$((\forall x_1, \ldots, \forall x_n)(s_1 = t_1 \& \ldots \& s_k = t_k)) \to (\forall x_1, \ldots, \forall x_n)(u = v).$$

By definition, formula (1) holds in $\mathcal{K}$ if the identity $u = v$ holds in every algebra from $\mathcal{K}$ which satisfies all identities of Σ, that is if the identity $u = v$ follows from identities of Σ in the class $\mathcal{K}$. The set of all formulas from $E\mathcal{K}$ of this form is called *the identity theory* of $\mathcal{K}$. For each of these (and other) theories one can ask, given a first order sentence, does it belong to this theory? The corresponding problem will be called the *elementary problem*, the *universal problem*, the *Q-problem*, the *identity problem*, etc. The identity problem for varieties is also called the Tarski-Mal'cev problem. The identity problem for classes of finite algebras in varieties will be called also the Rhodes problem.

If a class $\mathcal{V}$ is closed under homomorphisms, subalgebras and finite direct products (i.e. it is a *pseudovariety* [89]) then the universal problem for $\mathcal{V}$ is equivalent to the Q-problem of $\mathcal{V}$ (see [261], [277]) and both of them are equivalent to the so called *uniform word problem* which asks if there exists a uniform algorithm which solves the word problem simultaneously in all algebras from $FP(\mathcal{V})$. This means that given a finitely presented algebra $S \in FP(\mathcal{V})$ and a relation $u = v$, this algorithm decides if this relation holds in S. Thus the difference between the word problem and the uniform word problem is that the instance of the word problem is a pair of words (terms), and the instance of the uniform word problem is the triple: a pair of words and a set of defining relations. In particular the decidability of the word problem in a pseudovariety follows from the decidability of the uniform word problem of this pseudovariety. It is interesting that the converse implication does not hold for arbitrary algebras: A recent result by Mekler, Nelson, and Shelah [265] shows that there exists a variety $\mathcal{V}$ of universal algebras with finitely many operations which has solvable word problem and unsolvable uniform word problem. A similar example of a variety of algebras with infinitely many operations was constructed earlier by Wells [265]. It is known also that even if both the word problem and the uniform word problem are decidable in a variety, the uniform word problem is in general more complex. For example, the word problem in commutative semigroups may be solved in polynomial time, while any algorithm, solving the uniform word problem, needs at least exponential time (see [253], and Sections 2.8, 3.4.1, 7.3 of this survey). There exists an important connection between the uniform word problem in a pseudovariety and finite partial algebras. This connection was found by Evans (see [96]). Recall that a *partial universal algebra* is a set with a partial operations. If A is a partial universal algebra, B is a universal algebra of the same type, $A \subseteq B$ and every operation of A is a restriction of the corresponding operation of B then we say that the partial algebra A is *embedded* into the algebra B.

Connection 2.2 *Let $\mathcal{V}$ be a pseudovariety of universal algebras. The uniform word problem is solvable in $\mathcal{V}$ if and only if the set of finite partial algebras embeddable into algebras from $\mathcal{V}$ is recursive.*

The first example of a variety with an undecidable equational theory was found by Tarski in 1943-1944 and published in 1953 (see [69], [393], [394]). It was a variety of relational algebras. Mal'cev [234] found such examples among varieties of quasigroups and algebras with two unary operations. The undecidability of the equational problem in a variety is harder to prove than the undecidability of the word problem there. Indeed, the consequences of identities are harder to trace than the consequences of relations. But in fact, as G.McNulty has pointed out to the second author of this survey, every example of a finitely presented algebra with an undecidable word problem can be easily turned into an example of a finitely based variety with an undecidable equational problem.

Indeed, let us take an algebra S with an undecidable word problem. Suppose S is given by a finite set of generators X and a finite set of defining relations R inside a finitely based variety $\mathcal{V}$. Consider a new similarity type of universal algebras consisting of all operations of S and all elements of X as constants. Let $\hat{S}$ be the algebra S considered as an algebra of this new similarity type. The relations of the algebra S are identities of the algebra $\hat{S}$. The variety $\mathcal{U}$ generated by $\hat{S}$ is defined by the identities of R plus the identities of the variety $\mathcal{V}$, and $\hat{S}$ is a relatively free algebra of this variety! Since we cannot algorithmically decide whether or not a relation $u = v$ holds in S, we cannot algorithmically decide whether an identity $u = v$ holds in the variety $\mathcal{V}$. Therefore $\mathcal{U}$ has an undecidable equational problem.

The argument from the previous paragraph shows that there exists yet another connection between the equational problem and the word problem.

Connection 2.3 *The equational problem of a variety $\mathcal{V}$ is solvable if and only if the word problem in every free algebra in this variety is solvable.*

Therefore if the equational problem is unsolvable in a variety $\mathcal{V}$ then the word problem is also unsolvable there. Thus the Tarski variety of relational algebras was probably the first finitely based variety of universal algebras with an unsolvable word problem.

The connection between the equational problem and the identity problem is similar to that between the word problem and the uniform word problem. The identity problem in $\mathcal{V}$ is solvable if there exists a uniform algorithm which solves the equational problem in all finitely based subvarieties of $\mathcal{V}$ simultaneously. This means that given a subvariety of $\mathcal{V}$ defined by finitely many identities, and an identity α, this algorithm decides if α holds in this subvariety. But in the case of identities, it is not known whether or not the existence of the uniform algorithm is equivalent to the existence of all particular algorithms, i.e. whether or not the decidability of the identity problem in $\mathcal{V}$ is equivalent to the decidability of the equational problem for every finitely based subvariety of $\mathcal{V}$. Thus we formulate the following very interesting problem.

Problem 2.2 *Is there a finitely based variety $\mathcal{V}$ of universal algebras (groups, semigroups, associative and Lie algebras) with undecidable identity problem and such that every finitely based subvariety of $\mathcal{V}$ has decidable equational problem?*

Notice that the identity problem is equivalent to the *problem of coincidence of varieties*. Indeed, it is easy to show that the identity problem for a variety $\mathcal{V}$ is solvable if and only if there exists an algorithm which, given any pair of systems of identities, says if these systems define the same variety inside $\mathcal{V}$. Thus the role the identity problem plays in the study of varieties given by identities is the same as the role of the isomorphism problem in the study of algebras given by defining relations.

2.5 Algorithmic Problems for Pseudovarieties

Varieties may contain some weird infinite algebras. If we want to concentrate on finite algebras, then it is better to consider pseudovarieties of finite algebras — classes of finite algebras closed under homomorphic images, subalgebras and finite direct products.

The classes of all finite semigroups, of all finite groups, of all nilpotent semigroups (groups, associative or Lie algebras), all finite aperiodic semigroups[1], etc., are examples of pseudovarieties. For every variety $\mathcal{V}$ the class of all finite algebras from $\mathcal{V}$ is a pseudovariety, called the *finite trace* of $\mathcal{V}$ and denoted by $\mathcal{V}_{\mathrm{fin}}$. Every pseudovariety is a union of an increasing sequence of finite traces [89].

There are many interesting connections between pseudovarieties and the theory of profinite algebras, automata theory, and the theory of formal languages. The last two theories correspond to the theory of pseudovarieties of semigroups and this is one of the reasons why pseudovarieties of semigroups have been studied very intensively during the last 10 years. One of the main questions in the theory of formal languages (finite automata) is the following: Does a given language (automaton) belong to a class of languages (automata) which is constructed from some special kind of languages (automata) in a special way? This question may be formulated in terms of semigroup pseudovarieties in the following way: does a finite semigroup belong to a pseudovariety constructed from some special pseudovarieties in some special way. We do not explain the word "special" here – see Section 3.7.2 for more details. But one can see that we run into a kind of algorithmic problem again – the problem of decidability of membership in pseudovarieties.

It is interesting that, as was shown by Albert, Baldinger and Rhodes [9], these problems are very closely connected with the identity problem in pseudovarieties. See for example Theorem 3.18 in Section 3.3.2. This is actually how the identity problem for pseudovarieties first arose in works of Rhodes and his students; as a problem related to the theory of finite automata.

Unlike the identity problem, the uniform word problem (the Q-problem or the universal problem) for classes of finite algebras was introduced for purely algebraic reasons. Indeed finite algebras often come into the world given by generators and defining relations. The uniform word problem for classes of finite algebras was formulated for the first time by Yu.Gurevich in 1967 [133]. He proved that the class of all finite semigroups has undecidable uniform word problem. Later Slobodskoj proved the undecidability of this problem in the class of finite groups [378]. And only a few years later these results found their applications — in the theory of data bases

[1] A finite semigroup is called aperiodic if it does not contain non-trivial subgroups.

[134], [143]. Notice that Connection 2.2 shows a relationship between the uniform word problem for a pseudovariety $\mathcal{V}$ and the membership problem for the class of finite partial algebras embeddable into algebras in $\mathcal{V}$. This class of partial algebras is not a pseudovariety but it is closed under taking finite direct products and partial subalgebras.

2.6 Neighbors of Algorithmic Problems

The algorithmic direction in the theory of varieties is not an isolated point in the space of directions. There is a close and mutually beneficial connection between this direction and the rest of the theory.

The connection between the decidability of the word problem in a variety and the residual finiteness of its finitely presented algebras is the most celebrated one. Recall that an algebra is called *residually finite* if it has enough homomorphisms onto finite algebras to separate every pair of distinct elements.

Connection 2.4 *Let A be an algebra finitely presented in a finitely based variety $\mathcal{V}$. If A is residually finite then A has a decidable word problem.*

Proof. The algorithm for solving the word problem essentially belongs to McKinsey [261]. Let $A =< X >$ be given by a finite system of identities and relations Σ and u and v be two terms over X. In order to check if $u = v$ in A let us start two enumeration processes. The first process lists, one by one, all relations which follow from Σ. The second process lists all finite algebras generated by X which satisfy Σ. If $u = v$ in A then the first process will give us this equality. If $u \neq v$ then this equality does not hold in one of the finite algebras because A is residually finite, and the second process will give us this finite algebra.

Therefore, after a finite number of steps, one of these processes will end and we will decide if $u = v$ in A.[2]

Thus if, in a variety $\mathcal{V}$, all finitely presented algebras are residually finite then the word problem is solvable in $\mathcal{V}$. Moreover the McKinsey algorithm is "uniform": it does not depend on the presentation of an algebra and so it solves the uniform word problem also. By virtue of a result by Mekler, Nelson, Shelah, and Wells [265], this implies that the decidability of the word problem is weaker than residual finiteness: the varieties constructed in [265] have solvable word problem and unsolvable uniform word problem. There are also examples of varieties of groups, associative and Lie algebras where the word problem is solvable but not all finitely presented algebras are residually finite. Semigroup varieties are an exception: we do not know examples of semigroup varieties with solvable word problem which contain non-residually finite

[2]The idea of McKinsey's algorithm (the simultaneous listing of finite homomorphic images and all consequences of the system of relations) is applicable not only to the word problem. The same idea works for the conjugacy problem [236]. In this case, instead of residual finiteness we have to assume residual finiteness with respect to conjugacy.

finitely presented semigroups. Moreover there are strong reasons to believe that such examples do not exist (see details in Section 3).

A similar connection exists between residual finiteness and the solvability of the identity problem.

Connection 2.5 *If every relatively free algebra in every subvariety of a variety $\mathcal{V}$ is residually finite then the identity problem in $\mathcal{V}$ is solvable.*

Indeed, as we mentioned before, the identity problem is a "uniformization" of the equational problem, and the McKinsey algorithm is uniform.

Residual finiteness has close ties also with the solvability of the uniform word problem for finite algebras in a variety.

Connection 2.6 *If $\mathcal{V}$ is a finitely based variety and every finitely presented algebra in $\mathcal{V}$ is residually finite, then the uniform word problem is solvable in $\mathcal{V}_{fin}$.*

Proof. Indeed, let Σ be a finite set of relations over an alphabet X, u and v be two terms over X. Suppose we want to know if $u = v$ in every finite algebra $A =< X >$ from $\mathcal{V}$ which satisfies Σ. Take the algebra $S =< X >$ given by Σ inside $\mathcal{V}$. This algebra is residually finite. Hence it has solvable word problem (see Connection 2.4). Thus we can decide if $u = v$ in S. If u and v are equal in S then they are equal in any algebra $A =< X >$ from $\mathcal{V}$ satisfying Σ. On the other hand, if $u \neq v$ in S then, since S is residually finite, there exists a finite homomorphic image $A =< X >$ of S where these terms also represent different elements. Therefore Σ implies $u = v$ in $\mathcal{V}_{\mathrm{fin}}$ if and only if $u = v$ in S. This gives us the algorithm solving the uniform word problem in the class of finite algebras from $\mathcal{V}$. The Connection is established.

There is a connection between residual finiteness and the isomorphism problem. This connection, first noticed by Pickel [296], is based on the following observation. Let us call two algebras *quasi-isomorphic* if they have the same set of finite homomorphic images. It is clear that if two algebras are isomorphic then they are quasi-isomorphic. The converse implication does not hold [430]. Suppose that a finitely presented algebra A is such that there are only finitely many finitely presented algebras $\{A_1, \ldots, A_n\}$ which are quasi-isomorphic to A. Take another finitely presented algebra $B = < X|\Sigma >$. Let us start three parallel processes. The first process lists all consequences of the set of defining relations Σ, and of the sets of defining relations of the algebras A_i, $i = 1, \ldots, n$. The second process lists all finite homomorphic images of B and checks if they belong to the set of finite homomorphic images of A. The third process lists all finite homomorphic images of A and checks if they are homomorphic images of B: it is clear that the set of finite homomorphic images of any finitely presented algebra is recursive. Then either the first process will tell us that one of the algebras A_i is isomorphic to B (all relations of A_i are consequences of relations of B and vise versa) or the second process will find a finite homomorphic image of B which is not a homomorphic image of A, or the third process will find a finite homomorphic image of A which is not a homomorphic image of B. In the first case B is isomorphic to A if A_i is A, and B is not isomorphic to A if A_i is not A. In the second case and in the third case B is not isomorphic to A. Therefore for every such A there is an algorithm which decides if a finitely presented algebra B is isomorphic to A.[3]

There are some important cases (for example the case of finitely generated nilpotent groups [296]) when this method works well (see Section 6). If we want to use Pickel's method to solve the ordinary isomorphism problem in a variety $\mathcal{V}$ then we must have two conditions:

1. For every algebra A finitely presented in $\mathcal{V}$ there are only finitely many finitely presented algebras $A_1, \ldots, A_n$ in $\mathcal{V}$ which are quasi-isomorphic to A,

2. These algebras $\{A_1, \ldots, A_n\}$ must be constructed effectively, given a presentation of A.

If $A \in \mathcal{V}$ is a finite algebra and every finitely presented algebra in $\mathcal{V}$ is residually finite then there are only finitely many finitely presented algebras in $\mathcal{V}$ which are quasi-isomorphic to A. Therefore we have the following connection.

Connection 2.7 *If every finitely presented algebra in a variety $\mathcal{V}$ is residually finite, then for every finite algebra $A \in \mathcal{V}$ there exists an algorithm which decides if a finitely presented algebra from $\mathcal{V}$ is isomorphic to A. In particular, the triviality problem is solvable in $\mathcal{V}$*

We expect a strong connection between residual finiteness and the decidability of the finiteness problem.

Problem 2.3 *Suppose that all finitely presented algebras of a finitely based variety $\mathcal{V}$ are residually finite. Is the finiteness problem for $\mathcal{V}$ decidable?*

There is a strong connection between residual finiteness and the equational problem. As was mentioned above (see Connection 2.3) this problem is equivalent to the word problem in relatively free algebras. Thus if all relatively free algebras in a given variety are residually finite (or, equivalently, this variety is generated by its finite algebras) then the equational problem is solvable in this variety and in the corresponding finite trace — the set of all finite members of this variety.

Another neighbor of algorithmic properties is the *Higman property*. A variety is said to have the Higman property (to be a Higman variety) if every recursively presented algebra in this variety is embeddable into a finitely presented algebra from the same variety. Recall that an algebra is called *recursively presented* if it can be given by a finite set of generators and a recursively enumerable set of defining relations. The variety of all groups was the first one which was proved to be a Higman variety (Higman, [144], a simpler proof in Aanderaa [1]).

[3]This proof is yet another implementation of the McKinsey's idea.

Connection 2.8 *If a Higman variety $\mathcal{V}$ contains any algebra given by an infinite independent set of defining relations (no one relation follows from others) then the word problem is not solvable in this variety.*

Proof. Indeed, this set of defining relations contains a recursively enumerable but non-recursive subset. An algebra given by this subset of relations inside $\mathcal{V}$ is embeddable into a finitely presented algebra from the same variety. The word problem in the last algebra is undecidable, so $\mathcal{V}$ has undecidable word problem. The Connection is established.

However the Higman property does not always imply undecidability of the word problem. In particular, if every finitely generated algebra from a variety is finitely presented there (let us call this property $FG = FP$) then this variety clearly satisfies the Higman property. For example the variety of all Abelian groups satisfies $FG = FP$ and so it is a Higman variety! The variety of all commutative rings, the variety of metabelian groups and the variety of commutative semigroups are perhaps the most famous varieties which satisfy $FG = FP$. Notice that this property is equivalent to the ascending chain condition for congruences on every finitely generated algebra in a variety. The class of varieties of associative rings was the first one where the $FG = FP$ property was intensively studied (see Section 4.7).

The next neighbor is at first glance very unexpected while it is one of the oldest inhabitants in the theory of varieties. We mean the "finite basis property." There exists the following connection between this property and the decidability of the identity problem.

Connection 2.9 *If all subvarieties of $\mathcal{V}$ are finitely based then the equational problem in the finite trace $\mathcal{V}_{fin}$ is solvable.*

Proof. Indeed, consider the subvariety $\mathcal{U}$ generated by all finite members of $\mathcal{V}$. This subvariety is finitely based by the condition. Since it is generated by finite algebras, all its free objects are residually finite and so they have solvable word problems. Now consider an identity $u = v$. Obviously, $u = v$ holds in $\mathcal{V}_{\text{fin}}$ iff it holds in $\mathcal{U}$ iff u coincides with v in the free algebra of $\mathcal{U}$ with the corresponding number of generators. The Connection is established.

Notice that, by definition, the solvability of an algorithmic problem means that the algorithm solving the problem *exists*, but it does not mean that we can actually find it in all cases. In particular, Connection 2.9 means that the algorithm solving the equational problem exists, but does not say how to find it.

There is yet another direction in the theory of varieties whose connection with algorithmic problems was discovered only recently. This connection appears to be even stronger than that with residual finiteness and the Higman property. We mean the direction concerned with Burnside properties of algebras. By Burnside properties we understand those that figure in the three Burnside problems about groups: bounded, unbounded and restricted Burnside problems. Similar problems exist in all classical types of algebras.

The bounded problem for general algebras asks if a finitely generated algebra is finite provided the orders of its one-generated subalgebras are finite and bounded by a natural number. The unbounded problem asks if a finitely generated algebra is finite provided all its one-generated subalgebras are finite. The restricted problem asks if there are only finitely many finite algebras with any given number of generators and any given bound for orders of one-generated subalgebras. In the case of linear algebras over an infinite field, we have to replace in these definitions the word "order" by the word "dimension". In the case of Lie algebras, where every 1-generated subalgebra is automatically finite, we have to consider 2-generated subalgebras instead.

The "Burnside trace" can be seen already in the paper by Murskii [276] where he constructed an example of a semigroup variety with undecidable equational problem. Murskii essentially used the fact that his periodic variety is not *locally finite*, that is it contains an infinite finitely generated semigroup. And the cube free Thue-Morse-Arshon sequence, which plays an important role in his proof, undoubtably is on the coat of arms of the family of Burnside properties (see Section 7.7.1 below for more details). This connection appeared most explicitly in the paper of the second author of this survey [346] where it is proved, in particular, that if the word problem is solvable in a nonperiodic variety of semigroups then all periodic semigroups in this variety are locally finite (see Section 3.3). This and many other results of such kind hint that the positive solutions of algorithmic problems are hardly possible in those varieties where Burnside type problems are solved negatively. For example, the following statement gives a connection between a positive solution of the so-called Restricted Burnside problem (also known as Magnus' problem) and the solvability of the uniform word and identity problems in a pseudovariety. Recall that one of the equivalent formulations of the restricted Burnside problem concerning a variety is: *Do all locally finite members in this variety form a subvariety?*

Connection 2.10 *If the locally finite members of a variety $\mathcal{V}$ form a subvariety and one can compute the order of a free algebra of this subvariety given a number of generators, then the uniform word (identity) problem is solvable in $\mathcal{V}_{fin}$.*

Proof. Let Σ be a set of relations (identities). One needs to find out if Σ implies a relation (identity) $u = v$ in $\mathcal{V}_{\mathrm{fin}}$. Let n be the number of letters which occur in $\Sigma \cup \{u = v\}$. Compute the order of the n-generated free algebra in the maximal locally finite subvariety of $\mathcal{V}$. Given this order, one can compute the multiplication tables of this algebra and its homomorphic images. Then it remains to check all these images, choose those which satisfy Σ and verify if they also satisfy $u = v$. The Connection is established.

Old and recent positive solutions of Burnside type problems in associative and Lie algebras, groups and semigroups are very important in the study of algorithmic problems. And we would like to express our gratitude for all these solutions.

We are especially grateful to Kaplansky's Theorem on algebraic PI-algebras [161], Shirshov's "height" Lemma [371], Kostrikin's Theorem on the local finiteness of Lie algebras with an Engel identity [202], [203], Zelmanov's Theorem about the global nilpotency of Lie algebras with an Engel identity (characteristic 0) [425], the Hall-Higman-Kostrikin-Zelmanov solution of the restricted Burnside problem for groups [139], [202], [203], [201], [428], [427], and the Bean-Ehrenfeucht-McNulty-Zimin-Sapir description of semigroup varieties with locally finite nil-semigroups [29], [431], [346].

2.7 The Influence of the Theory of Varieties

Of course the theory of algorithmic problems in varieties is a part of the general theory of varieties. The mere perception of this fact leads one to correct formulations of the final goals and strategy.

For every algorithmic problem and every class of varieties $\mathcal{C}$ the final goal is the "full and complete" description of all varieties from $\mathcal{C}$ where this problem is solvable (weakly solvable). Of course, the word "description" is not descriptive! There exist many ways to describe varieties (an attempt to classify those ways was made by Shevrin and Sukhanov [368]). We believe that it would be most natural to try to get algorithmic descriptions.

By an *algorithmic description* of varieties with a property α from a given class $\mathcal{C}$ we mean presenting an algorithm which, given a finite set of identities Σ that defines a variety from the class $\mathcal{C}$, says whether or not this variety satisfies α.

The first problems about algorithmic descriptions of varieties with different properties were raised by Tarski in his classic survey [392]. Every algorithmic description is certainly limited because it deals only with finitely based varieties (recall that a variety is *finitely based* if it can be defined by a finite number of identities). But when one studies algorithmic problems for finitely presented algebras this restriction is natural. Indeed, algorithmic problems can be considered only in constructively presented classes and it is clear that finitely based varieties are constructively presented. Of course, one can consider, say, recursively based varieties which seem to be constructively presented also. But in fact recursively presented varieties are much less constructive objects than finitely based ones. For example if a variety is finitely based then for any given finite algebra one can decide whether or not it belongs to this variety. For recursively based varieties this problem is in general undecidable. And it is worth noting here that algorithmic properties of varieties essentially depend on the presence or absence of certain finite algebras in these varieties (see, for example, [335]).

It is known (see, for example, the survey by Bakhturin and Ol'shanski [19]) that one of the most natural strategies in describing varieties with a certain property α consists in finding as many as possible *minimal non-α-varieties*. A variety is called minimal non-α if it is not an α-variety, but all its proper subvarieties satisfy α. If the property α is such that every subvariety of an α-variety also satisfies α, then each minimal non-α variety is very important. Indeed, then every variety containing this "minimal counterexample" does not satisfy α, so the area of search gets much smaller.

Even for properties α which are not hereditary for subvarieties (in particular, for the decidability of the word problem) minimal non-α varieties often reduce the area of search (see [343], [184]). Notice that the problem of finding minimal non-α varieties is interesting in itself as is every problem about boundaries between "Yes" and "No". The first decidability/undecidability boundary — a minimal variety with undecidable word problem — was found in the class of semigroups (see [268], for more details see other sections of this survey).

Another strategy has been used, for example, in the work by Groves [118]. It consists in searching for what we call "indicator" varieties. A variety $\mathcal{V}$ is called an *indicator* with respect to a property α if for all varieties $\mathcal{U}$, $\mathcal{U}$ satisfies α iff the intersection $\mathcal{V} \cap \mathcal{U}$ satisfies α. It is clear that if we managed to find a relatively simple indicator variety then the problem of describing all α-varieties becomes simpler. There are some reasons to believe that for many algorithmic properties one can find relatively simple indicator varieties.

2.8 Complexity of Algorithmic Problems

The solvability of an algorithmic problem does not mean that the problem can be solved in practice. First of all the existence of an algorithm does not mean that it is readily available (see the discussion after Connection 2.9). Another, more important, obstacle is that the algorithm can be too slow. For example, if we look at the McKinsey algorithm (Connection 2.4) from this point of view, then it will be clear that in general this algorithm is very slow. Even for small words u and v it would take a lot of time to decide, using this algorithm, if $u = v$ in a given residually finite algebra.

Thus the next thing to do, after we find out that a problem is decidable, is to find the computational complexity of this problem.

To make this more precise let us present here some concepts from Computational Complexity Theory.

Any decision problem D may be considered as a membership problem for elements of some set B_D in a subset S_D. For example if D is the uniform word problem for semigroups then B_D is the set of all triples (u, v, R) where u, v are words, R is a set of defining relations, and S_D is the subset of triples (u, v, R) such that $u = v$ in the semigroup defined by R.

With any element x in B_D one associates a number which is called the *size* of this element. Usually the size is roughly the minimal space which is needed to write x down. The size depends on the way we choose to represent the elements. For example if x is a natural number then we can represent x as x units. Then the size of x will be equal to x. If we represent x as a sequence of binary digits then the size of x will be approximately $log_2(x)$.

Algorithms may be realized by Turing machines that have one read/write tape and one head. We can assume that this machine is equipped with a voice synthesizer and can say two words "Yes" and "No". An algorithm solving the decision problem D starts working with an element x of B_D written on the tape of the machine. When it ends, it says "Yes" if $x \in S_D$ or "No" if $x \notin S_D$.

With every algorithm A solving the problem D one can associate two important functions: the time complexity function and the space complexity function.

The time complexity function $t_A(n)$ is the maximal number of steps of the machine needed for the algorithm to decide if an element x of size $\leq n$ is in S_D The space complexity function $s_A(n)$ is the maximal number of cells of the tape visited by the machine while it is working on an element x of size $\leq n$. If one wants to consider less than linear space algorithms, one has to consider more complicated computing devices [104].

The following connection is clear.

Connection 2.11 $s_A(n) \leq t_A(n)$.

Indeed, even if at every step the algorithm used a new cell of the tape, $s_A(n)$ would be only equal to $t_A(n)$. In reality $s_A(n)$ is always smaller than $t_A(n)$. On the other hand the following connection also holds.

Connection 2.12 $t_A(n) \leq c^{s_A(n)}$ *where c is some constant depending on the machine but independent of n.*

Indeed, at every particular moment the behavior of the machine depends only on three parameters:

1. the word written on the part of the tape which is visited by the head of the machine during its work; by definition, the length of this word can not exceed $s_A(n)$,

2. the state of the head; the number of states does not depend on n,

3. the position of the head of the machine; the number of such positions does not exceed $s_A(n)$.

Taking into account that there are no more than $|X|^{s_A(n)}$ possible words of length $s_A(n)$ in the alphabet X of the machine, we can conclude that there are at most $c^{s_A(n)}$ possible situations for some constant c. If one of the situations repeats during the machine's work, the algorithm will cycle indefinitely, which is impossible: it must finally say "Yes" or "No". Therefore the time complexity function cannot exceed $c^{s_A(n)}$.

If there exists an algorithm A which solves D and $t_A(n)$ is bounded from above by a polynomial (exponential) in n then we say that D *can be solved in polynomial* *time (exponential time)*. The solvability in polynomial space (exponential space) is defined similarly.

It is worth mentioning that if we modernize the Turing machine by, say, adding more tapes or heads, we won't change the complexity of the problem much. For example a non-polynomial time (space) problem cannot become polynomial as a result of that. The class of all problems which can be solved in polynomial time is denoted by P.

If in the definition of the time complexity, we replace the (deterministic) Turing machines by non-deterministic Turing machines, then we obtain definitions of the solvability in non-deterministic polynomial time, non-deterministic exponential time, etc. Recall that a non-deterministic Turing machine is more intelligent than a deterministic one: it does not blindly obey the commands of the program, but, at every step, guesses itself what the next step should be. Roughly speaking a problem D can be solved in non-deterministic polynomial time if for every element $x \in S_D$ there exists a proof that x belongs to S_D and the length of this proof is bounded by a polynomial of the size of x. The class of all problems which can be solved in polynomial time by a non-deterministic Turing machine is denoted by NP. It is not known if P=NP. This is one of the central problems in Theoretical Computer Science.

On the other hand the space complexity does not change much if we use non-deterministic Turing machines instead of deterministic Turing machines. In fact if a problem can be solved by a non-deterministic Turing machine with a space function $s(n) > \log_2(n)$ then it can be solved by a deterministic Turing machine with space function $cs(n)^2$ for some constant c [362]. In particular, a problem solvable in polynomial space by a non-deterministic Turing machine is automatically solvable in polynomial space by a deterministic Turing machine.

In order to prove that a problem is solvable in polynomial time it is enough to find a polynomial time algorithm solving this problem.

In order to prove that a problem D is not polynomial (more than exponential, etc.) one has to take a problem Q which is known to be "hard" and reduce it to D.

There are several kinds of reductions used in the Computer Science literature. One of them, polynomial reduction in the sense of Karp [104], is the following. A reduction of a problem Q to a problem D is a function ϕ from B_Q to B_D such that

- An element x from B_Q belongs to S_Q if and only if $\phi(x)$ belongs to S_D.
- The element $\phi(x)$ can be computed in polynomial time, in particular the size of $\phi(x)$ is bounded by a polynomial of the size of x.

It is clear that if Q is "hard" and Q can be reduced to D then D is "hard" as well.

A.Meyer was probably the first who raised questions about the computational complexity of decidable algorithmic problems in algebra. One of the first algorithmic problems considered by him and his students was the word problem in the variety of commutative semigroups.

A student of A.Meyer, E.Cardoza [68], noticed that the word problem can be solved in linear time. Mayr and Meyer [253] proved that the uniform word problem for commutative semigroups is *exponential space complete* which means that there exists an algorithm solving this problem, which needs an exponential amount of space on the tape, and every problem which requires exponential space may be reduced to the uniform word problem for commutative semigroups (see Sections 3.4.5, 7.3).

Notice that, when we prove the undecidability of a problem D, we usually also reduce a problem Q known to be "hard" (which in this case means undecidable), to D. In order to reduce Q to D we find a similar mapping ϕ, but we do not care about the size of $\phi(x)$ (see Sections 7.2.1, 7.5, 7.6 below).

It is clear for us that methods of reducing undecidable problems worked out in the theory of algorithmic problems in algebra must help in proving that this or that decidable problem is "hard". Conversely, methods of reducing "hard" problems worked out in Computer Science will certainly help in proving undecidability of algorithmic problems.

In fact, connections between Computer Science and algorithmic problems in algebra are already very strong. We have mentioned connections between data bases and the uniform word problem for finite semigroups and groups. One of the reasons to study the complexity of the uniform word problem for commutative semigroups was a need of the theory of Petri nets, yet another part of Computer Science. Petri nets are used in constructing and analyzing information networks. It is an instrument in describing the information flow in complicated systems. We will not give the Computer Science definition of Petri nets, and we will not formulate "real life" problems related to Petri nets (see [314], [63]). Let us only notice that from the algebraic point of view Petri nets relate to presentations of commutative semigroups just like semi-Thue systems relate to semigroup presentations (Thue systems). In Thue systems we can apply relations both from the left to the right and from the right to the left; in semi-Thue systems only left-right applications are allowed. Recall that to apply a relation $u = v$ to a word w means to represent w in the form puq where p and q are words and then replace u by v in this presentation. It is easy to see that semi-Thue systems with symmetric sets of relations (that is *reversible semi-Thue systems*) are just Thue systems. Reversible Petri nets are precisely presentations of semigroups in the variety of commutative semigroups. Petri nets have been extensively studied during the last 20+ years (there are more than 500 publications[4]).

Another connection between Computer Science and algorithmic problems in algebra was found by R.Fagin [98]. He proved, in particular, the following amazing

[4]These are only the "open" publications. We can only guess how much classified information about Petri nets related to military communication systems have been produced during the Cold War. Perhaps, after this information is declassified, we will learn some sensational facts about finitely generated commutative semigroups.

model-theoretic characterization of classes of finite algebras whose membership problem can be solved in non-deterministic polynomial time.

Connection 2.13 *The membership problem for an abstract (closed under isomorphisms) class of finite algebraic systems is in NP if and only if it is the class of all finite models of a second-order formula of the following type:*

$$\exists Q_1 \exists Q_2 \ldots \exists Q_n (\Theta)$$

where Q_i is a predicate, and Θ is a first-order formula.

Basically this Theorem means that the membership problem of a class of finite algebras is in NP if and only if we can describe the structure of algebras of this class in terms of functions and relations. Since all known methods of studying the structure of algebras are based on studying functions (endomorphisms, polynomial functions, etc.) and relations (congruences, etc.) we can conclude that for a class of finite algebras, having a membership problem in NP is equivalent to admitting a reasonable structure description.

Classes where the membership problem has other types of computational complexity have similar model theoretic characterizations. Classes with membership problem in P have been characterized by Immerman [152], Sazonov [364], [363] and Vardi [407]. Classes with exponential time membership problem were characterized by Fagin [98]. Classes with non-deterministic exponential time membership problem have been characterized by Fagin [98] and Jones and Selman [158]. A detailed survey of these results has been recently published by Fagin [97].

It seems obvious that these Connections will have many applications to the theory of pseudovarieties of finite algebras.

Finally let us mention some problems which arise naturally when one tries to analyze the complexity of algorithms related to varieties.

Problem 2.4 *(Sapir) For every finite universal algebra (semigroup, group, ring) A find the computational complexity of the following problem:*

Input. *An identity $u = v$.*

Task. *Check if $u = v$ does not hold in A.*

In particular, is there a finite universal algebra (semigroup, group, ring) A such that this problem cannot be solved in polynomial time?[5]

It is clear that this problem is in NP and can be solved in linear space. The similar problem for finite dimensional linear algebras is also in NP if basic operations of the

[5]The referee noticed that if A is the two element Boolean algebra $\{0, 1\}$ then the identity $u = 0$ holds in A iff the formula u is identically false. It is known that the problem of verifying whether a formula is true or false is NP complete [104]. Thus the Problem 2.4 is NP complete for the two element Boolean algebra. For groups and semigroups this problem is still open.

ground ring K (the addition and the multiplication) are computable in polynomial time, say, if K is the field of rational numbers or the ring of integers. As far as we know there are no examples of finite dimensional algebras over **Q** or **Z** for which this problem cannot be solved in polynomial time. Problem 2.4 is important because it happens very often that algorithmic descriptions of varieties are formulated in terms of "forbidden algebras" (see Section 2.7).

Problem 2.5 *(Margolis, Sapir). For every finite universal algebra A find the computational complexity of the following problem:*

Input. *A finite algebra B.*

Task. *Check if B belongs to the variety generated by A.*

In particular, is there an algebra A for which this problem cannot be solved in polynomial time or in polynomial space?

2.9 Algorithmic Problems and Rewrite Systems

Rewrite systems participate crucialy in many solutions of algorithmic problems and in many practical implementations of these solutions. There are a huge number of papers, surveys and books devoted to these systems. See, for example, [223], [81], [34], [47], [30]. Here we would like to mention some general connections between the theory of rewrite systems and algorithmic problems in varieties. Our presentation here follows paper [338] by the second author of this survey.

Let k be a natural number, X be a k-element alphabet, and let F be an algebra generated by X. Suppose that we are studying finitely generated congruences on F. For example, F can be the free algebra over X in some variety $\mathcal{V}$. Then finitely generated congruences on F correspond to finitely presented algebras in $\mathcal{V}$. The word problem for finitely presented algebras in $\mathcal{V}$ corresponds to the membership problem for pairs from $F \times F$ in finitely generated congruences.

One of the possible ways to study a congruence σ on F is to find, in some sense, the unique normal form in each congruence class (we will discuss an alternative approach in Section 7.1). Given an algorithm for finding these normal forms, we can describe the congruence by saying that two elements are in the same class if and only if they have the same normal forms. From the algebraic point of view, these normal forms and algorithms for finding them are the main personages of the theory of rewrite systems.

In the most general form a rewrite system is just an oriented graph G, where vertices are called *objects*, and an edge (a, b) means that the object b is obtained from a by an *elementary transformation.* If we forget for a moment about the orientation of the edges, we can break the graph G into a disjoint union of connected components. This partition is called the *partition generated by the rewrite system G.*

In practice we are usually given a partition of a set G and the goal is to find a rewrite system which generates the same partition and helps us to find the normal forms in the way described below.

For example, objects may be elements of F and elementary transformations may be the left-right applications of relations from some fixed set of relations (see [198]). Recall that if $w, p, q \in F$ then in order to apply the relation (p, q) to w we have to represent w as a term t in generators of F, find a subterm of t which represents p, and replace this term by a term which represents q. The element represented by the resulting term is the *result of the application.* Sometimes, for example in the case of rings, we need more restrictive definitions of an application of a relation (see Section 4.10). Thus with every set of pairs $\Sigma \subseteq F \times F$ one can associate a rewrite system which we will denote by $\Omega(F, \Sigma)$. The connected components of this rewrite system are congruence classes of the congruence generated by Σ.

Thus every semi-Thue system and every Petri net are rewrite systems. The objects of a semi-Thue system are words, i.e. elements of the absolutely free semigroup X^+, and the objects of a Petri net are elements of the free commutative semigroup, that is commutative words. In the Computer Science literature semi-Thue systems are called "string-rewriting systems". There are several surveys and books devoted to string-rewriting systems. See, for example, Book and Otto [47].

Given a rewrite system which generates a partition σ, we can try to find the normal form of objects in a connected component by taking an element a in this component and by applying elementary transformations to this object until we get an object which cannot be further transformed by the elementary transformations. This procedure is usually (though not always) very fast and efficient.

It is clear, however, that we have to avoid two fundamental difficulties. First of all, the process of applying elementary transformations to our object a might never end. Then we would get an infinite directed path in our graph G:

$$a \to a_1 \to a_2 \to \ldots \to a_n \to \ldots .$$

Second, even if the process ends, we are not guaranteed that the element that we finally get, is unique in the connected component (the normal form in a connected component must be unique, of course). Indeed, for example, the graph G can have the following form

Then a and c are in the same connected component but our procedure applied to a and c gives different results.

A rewrite system G is called *terminating* if every directed path on G is finite. A rewrite system G is called a *Church-Rosser system* if for every two elements a and c from the same connected component of G there exist directed paths $a \to a_1 \to a_2 \ldots \to a_n$ and $c \to c_1 \to c_2 \to \ldots \to c_m$ with $a_n = c_m$. It is clear that if a rewrite system is terminating and Church-Rosser then the above mentioned procedure works and gives the desired normal form in every connected component.

Terminating Church-Rosser rewrite systems were first formally introduced in [70] and [280] (see also [147]).

Now let us consider again an algebra $F =< X >$, a set of pairs $\Sigma = \{(p_i, q_i) \mid i = 1, 2, \ldots\}$ from $F \times F$ and the rewrite system $\Omega(F, \Sigma)$ with elements of F as objects

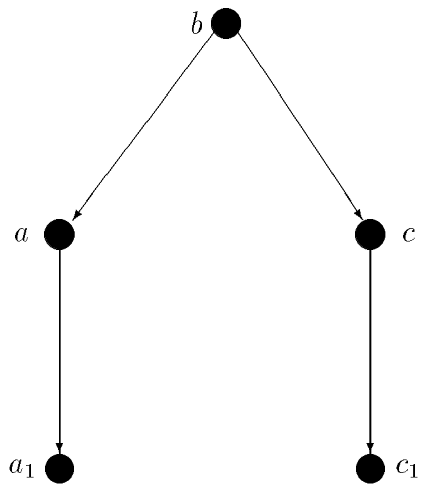

Figure 1:

and left-right applications of relations $p_i = q_i$ as elementary transformations.

Of course, if we want to use the rewrite system $\Omega(F, \Sigma)$ in order to actually compute the normal forms in the congruence classes, we need the decidability of the following two problems in F:

1. The word problem;

2. The "applicability" problem: given an element $w \in F$ and a pair (p, q) decide if the relation (p, q) is (left-right) applicable to w and if it is applicable then write down the result of the application.

We will call the set of relations $\Sigma \subseteq F \times F$ terminating (resp. Church-Rosser) if the corresponding rewrite system $\Omega(F, \Sigma)$ possesses this property. There is an almost obvious but very important connection between the terminating Church-Rosser presentations and the word problem.

Connection 2.14 *Let the word problem and the applicability problem be decidable in F, and let a congruence γ on F be generated by a finite terminating Church-Rosser system $\Omega(F, \Sigma)$. Then the word problem in F/γ is decidable.*

Proof. Indeed, let u and v be two elements in F. In order to decide if $(u, v) \in \gamma$ one has to construct two sequences:

$$u = u_0 \to u_1 \to u_2 \to \ldots,$$

$$v = v_0 \to v_1 \to v_2 \to \ldots,$$

where each arrow $\to$ means an application of a relation from Σ (from left to right). If at some step we can apply several relations, chose one of them. These processes eventually end since the presentation is terminating. The Church-Rosser property guarantees that $(u, v) \in \gamma$ if and only if the final word in the first sequence coincides with the final word in the second sequence. The connection is established.

If we want to use Church-Rosser presentations in order to study finitely presented algebras in a variety $\mathcal{V}$, then it seems natural to take the free algebra $F = F_X(\mathcal{V})$ and to try to find terminating Church-Rosser rewriting systems which generate finitely generated congruences on F. This strategy works for the variety of all commutative semigroups or rings (see Sections 3.5.2, 4.10). Sometimes, however, the free algebras of $\mathcal{V}$ are not the best choice. As in [338], we shall call a finitely generated algebra F a *pseudo-free algebra for* $\mathcal{V}$ *of rank* n if the free n-generated algebra of $\mathcal{V}$ is isomorphic to a factor-algebra of F over a finitely generated congruence. Notice that F itself may not belong to $\mathcal{V}$. This may be, for example, a free algebra in some bigger (and better) variety (see Sections 3.5.2 and 4.10 for examples). If F is a pseudo-free algebra for $\mathcal{V}$ of rank n then every finitely presented n-generated algebra A in $\mathcal{V}$ is a factor-algebra of F over a finitely generated congruence γ. We shall say that A *has a terminating (resp. Church-Rosser) presentation with respect to* F if γ is generated by a terminating (resp. Church-Rosser) rewrite system $\Omega(F, \Sigma)$.

We call a variety $\mathcal{V}$ a *Church-Rosser variety* if for every n there exists a pseudo-free algebra F_n of rank n such that F_n has solvable word and applicability problems and every n-generated finitely presented algebra in $\mathcal{V}$ has a terminating Church-Rosser presentation with respect to F_n.

There are many Church-Rosser varieties of algebras. For example locally finite varieties of algebras with finitely many operations are Church-Rosser. Indeed, the union of multiplication tables of any finite algebra gives a terminating Church-Rosser presentation of this algebra with respect to the corresponding free algebra in the variety.

There are also non-locally finite Church-Rosser varieties. The variety of commutative associative algebras over a field, say, K, was one of the first examples: in 1964 Hironaka [145] and in 1965 Buchberger [61] proved, in our terminology, that every finitely generated commutative associative algebra possesses a finite terminating Church-Rosser presentation. The free algebra in this variety is the algebra of multivariate polynomials $K[X]$. Commutative algebras correspond to the ideals of $K[X]$, presentations of algebras correspond to bases of these ideals. The bases which correspond to terminating Church-Rosser presentations are called *Gröbner bases* (see more details in Section 4.10). In Section 3.5.2 and 4.10 we shall show that there are many other non-locally finite Church-Rosser varieties. In fact we do not know even the answer to the following problem.

Problem 2.6 *Is there a (finitely based) non-Church-Rosser variety of universal algebras with decidable word problem?*

There are finitely presented algebras, even semigroups, with solvable word problem which do not have a finite terminating Church-Rosser presentation with respect to the absolutely free algebra (see, for example, [380]). But these algebras generate varieties with undecidable word problem.

Church-Rosser presentations are not only used to solve the word problem. For example all calculations with polynomials (solving systems of polynomial equations, integration of rational functions, etc.) in all symbolic computation packages like Mathematica or Maple are based on using Gröbner bases. Church-Rosser presentations of finite and infinite groups are used in software packages described in [146].

Unfortunately even if an algebra has a finite terminating Church-Rosser presentation with respect to a pseudo-free algebra F, it does not necessarily come with one. Thus the problem is: given any finite rewrite system that generates a congruence γ on F, find a finite terminating Church-Rosser rewrite system which generates the same congruence.

Often it is relatively easy to find a terminating rewrite system. Indeed, usually the free algebra F has a natural order which satisfies the descending chain condition and is stable under the operations of F. If this is the case, we will call F *ordered*. For example this is true for finitely generated free semigroups and free commutative semigroups. One can order words (commutative words) first by the length and then the words of the same length by the lexicographic order. This order is called the *ShortLex* order. If F is ordered, then without loss of generality one can assume that in every set of pairs $\Sigma = \{(p_i, q_i) \mid 1 \leq i \leq n\}$ we have $q_i \leq p_i$ (indeed, if $p_i < q_i$ then we can interchange p_i and q_i in this pair, this won't change the congruence generated by Σ). Therefore when we apply any relation from Σ to an element $w \in F$, we get a smaller term.

If F is ordered, then one can try to produce a Church-Rosser presentation by using the so-called Knuth-Bendix procedure and its variations. This procedure starts with a terminating rewrite system Σ and creates iteratively new finite rewrite systems $\Sigma_1 = \Sigma, \Sigma_2, \ldots$. If the procedure halts then the last rewrite system, say, Σ_n is a finite terminating Church-Rosser rewrite system. If it never halts, it produces an infinite Church-Rosser rewrite system. Of course, this system may be non-recursive, and thus useless. The system Σ_{i+1} is obtained from Σ_i by adding a new pair (c_1, a_1), $c_1, a_1 \in F$, $c_1 > a_1$, to Σ_i if there exists an element b which has two different descendants a_1 and c_1 such that pairs from Σ_i do not apply to a_1 and c_1 (like in the Figure 1). If we cannot find such a_1 and c_1, the procedure halts.

In practice, one does not have to check all (infinitely many) elements b, only the "minimal" ones. For example, in the case of congruences on the free semigroup, it is enough to consider words b which can be written in one of the following two ways: $b = u_1x = yu_2$ or $b = u_1 = xu_2y$ for some words x and y and some relations $u_1 = v_1$ and $u_2 = v_2$ from Σ_{i-1}. If $b = u_1x = yu_2$ then we can assume, in addition, that these occurrences of u_1 and u_2 overlap. It is clear that the number of such words b is finite. The pair of words v_1x and yv_2 (resp. v_1 and xv_2y), which is obtained from b by applying relations $u_1 = v_1$ and $u_2 = v_2$, is called a *critical pair*. There is a concept of a critical pair in the case of commutative rings and in other important cases where one can apply the Knuth-Bendix algorithm. Notice, that the Knuth-Bendix algorithm in an ordered algebra F is completely determined by the following two components:

1. The definition of the applicability of a relation (pair) to an element of F;

2. The definition of a critical pair.

The concept of a critical pair and the Knuth-Bendix algorithm in the case of commutative rings was introduced in 1965 by Buchberger [61]. The finiteness of the number of critical pairs is an important prerequisite for the Knuth-Bendix procedure. Moreover the Knuth-Bendix procedure can be effectively applied only if the following *critical pair problem* is decidable in F: "Given two pairs (p_1, q_1), (p_2, q_2) in $F \times F$, list all critical pairs of these relations".

It is also interesting to study terminating Church-Rosser systems of identities. Perhaps the first terminating Church-Rosser systems of identities were investigated by T. Evans in the cases of loops and quasigroups ([95], [96]). In fact Evans used a variant of the algorithm which later was called the Knuth-Bendix procedure. Evans used these Church-Rosser presentations in order to get the first solutions of the equational problem for loops and quasigroups. The original article by Knuth and Bendix [198] was devoted to identities rather than relations. Some infinite terminating Church-Rosser bases of identities of some important semigroup varieties were found in [281], [79]. But as far as we know, nobody has conducted a comprehensive study of such systems in the cases of semigroups, groups, Lie and associative algebras.

2.10 Basic Definitions

Classes of algebras

A *variety* is a class of algebras closed under taking subalgebras, direct products and homomorphic images. Equivalently a variety is a class of algebras given by a set of identities.

A *pseudovariety* is a class of algebras closed under taking subalgebras, finite direct products and homomorphic images.

The *finite trace* $\mathcal{V}_{\text{fin}}$ of a variety $\mathcal{V}$ is the set of all finite algebras in $\mathcal{V}$. Every finite trace is a pseudovariety.

Operations on Varieties and Pseudovarieties

For every class $\mathcal{C}$ of algebras the (pseudo-)variety generated by C, written (p)var$\mathcal{C}$ is the class of homomorphic images of subalgebras of (finite) direct products of algebras of $\mathcal{C}$.

A *join* of two (pseudo-) varieties $\mathcal{A}$ and $\mathcal{B}$, written $\mathcal{A}+\mathcal{B}$ is the (pseudo-) variety generated by direct products $A \times B$ where $A \in \mathcal{A}$, $B \in \mathcal{B}$.

A *Mal'cev product* $\mathcal{AB}$ of two (pseudo-) varieties $\mathcal{A}$ and $\mathcal{B}$ is the (pseudo-) variety generated by all algebras C which have congruences σ such that the factor-algebras C/σ belong to $\mathcal{B}$, and all congruence classes of σ which are subalgebras belong to $\mathcal{A}$.

Algebras

An algebra is called *finitely presented in a variety* $\mathcal{V}$ if it can be given by a finite number of generators, a finite number of defining relations plus all of the identities of the variety $\mathcal{V}$.

Class $FP(\mathcal{V})$ is the class of all algebras finitely presented in the variety $\mathcal{V}$.

Class $FP(\mathcal{U}) \cap \mathcal{V} = \mathcal{FP} \cap \mathcal{V}$ is the class of all algebras finitely presented in some bigger variety $\mathcal{U}$ which belong to $\mathcal{V}$.

An algebra A is called *residually finite* if for any two distinct elements $u, v \in A$ there exists a homomorphism $\phi : A \to F$ such that $\phi(u) \neq \phi(v)$ and F is finite. For groups this is equivalent to the property that the intersection of all normal subgroups of finite index is trivial. In the cases of Lie (associative) algebras over infinite fields the role of finite algebras is played by finite dimensional algebras. An algebra is called residually finite if the intersection of all its ideals of finite co-dimension is finite.

An algebra is called *Hopfian* if it is not isomorphic to any of its proper homomorphic images. Every finitely generated residually finite algebra is Hopfian.

An algebra A is called *locally finite* (*locally residually finite*, *locally Hopfian*, etc.) if every finitely generated subalgebra of it is finite (residually finite, Hopfian, etc.).

Theories

The *elementary theory* of a class $\mathcal{C}$ is the set $E\mathcal{C}$ of all first order formulas of the corresponding similarity type which hold in all algebras of this class.

The *universal theory* of a class $\mathcal{C}$ is the set $A\mathcal{C}$ of all universal formulas (i.e. first order sentences without existential quantifiers) from $E\mathcal{C}$.

The *Q-theory* of the class $\mathcal{C}$ is the set $Q\mathcal{C}$ of all quasi-identities, that is formulas of the type

$$(\forall x_1, \ldots, \forall x_n)(s_1 = t_1 \& \ldots \& s_k = t_k \to s = t)$$

where s_i, t_i, s, t are terms of the type of $\mathcal{C}$) from $A\mathcal{C}$.

The *equational theory* of a class $\mathcal{C}$ is the set $I\mathcal{C}$ of all identities of $\mathcal{C}$.

The *identity theory* of a class $\mathcal{C}$ is the set of all formulas from $E\mathcal{C}$ of the type

$$\&\Sigma \to u = v,$$

where Σ is a (finite) set of identities and $u = v$ is an identity.

Algorithmic problems

The *word problem* for an effectively given algebra A asks if there exists an algorithm deciding for any two terms in the alphabet of generators of A if they are equal in A.

The *isomorphism problem* in the class $\mathcal{V}$ asks if there exists an algorithm deciding for any two algebras in $FP(\mathcal{V})$ if they are isomorphic.

The *finiteness problem* asks for an algorithm to decide if an algebra in $FP(\mathcal{V})$ is finite.

The *triviality problem* asks for an algorithm to decide if an algebra in $FP(\mathcal{V})$ is trivial (that is 1-element).

An algorithmic problem is said to be *solvable* in a variety $\mathcal{V}$ if it is solvable for all algebras in $FP(\mathcal{V})$.

An algorithmic problem is said to be *weakly solvable* in a variety $\mathcal{V}$ if it is solvable for all algebras in $FP \cap \mathcal{V}$.

The *elementary* (*universal, Q-, equational, identity*) *problem* asks for the existence of an algorithm to decide, given a first order sentence, whether it belongs to the corresponding theory. The identity problem for varieties is also called the *Tarski-Mal'cev problem.* The identity problem for finite traces of varieties is also called the *Rhodes problem.* The universal problem is also called the *uniform word problem.*

Neighbors of algorithmic problems

A variety is said to have the *Higman property* (to be a Higman variety) if every recursively presented algebra in this variety is embeddable into a finitely presented algebra from the same variety. An algebra is called recursively presented if it can be given by a finite set of generators and a recursively enumerable set of defining relations.

The property $FG = FP$ means that every finitely generated algebra from a variety is finitely presented there.

A variety is called *finitely based* if it can be defined by a finite number of identities.

A variety is called *hereditary finitely based* if all its subvarieties are finitely based.

If α is a property of varieties, then a variety $\mathcal{V}$ is called a *minimal non-α* variety if it does not have property α but all its subvarieties have this property.

A variety $\mathcal{V}$ is called an *indicator with respect to a property* α if for all varieties $\mathcal{U}$, $\mathcal{U}$ satisfies α iff the intersection $\mathcal{V} \cap \mathcal{U}$ satisfies α.

3 Semigroups

3.1 Basic Definitions

For basic definitions of semigroup theory we refer the reader to Clifford and Preston [73]. Surveys Shevrin and Volkov [369] and Shevrin and Sukhanov [368] provide an excellent introduction to the theory of varieties of semigroups.

We shall need the following basic definitions.

A *monoid* is a semigroup with a unit.

A semigroup where every element is an *idempotent*, that is $x^2 = x$ holds identically, is called a *band*.

A commutative band is called a *semilattice*.

A semigroup with identity $xy = x$ (resp., $xy = y$, $xyx = x$, $xy = zt$) is called a *left zero semigroup* (resp., *right zero semigroup*, *rectangular band*, *zero semigroup*).

A semigroup S is called a *band of semigroups* S_α, $\alpha \in A$ if S is a disjoint union of S_α, and the corresponding partition is a congruence. The factor of S over this congruence is a band. If this band is commutative (rectangular), then S is called a *semilattice* (resp., a *rectangular band*) of semigroups S_α.

An subset I of a semigroup S is called an *ideal* if I is stable under multiplication by elements of S from the left and from the right.

The *free semigroup* over a set X is the set of all words in X with the operation of concatenation. This semigroup is denoted by X^+.

A *semigroup identity* is a formal equality $u = v$ where u and v are words. An identity $u = v$ holds in a semigroup S if this equality holds for every substitution of elements of S for letters of u and v.

A semigroup S is *periodic* if all its one-generated subsemigroups are finite, equivalently if for every element $x \in S$ there exist two different numbers m_x and n_x such that $x^{m_x} = x^{n_x}$.

A *nil-semigroup* is a semigroup such that a power of every element is equal to zero.

A *nilpotent semigroup of degree* n is a semigroup where any product of n elements is zero. Every finite nil-semigroup is nilpotent (see Clifford and Preston [73]).

3.2 Overview

The case of semigroups is the luckiest among other cases considered in this survey. We have an almost complete map of varieties with solvable word problem, locally residually finite varieties, etc. Basically there is only one big white spot — varieties of periodic groups. And these varieties hardly belong to the theory of semigroups at all.[6] Varieties of periodic groups form the intersection of the universe of semigroup varieties and the universe of group varieties. It is very hard to prove that a certain variety of periodic groups is not locally finite (see Novikov and Adian [286], Adian [4], Ol'shanskii [288]). It is clearly even harder to study algorithmic problems there.

One of the main features discovered in the process of studying semigroup varieties is that many different properties of semigroup varieties are equivalent or almost equivalent. There are very few different equivalence classes. We will call them Clubs, because each class is not a formal collection of equivalent properties, but rather an informal association of them.

Clubs of the properties that we consider in this survey form a partially ordered set: A Club with weaker properties is higher than a Club with stronger properties. An interesting thing is that this partially ordered set is a chain.

The highest Club is that of Burnside-type properties. The solvability of the Tarski-Mal'cev problem and the solvability of the Rhodes problem are members of this Club as well.

The next Club is that of the decidable word problem. The decidability of the uniform word problem, the weak decidability of the word problem and many others belong to this Club, as well as the residual finiteness of finitely presented semigroups. The isomorphism problem seems to belong to this Club also, but we are not quite sure about it.

Just below is the Club of local residual finiteness. The property "To be locally representable by matrices", and many other properties belong to this Club.

The lowest Club is that of solvability of the elementary theory. Residual finiteness and other properties also belong to this Club.

We will describe these Clubs from the top to the bottom.

3.3 The Club of Burnside Problems

Permanent members: The Burnside problems (the analogs of the three well known Burnside problems for groups), the Tarski-Mal'cev problem (the identity problem for varieties), the Rhodes problem (the identity problem for finite traces).

Associated members: The equational problem, the finite basis problem.

[6] We do not want to abandon these difficult varieties, though. We are patiently trying to include them in the happy family of other varieties of semigroups (see Section 7.2.9 below).

Undecided membership: The Higman property.

3.3.1 The Burnside Problems

The description of varieties where periodic semigroups are locally finite (see Theorems 3.7 and 3.8 below) plays an exceptional role in the study of algorithmic problems in semigroup varieties. Most of the results about algorithmic problems in varieties would be impossible to obtain without it.

The first result about Burnside-type problems in semigroup varieties was published by Morse and Hedlund [274]. The result was the following:

Theorem 3.1 *(Morse and Hedlund, [274]). There exist an infinite 3-generated semigroup that satisfies the identity*[7] $x^2 = 0$ *and an infinite 2-generated semigroup that satisfies the identity* $x^3 = 0$.

Morse and Hedlund used certain infinite words W_1 and W_2 over a 2-letter alphabet and a 3-letter alphabet, which *avoid* words x^3 and x^2 respectively. In general if u is a word and ϕ is an endomorphism of a free semigroup then $\phi(u)$ is called a *value* of u. A word u is called *avoidable* by a word W if W does not contain values of u. A word u is called *avoidable* if it is avoided by an infinite word over a finite alphabet.

We present the Thue construction of the word W_1 in Section 7.2.5.

Infinite words W_1 and W_2 had been around long before Morse and Hedlund. It is a common opinion that the first paper where these words were constructed was Thue [398] (see also Arshon [13]). But recently George McNulty informed us that this is not correct at least as far as the word W_1 is concerned. This infinite word was implicitly constructed in 1851 by M.E.Prouhet [307]. He considered a partition of natural numbers into two sets that satisfies a nice number theoretic property. If we list all natural numbers, and then replace the numbers from the first set of the partition by a, and numbers of the second set by b, then we will get the word W_1. We won't be surprised if it eventually turns out that some of the missing books by Diophantus and Pythagoras contain these infinite words. And who knows, maybe it is worth studying again the paintings of prehistoric peoples and the notation of ritual dances of some tribes.

There is a natural correspondence between infinite words over a finite alphabet and finitely generated semigroups (see Morse and Hedlung [274], comments after Theorem 3.12 below, and Section 7.2.5). This correspondence implies the following connection between the avoidability of words and the Burnside-type properties.

Theorem 3.2 *(Bean, Ehrenfeucht, McNulty [29]). A word u is avoidable if and only if the variety given by the identity $u = 0$ is not locally finite.*

[7] Here and later we will use the short expression $u = 0$ for the pair of identities $ux = u, xu = u$ where x does not occur in the word u.

In 1952, Green and Rees considered identities of the form $x = x^n$ [115].

Theorem 3.3 *(Green, Rees, [115]). A semigroup satisfying an identity $x = x^n$ is locally finite if and only if all its subgroups are locally finite.*

In particular, a variety given by an identity $x = x^n$ is locally finite if $n = 2, 3, 4, 5, 7$, because every group satisfying this identity is locally finite (see Ol'shanskii [288]).

A semigroup satisfying such an identity is a union of groups, because every one-generated subsemigroup is a group. Semigroups which are unions of groups are called *completely regular*. Completely regular semigroups were described by Clifford in 1941 [72]. He proved that a completely regular semigroup has the following very special structure.

Theorem 3.4 *(Clifford, [72]). Every completely regular semigroup is a semilattice of rectangular bands of groups.*

Now Theorem 3.3 follows from Theorem 3.4, the trivial fact that semilattices and rectangular bands are locally finite, and the following important result of Brown [58], [57].

Theorem 3.5 *(Brown, [58]). If a semigroup S has a homomorphism onto a locally finite semigroup T and every pre-image of an idempotent is a locally finite semigroup then S is locally finite.*

A particular case of this result where T is an idempotent semigroup was obtained earlier by L.N.Shevrin [367]. Thus Theorem 3.3 also follows from Clifford's theorem and Shevrin's result.

The next step was made by Bean, Ehrenfeucht, and McNulty [29], and independently by Zimin [431] (an announcement of Zimin's result was published in 1978 [433]). They found algorithms for checking if a word is avoidable. In order to formulate their results we need some more definitions.

Let u be a word over an alphabet X. Let $Y, Z \subseteq X$. We will say that Y and Z form a *fusion* in u if for every 2-letter subword yz of u we have:

$$y \in Y \text{ if and only if } z \in Z.$$

If Y, Z form a fusion in u then any subset of $Y \backslash Z$ is called a *free set* in u. Given a set of letters A, we can delete all letters belonging to A from a word u. This operation is called a *deletion*. A deletion of a free subset in u is called a *free deletion*.

We also need the definition of the words Z_n (Zimin words in our terminology):

$$Z_1 = x_1, \ldots, Z_{n+1} = Z_n x_{n+1} Z_n.$$

Zimin words have also been around for quite some time. Zimin was the first to discover their crucial role for the Burnside-type problems in varieties (see [431] and

Theorem 3.6 below). But the same words appear as an example in the paper [29]. In 1966, thirteen years earlier, these words appeared in a paper by Coudrain and Schützenberger [75] as $\tilde{k}_n$. Coudrain and Schützenberger proved that every infinite word over a finite alphabet contains a value of Z_n. The same result is implicitly contained in the classic book by Jacobson [157] published ten years earlier, in 1956. In [157] values of Z_n are called m-sequences. We will return to the discussion of the antiquity of the Zimin words at the end of this subsection.

The following theorem is a translation of results from Bean, Ehrenfeucht, McNulty [29] and Zimin [431] into the language of varieties.

Theorem 3.6 *(Bean, Ehrenfeucht, McNulty [29], Zimin, [431]). Let $\mathcal{V}$ be a variety of semigroups given by a (possibly infinite) set of identities $\{u = 0 \mid u \in \Sigma\}$. Assume that the number of variables occurring in words of Σ is n. Then the following conditions are equivalent.*

1. *$\mathcal{V}$ is locally finite.*
2. *Z_n contains a value of some word u in Σ.*
3. *$\mathcal{V}$ satisfies the identity $Z_n = 0$.*
4. *There exists a word $u \in \Sigma$ which can be reduced to the empty word by a sequence of free deletions.*

This theorem gives an algorithmic description of locally finite varieties defined by identities of the form $u = 0$. Notice that all these varieties consist of nil-semigroups.

The next result by the second author of this survey [346] gives an algorithmic description of arbitrary varieties where nil-semigroups are locally finite. Recall that if F is the free semigroup over a set X of generators then a *substitution* is a map from X into F. Every substitution is extendable to an endomorphism of F.

Theorem 3.7 *(Sapir, [346]). Let $\mathcal{V}$ be a variety of semigroups given by a (possibly infinite) set of identities Σ. Assume that the number of variables occurring in words of Σ is n. Then the following conditions are equivalent.*

1. *All nil-semigroups from $\mathcal{V}$ are locally finite.*
2. *All semigroups from $\mathcal{V}$ with the identity $x^2 = 0$ are locally finite.*
3. *There exists an identity $u = v \in \Sigma$ such that Z_{n+1} contains a value $\phi(u)$ (resp. $\phi(v)$) for some substitution ϕ where $\phi(u) \neq \phi(v)$.*[8]

[8] In [346], Z_n was incorrectly used instead of Z_{n+1}. The cause of this error was a mistake in the proof of Lemma 4.8 of [346]. R.McKenzie and G.McNulty notified the author about this mistake. It was corrected in [347].

4. *$\mathcal{V}$ satisfies a non-trivial identity with one side equal Z_{n+1}.*

5. *There exists an identity $u = v \in \Sigma$ and a sequence of words $u_1, \ldots, u_k, v_1, \ldots, v_k$ such that*

   (a) *$u = u_1$, $v = v_1$;*

   (b) *u_i is obtained from u_{i-1} by a free deletion σ_i;*

   (c) *v_i is obtained from v_{i-1} by the same deletion σ_i;*

   (d) *u_k is an empty word;*

   (e) *For some i with $1 \leq i \leq k$ there exists a fusion in u_i (resp. v_i) which is not a fusion in v_i (resp. u_i).*

Condition 5 of Theorem 3.7 did not appear in [346] but can be deduced from the proof of Theorem N of [346]. This condition is easier to verify than the other conditions of Theorem 3.7. We still don't know the computational complexity of any of these conditions though. The second author can show (unpublished) that there is a polynomial time algorithm verifying whether a word is avoidable (unavoidable). Thus there is a polynomial time algorithm verifying that a variety given by finitely many identities of the type $u = 0$ is locally finite.

The next theorem, also from Sapir [346], describes varieties where all periodic semigroups are locally finite in the class of varieties with "good" groups and in the class of non-periodic varieties[9].

Theorem 3.8 *(Sapir, [346]). Let $\mathcal{V}$ be a variety of semigroups given by a (possibly infinite) set of identities Σ. Assume that the number of variables occurring in words of Σ is n. Assume also that the variety $\mathcal{V}$ either is non-periodic or contains no non-locally finite groups of finite exponent. Then the following conditions are equivalent.*

1. *All periodic semigroups from $\mathcal{V}$ are locally finite.*

2. *All nil-semigroups from $\mathcal{V}$ are locally finite.*

3. *There exists an identity $u = v \in \Sigma$ such that Z_{n+1} contains a value $\phi(u)$ (resp. $\phi(v)$) for some substitution ϕ but $\phi(u) \neq \phi(v)$.*

The first announcement of Theorems 3.7 and 3.8 was published in Sapir [334].

These two theorems have many interesting corollaries and applications (see [346], [348] and other papers by the second author). Let us present just four of them immediately. Others will be discussed later.

[9] Recall that a variety of semigroups is called *non-periodic* if it contains a non-periodic semigroup (equivalently, if it contains the additive semigroup of natural numbers).

Theorem 3.9 *(Sapir, [346]). A finitely based periodic semigroup variety is locally finite if and only if all its groups and all its nil-semigroups are locally finite.*

Theorem 3.10 *(Sapir, [346]). If all subvarieties of a semigroup variety $\mathcal{V}$ are finitely based then all nil-semigroups in $\mathcal{V}$ are locally finite.*

Theorem 3.11 *(Sapir, [346]). A locally finite variety of semigroups is not finitely based provided it contains the following Brandt monoid B_2^1 of 2×2-matrices:*

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}. \qquad (2)$$

A finite algebra which cannot belong to a locally finite finitely based variety is called *inherently non-finitely based* [293]. Theorem 3.11 actually states that B_2^1 is an inherently non-finitely based semigroup. See Section 3.7 for an algorithmic description of finite inherently non-finitely based semigroups. Theorem 3.11 answered a question by G.McNulty and C.Shallon [262]. The next application answered a question by S.Eilenberg and M.P.Schützenberger [90]. It has been proved in [348].

Theorem 3.12 *(Sapir, [348]). If a finite semigroup has a finite basis of identities in the class of finite semigroups then it has a finite basis of identities in the class of all semigroups.*

We do not want to discuss the proofs of Theorems 3.7 and 3.8 in all their details. But we should mention one important detail. As far as we know the paper [346] was the first place where the following connection between symbolic dynamical systems and Burnside-type problems was explored. A *symbolic dynamical system* is a closed subset of the Tikhonov product $X^{\mathbf{Z}}$, where X is a finite set with the discrete topology, which is stable under the shift homeomorphism T (this homeomorphism shifts every sequence from $X^{\mathbf{Z}}$ one position to the right).

There exists the following correspondence between semigroups and symbolic dynamics (see [346], [340], [339], [246]).

Let $S =< X >$ be an infinite finitely generated semigroup (the same argument may be applied for any universal algebra). Then there is an infinite set T of words over X such that every element of S represented by a word of T cannot be represented by words over X of lesser length. Such words will be called *geodesic words*: they label geodesics on the Cayley graph of the semigroup. It is clear that every subword of a geodesic word is also geodesic. Now, in every word of T, mark a letter which is closest to the center of this word. There is an infinite subset $T_1 \subseteq T$ of words which have the same marked letters, an infinite subset $T_2 \subseteq T_1$ of words which have the same subwords of length 3 containing the marked letters in the middle, ..., an infinite subset $T_n \subseteq T_{n-1}$ of words which have the same subwords of length $2n-1$

with the marked letters in the middle, and so on. Therefore there is an infinite (in both directions) word W such that every subword of W is a subword of a word from T. Thus every subword of W is a geodesic word. Infinite words with this property will be called *infinite geodesic words*. The set $D(S)$ of all infinite geodesic words is a symbolic dynamical system because it is stable under the shift (obviously) and is closed in the Tikhonov topology (this can be easily proved). Conversely, with every symbolic dynamical system D one can associate a semigroup $S(D)$ as follows: $S(D)$ consists of all finite subwords of infinite words from D, and 0. If u and v belong to $S(D)$ then $u \cdot v$ is equal to uv if uv belongs to $S(D)$, or 0 otherwise. It is easy to show that $S(D)$ is a semigroup. It is interesting that for every symbolic dynamical system D we have that $D(S(D)) = D$.

This correspondence allows one to show that some important properties of the theory of semigroups and that of the theory of symbolic dynamical systems are in fact closely connected. For example:

- *The semigroup S is infinite if and only if $D(S)$ is not empty;*
- *If a semigroup S is periodic then the symbolic dynamical system $D(S)$ does not have cyclic trajectories.*

One of the important concepts of the theory of symbolic dynamical systems is the concept of uniformly recurrent word.

An infinite word U is called *uniformly recurrent* if for every finite subword u of U there exists a number $N_U(u)$ such that every subword of U of length $N_U(u)$ contains u as a subword. It is an easy corollary from [103] (see [346] for details) that for every infinite word U there exists a uniformly recurrent word U' such that every subword of U' is a subword of U. It is easy to see that if U belongs to $D(S)$ then U' also belongs to $D(S)$.

Therefore for every infinite finitely generated semigroup $S =< X >$ there exists a uniformly recurrent geodesic word over X (see [346]).

Uniformly recurrent words are much more convenient than arbitrary infinite words. For example, in the proof of Theorem 3.7, one has to show that every finitely generated nil-semigroup S satisfying a non-trivial identity $Z_{n+1} = W$ is finite. In the language of symbolic dynamical systems this means that the symbolic dynamical system $D(S)$ is empty. Suppose to the contrary, that it is not empty. Then it must contain a uniformly recurrent word U. For the sake of simplicity assume that W contains x_1^2. Then a simple argument shows that for arbitrary letter a occurring in U there exists a finite subword u in U such that $u = pa^2q$ (Mod $Z_{n+1} = W$) for some words p and q. Since U is uniformly recurrent it may be represented in the form $\ldots uv_1uv_2u\ldots$ where the lengths of words v_i are bounded by $N_U(u)$. Then, applying our identity $Z_{n+1} = W$ we can transform this word into $\ldots pa^2qv_1pa^2qv_2pa^2q\ldots$. Now introduce a new finite alphabet $x_0, x_1, \ldots$ and replace in U, a^2 by x_0, qv_1p by x_1, qv_2p by x_2, and so on. The word U_1 may not be uniformly recurrent. Let us consider the symbolic dynamical system generated by U_1, i.e. the closure of the set $\{T^n(U_1) \mid n \in \mathbf{Z}\}$. This symbolic dynamical system contains a uniformly recurrent word U_1'. It is clear that all finite subwords of U_1' are subwords of U_1. Then we can find a finite subword u_1 in U_1' which is equal to $p_1 x_0^2 q_1$ modulo the identity $Z_{n+1} = W$. Replacing letters x_i by the words labeled by these letters, we will get a subword u_2 in U which is equal to $p_3 a^4 q_3$ modulo $Z_{n+1} = W$. Continuing this process we will generate bigger and bigger powers of the letter a. This leads us to a contradiction because S is a nil-semigroup.

There are many other applications of this technique. In particular, using it, one can very easily establish the theorem of Brown (Theorem 3.5 above). The same idea has been used in [76] to prove other finiteness conditions for semigroups. In Sapir [339] and Margolis and Sapir [246], it has been used to prove some properties of varieties generated by finite inverse semigroups, and quasi-varieties generated by finite semigroups. See also Ufnarovsky [404] where uniformly recurrent words are used in the Lie algebra situation in order to prove the so called "sandwich Lemma", the key lemma in Kostrikin's solution of the restricted Burnside problems for groups of prime exponents.

Applications of uniformly recurrent words are very effective, but they are not constructive. Indeed, there is no algorithm to find the number $N_U(u)$.

The proofs of Theorems 3.7 and 3.8 have been made constructive in Sapir [347] where the analog of the restricted Burnside problem for semigroup varieties is discussed. Recall that this problem asks if there are only finitely many finite semigroups with given number of generators in a given variety $\mathcal{V}$. Zelmanov's solution of the original restricted Burnside problem (modulo the Classification of Finite Simple Groups) [428], [427], and results from [346] have been used to find the following complete algorithmic description of semigroup varieties where the analog of the restricted Burnside problem has a positive solution.

Theorem 3.13 *(Sapir, [347]). For an arbitrary finitely based variety $\mathcal{V}$ of semigroups the following conditions are equivalent.*

1. *There are only finitely many finite semigroups in $\mathcal{V}$ with any given number of generators.*

2. *There is a recursive function $f(n)$ such that the order of every n-generated semigroup in $\mathcal{V}$ does not exceed $f(n)$.*

3. *Locally finite semigroups in $\mathcal{V}$ form a variety.*

4. *$\mathcal{V}$ is a periodic variety and all nil-semigroups in $\mathcal{V}$ are locally finite.*

5. *$\mathcal{V}$ is periodic and satisfies a non-trivial identity of the form $Z_n = W$.*

This theorem plays a crucial role in the study of algorithmic properties of finite semigroups in varieties.

Theorems 3.8, 3.9, 3.10, 3.13 show that properties of varieties depend very much on the properties of the nil-semigroups in these varieties. We will meet this phenomenon many times later.

Finishing our discussion of Burnside properties in semigroup varieties we would like to mention a stunning similarity between Burnside properties of semigroup and group varieties, which has been recently found by E.Zelmanov [426].

Let us define the Zelmanov word $\mathcal{Z}_n$ by the following rule:

$$\mathcal{Z}_1 = x_1, \ldots, \mathcal{Z}_{n+1} = (\mathcal{Z}_n, x_{n+1}, \mathcal{Z}_n)$$

where round brackets denote the group commutator: $(x, y) = x^{-1}y^{-1}xy$, $(x, y, z) = ((x, y), z)$. One can easily see that the Zelmanov word is precisely the Zimin word where the multiplication is replaced by the group commutator. The following theorem is proved in [426]. It solves a long-standing problem by B.H.Neumann [279] in the case of prime exponents.

Theorem 3.14 *(Zelmanov, [426]). For every prime number p there exists a natural number n such that a group of exponent p is locally finite if and only if it satisfies the identity $\mathcal{Z}_n = 1$.*

Now compare this theorem and Theorem 3.6 (see especially Condition 3 of Theorem 3.6). We are sure that this is not just a coincidence, and if indeed there was a word at the Beginning of the universe it was a Zimin word. We don't think this was a Zelmanov word because commutators are difficult to pronounce.

3.3.2 The Identity Problem and Related Problems

The following result is well known.

Theorem 3.15 *Every non-periodic variety of semigroups has a decidable equational problem.*

Indeed, every identity of a non-periodic semigroup variety is *homogeneous*, that is every letter occurs the same number of times in each side of this identity. Now let F be a relatively free semigroup in a non-periodic variety. Let I_n be the ideal generated by all words of length n in generators. Then the intersection of all these ideals is empty and each of them has finite index (F/I_n is finite). Therefore F is residually finite, so it has a solvable word problem. It remains to use Connection 2.3.

Notice that a similar argument will be used later (see Section 4.3), in the case of associative algebras over an infinite field. Free algebras in these varieties have a similar structure. This, by the way, justifies the general belief that non-periodic semigroup varieties are semigroup analogs of varieties of associative algebras over an infinite field.

The identity (Tarski-Mal'cev) problem is decidable in any variety all of whose subvarieties have decidable word problem. Therefore it is decidable in any variety covered by Theorems 3.28 and 3.29 below.

The undecidability of the Tarski-Mal'cev problem in the class of all semigroups was shown by Murskii in [276].

The following theorem, which is yet another application of Theorem 3.7, contains an algorithmic description of non-periodic semigroup varieties and varieties with "good" groups where the Tarski-Mal'cev problem is decidable. This theorem was proved by the second author of this survey in [340].

Theorem 3.16 *(Sapir, [340]). Let $\mathcal{V}$ be a finitely based semigroup variety which is either a non-periodic variety, or a periodic variety with all groups locally finite. Then the following conditions are equivalent.*

1. *The identity problem for variety $\mathcal{V}$ is decidable.*
2. *All nil-semigroups in $\mathcal{V}$ are locally finite.*

The Rhodes problem for finite traces of varieties is decidable for any variety where the analog of the restricted Burnside problem has a positive and constructive solution (see the Introduction). Therefore it is decidable in any variety which is covered by Theorem 3.13. The following theorem shows that there are no other varieties where the Rhodes problem is decidable. This theorem was also proved by the second author of this survey in [340]. The undecidability of the Rhodes problem in the class of all finite semigroups was shown by Albert, Baldinger and Rhodes in [9].

Theorem 3.17 *(Sapir, [340]). For an arbitrary finitely based semigroup variety $\mathcal{V}$ the following conditions are equivalent.*

1. *The Rhodes problem in $\mathcal{V}$ is decidable.*
2. *All nil-semigroups in $\mathcal{V}$ are locally finite.*
3. *There is a recursive function $f(k,m,n)$ such that the order of every n-generated semigroup from $\mathcal{V}$ satisfying the identity $x^m = x^{m+n}$ does not exceed $f(k,m,n)$.*

The following statement gives a connection between the Rhodes problem and the membership problem for pseudovarieties. It may be found in [340]. A particular case was obtained by Albert, Baldinger and Rhodes (see Theorem 11 in [9]).

Theorem 3.18 *If the Rhodes problem is undecidable in a finitely based variety of semigroups $\mathcal{V}$ then there exists a finitely based subvariety $\mathcal{E} \subseteq \mathcal{V}$ such that the join of $\mathcal{E}_{fin}$ and the pseudovariety of finite commutative semigroups has an undecidable membership problem.*

Recall that the join of two pseudovarieties $\mathcal{U}$ and $\mathcal{V}$ consists of homomorphic images of subalgebras of direct products $U \times V$ where $U \in \mathcal{U}$, $V \in \mathcal{V}$. Thus Theorem 3.18 is interesting because the membership problem is easily decidable in the finite trace of any finitely based variety, and the join of two varieties seems easy to construct.

3.3.3 The Equational Problem in Completely Regular Varieties

As we have mentioned in the Introduction, the equational problem in a variety is equivalent to the word problem in free algebras of this variety. A solution of the equational problem is often the first step in any investigation of a variety, regardless the question we are actually interested in. Thus there are lots of results solving the equational problem in particular varieties. We will mention only series of results leaving sporadic results for other surveys.

There are many important results concerning the equational problem in varieties of completely regular semigroups, that is semigroups which are unions of subgroups. The interest in these varieties is inspired by the role which completely regular semigroups play in the theory of semigroups. We have already mentioned that properties of varieties depend very much on properties of their nil-semigroups. If all nil-semigroups in a variety $\mathcal{V}$ are nilpotent of degree n then we call $\mathcal{V}$ a variety of *index* n. See the paper by Sapir and Sukhanov [355] where the hierarchy of varieties of finite index is described. Many properties of semigroup varieties imply the property "to be of finite index" (see Shevrin and Sukhanov [368]). Varieties of completely regular semigroups are exactly the varieties where all nil-semigroups are trivial, that is they are exactly the varieties of index 1.

Every completely regular semigroup has a partition into subgroups [73]. Therefore there is a unary operation $^{-1}$ in every completely regular semigroup. It takes every element to the inverse of this element in the corresponding subgroup. In particular, if a completely regular semigroup satisfies the identity $x = x^n$, $n > 2$, then $x^{-1} = x^{n-2}$. It is much more natural to consider completely regular semigroups as algebras with two operations than as semigroups, i.e. as algebras with one binary operation. If we add this unary operation to the signature of completely regular semigroups, then the class $\mathcal{CR}$ of all completely regular semigroups becomes a variety given by the following identities: $(x^{-1})^{-1} = x, xx^{-1}x = x, xx^{-1} = x^{-1}x$. The variety of all groups is a subvariety of $\mathcal{CR}$. The intersection of the set of subvarieties of $\mathcal{CR}$ and the set of varieties of semigroups is the set of periodic varieties of completely regular semigroups, i.e. the class of varieties satisfying identities of the form $x = x^n$ (varieties given by such identities have been discussed above: see Theorem 3.3). If we fix a variety of groups $\mathcal{V}$ then the class of all completely regular semigroups with subgroups from $\mathcal{V}$ is also a variety, denoted by $\mathcal{CR}(\mathcal{V})$. By a theorem of Clifford (Theorem 3.4 above) this variety is equal to the following Mal'cev product: $(\mathcal{V}(\mathcal{L} \times \mathcal{R}))\mathcal{I}$ where $\mathcal{L}$ (resp. $\mathcal{R}$, $\mathcal{I}$) is the variety of left zero semigroups (resp. right zero semigroups, commutative idempotent semigroups), and $\mathcal{L} \times \mathcal{R}$ is the class of all direct products of left zero semigroups and right zero semigroups.

In particular, the variety given by the identity $x = x^n$ is equal to $\mathcal{CR}(\mathcal{B}_{n-1})$, where $\mathcal{B}_{n-1}$ is the variety of groups of exponent $n - 1$. Other important subvarieties of $\mathcal{CR}$ are the following:

- $\mathcal{OCR}(\mathcal{V})$, the variety of orthodox completely regular semigroups with subgroups from $\mathcal{V}$. A completely regular semigroup is called *orthodox* if its idempotents form a subsemigroup.

- $\mathcal{BCR}(\mathcal{V})$, the variety of bands of groups from $\mathcal{V}$. This variety is the Mal'cev product of the variety $\mathcal{V}$ and the variety of all idempotent semigroups.

- $\mathcal{CS}(\mathcal{V})$, the variety of all completely simple semigroups with subgroups from $\mathcal{V}$. A completely regular semigroup S is called *completely simple* if it does not contain ideals distinct from S, equivalently, a completely simple semigroup is a rectangular band of groups (see the Clifford description of completely regular semigroups, Theorem 3.4). This variety is the Mal'cev product of the variety $\mathcal{V}$ and the variety $\mathcal{L} \times \mathcal{R}$).

The equational problem in the variety $\mathcal{CR}$ has been solved independently by Gerhard [105], [106] and Trotter [401], while the word problem in the relatively free semigroup with 2 generators from $\mathcal{CR}$ has been solved by Clifford in [71].

Algorithms, found by Gerhard and Trotter, are very complex. Later Kaďourek and Polák [169] found an easier algorithm.

Then Pastijn and Trotter [291] generalized the Gerhard-Trotter result and the result of Kaďourek and Polák, cited above. They proved that if the variety of groups $\mathcal{V}$ has solvable equational problem then the equational problem is solvable in $\mathcal{CR}(\mathcal{V})$. Their algorithm is a generalization of the algorithm from [169]. They also proved that if relatively free groups in $\mathcal{V}$ are residually finite then the relatively free semigroups in $\mathcal{CR}(\mathcal{V})$ are residually finite also. An important particular case, when $\mathcal{V}$ is a Burnside variety of groups $\mathcal{B}_n$, has been considered in Kaďourec and Polák [168]. The paper [168] was published two years later than Pastijn and Trotter [291], although it was written, and submitted to a journal as early as 1986. The second author of this survey personally saw the manuscript in 1988 when he was visiting Polák in Czechoslovakia.

We present here the Pastijn-Trotter-Kaďourek-Polák theorem which contains an algorithm solving the equational problem in $\mathcal{CR}(\mathcal{V})$, to show the non-trivial combinatorial objects which arise in the study of completely regular semigroups. In order to formulate this theorem we need some definitions.

Let $\mathcal{U}$ be the variety of all unary semigroups, that is semigroups equipped with a unary operation (which we will denote by $^{-1}$). The variety $\mathcal{CR}$ is a subvariety of $\mathcal{U}$. Let X be a set and let $F\mathcal{U}_X$ be the free unary semigroup over X. Semigroup $F\mathcal{U}_X$ consists of words in the alphabet $X \cup \{(,),^{-1}\}$ where symbols (,), $^{-1}$ must obey the natural rules. For example, $ab(a(ab)^{-1}cd)^{-1}$ is a correct word, but $ab(cd(^{-1}))$ is not correct. A *segment* of a word $u \in F\mathcal{U}_X$ is a subword of u thought of as a word in this extended alphabet. Notice that a segment of a word from $F\mathcal{U}_X$ may not belong to $F\mathcal{U}_X$. For example, $ab($ is a segment of $ab(cd)^{-1}$.

Let u be a word from $F\mathcal{U}_X$. By $c(u)$ we denote the set of variables occurring in u.

If d is a segment of a word $u \in F\mathcal{U}_X$, and $c(d) \geq 1$ then by $\hat{d}$ we denote the word which is obtained from d by deleting unmatched parentheses.

Let $u \in F\mathcal{U}_X$ and $|c(u)| > 1$.

Put $o(u) = \hat{a}$ where a is the largest initial segment of u such that $|c(a)| = |c(u)| - 1$. Likewise put $\ell(u) = \hat{b}$ where b is the largest final segment of u such that $|c(b)| = |c(u)| - 1$.

Now let $[u]$ denote the *characteristic sequence* of u invented by Kadourek and Polák in [169]: $[u] = (\epsilon_0 u_0, \ldots, \epsilon_{h+1} u_{h+1})$. In this sequence $\epsilon_i \in \{-1, 1\}$ and $u_i \in F\mathcal{U}_X$ is such that $|c(u_i)| = |c(u)| - 1$ for each i. The construction of $[u]$ is inductive.

Suppose u has no segment $(q)^{-1}$ where $c(q) = c(u)$. Select successively, from the left, the largest segments $b_0, \ldots, b_{h+1}$ of u such that $|c(b_i)| = |c(u)| - 1$ for each i. Put $\epsilon_i = 1$ and $u_i = \hat{b}_i$.

Suppose $u = p(q)^{-1}r$ where $c(q) = c(u)$. Define inductively

$$[u] = (< po(q) >, -[\ell(q)qo(q)], < \ell(q)r >)$$

where

$$< b >= \begin{cases} b & \text{if } |c(b)| = |c(u)| - 1; \\ [b] & \text{if } c(b) = c(u). \end{cases}$$

and if $[v] = (\eta_0 v_0, \ldots, \eta_k v_k)$ then $-[v] = (-\eta_0 v_0, \ldots, -\eta_k v_k)$.

For example, let $u = x(y((z)^{-1}zx)^{-1}y)^{-1}$. Then $u = p(q)^{-1}r$ where $p = x$, $q = y((z)^{-1}zx)^{-1}y$, and r is the empty word. Now $o(q) = y(z)^{-1}z$ and $\ell(q) = xy$. Hence

$$< po(q) >= (xy, y(z)^{-1}z), \quad < \ell(q)r >= xy$$

and

$$[\ell(q)qo(q)] = (xy^2, y^2(z)^{-1}z, ((z)^{-1}zx)^{-1}x, xy^2, y^2(z)^{-1}z).$$

Thus

$$\begin{aligned}[u] = \; & (xy, y(z)^{-1}z, -y^2(z)^{-1}z, -xy^2, -((z)^{-1}zx)^{-1}x, \\ & -y^2(z)^{-1}z, -xy^2, xy).\end{aligned}$$

Theorem 3.19 *Let $\mathcal{V}$ be a variety of groups, and let $u, v \in F\mathcal{U}_X$. Let*

$$[u] = (\epsilon_0 u_0, \ldots, \epsilon_{h+1} u_{h+1}), \; [v] = (\eta_0 v_0, \ldots, \eta_{k+1} v_{k+1}).$$

Then $u = v$ is an identity in $\mathcal{CR}(\mathcal{V})$ if and only if the following are satisfied.

1. *$c(u) = c(v)$.*
2. *If $|c(u)| = 1$ then $u = v$ is an identity in $\mathcal{V}$.*
3. *$o(u) = o(v)$ and $\ell(u) = \ell(v)$ are identities of $\mathcal{CR}(\mathcal{V})$.*
4. *Let ϕ be a mapping from the set $\{u_0, \ldots, u_{h+1}, v_0, \ldots, v_{k+1}\}$ into X which glues words together if and only if these words form an identity in $\mathcal{V}$. Then the equality*

$$(u_1\phi)^{\epsilon_1} \cdots (u_h\phi)^{\epsilon_h} = (v_1\phi)^{\eta_1} \cdots (v_k\phi)^{\eta_k}$$

is an identity of $\mathcal{V}$.

This theorem shows that the equational problem in $\mathcal{CR}(\mathcal{V})$ is reduced to the equational problem in $\mathcal{V}$, but in a very non-trivial way.

From Theorem 3.19, it follows that maximal subgroups of the relatively free semigroups from $\mathcal{CR}(\mathcal{V})$ are subgroups of relatively free groups of $\mathcal{V}$ (it is not known if they are free themselves). The same fact holds in varieties of orthodox completely regular semigroups $\mathcal{OCR}(\mathcal{V})$, and varieties of completely simple semigroups $\mathcal{CS}(\mathcal{V})$. The equational problem in the variety of all orthodox completely regular semigroups was solved by Gerhard and Petrich [109]. The word problem for $\mathcal{OCR}(\mathcal{V})$ was reduced to the equational problem in $\mathcal{V}$ by Rasin in [311]. These results have been simplified and generalized to many other varieties of completely regular semigroups by Polák [300], [301], [302]. See also [71].

The equational problem for the variety $\mathcal{CS}(\mathcal{V})$ has been reduced to the equational problem in $\mathcal{V}$ by Rasin [310], and Jones [159]. See also [108].

The situation in the variety $\mathcal{BCR}(\mathcal{V})$ of bands of groups is more complex because the maximal subgroups in the relatively free semigroups of this variety are not necessarily relatively free. They are not free even in the case when $\mathcal{V}$ is the variety of all groups (see Trotter, [402]). So in order to solve the equational problem in these varieties one has to first of all find defining relations of these groups, and then solve the word problem there. This is very non-trivial and has been done by Kaďourek in [165]. In fact, Kaďourek solved the equational problem in a class of completely regular varieties which is much bigger than the class of varieties of bands of groups. A particular case, when $\mathcal{V}$ is Abelian, is simpler. It has been considered in Pastijn and Trotter [290].

We can conclude that the equational problem is decidable in every reasonable variety of completely regular semigroups. Nevertheless, to the best of our knowledge, the following question is still open.

Problem 3.1 *Is there a finitely based variety of completely regular semigroups $\mathcal{C}$ such that the equational problem in $\mathcal{C}$ is undecidable but the equational problem in the maximal group subvariety of $\mathcal{C}$ (which is also finitely based, of course) is decidable?*

Notice that we gave only a sketch of the theory of varieties of completely regular semigroups. For a more complete presentation of this history see the introductions to papers [291], [165], [403], and the survey [368].

If we move from the varieties of completely regular semigroups higher, to varieties of index 2, the next level in the Sapir-Sukhanov hierarchy, then the situation becomes more difficult.

Varieties of index 2, that is varieties where all nil-semigroups have zero product, have been described in Golubov and Sapir [112]: A semigroup variety has index 2 if and only if it satisfies an identity of one of the following forms:

$$xy = (xy)^n,\ xy = x^n y,\ xy = xy^n,\ n > 1.$$

Ershova and Volkov [94] studied varieties given by the first of these identities, that is varieties of semigroups with completely regular squares. They obtained results similar to the results of Pastijn and Trotter (see Theorem 3.19 above). Namely, it turned out that the equational problem in the variety given by the identity $xy = (xy)^n$ may be reduced to the equational problem for the corresponding Burnside variety of groups of exponent $n - 1$.

The answer to the following question is not known yet.

Problem 3.2 *Suppose the equational problem in the Burnside group variety $\mathcal{B}_{n-1}$ is decidable, but $\mathcal{B}_{n-1}$ is not locally finite. Is the equational problem decidable in the variety given by the identity $xy = xy^n$ (resp. $xy = x^n y$) ?*

If $\mathcal{B}_{n-1}$ is locally finite then the variety given by the identity $xy = xy^n$ (resp. $xy = x^n y$) is locally finite [355] and so its equational problem is obviously decidable.

Varieties of arbitrary finite index r satisfy identities of the form

$$x_1 \dots x_{i-1}(x_i \dots x_j)x_{j+1} \dots x_r = x_1 \dots x_{i-1}(x_i \dots x_j)^n x_{j+1} \dots x_r \tag{3}$$

for some i, j, n, $1 \le i \le j \le r$ (see [355], [40]).

Problem 3.3 *Suppose the equational problem in the Burnside group variety $\mathcal{B}_{n-1}$ is decidable, but $\mathcal{B}_{n-1}$ is not locally finite. Is the equational problem decidable in the variety given by the identity (3)?*

The answer to this question is not known for any $r > 2, i, j$, while we think that this problem is doable and the answer is positive. Indeed, we know almost complete information about the structure of semigroups in these varieties (see Theorem 2 in [355]). Again, if $\mathcal{B}_{n-1}$ is locally finite then the identity (3) defines a locally finite variety [355].

3.3.4 The Equational Problem in Burnside Semigroup Varieties

The equational problem in Burnside varieties of semigroups became very popular a few years ago. We will call relatively free semigroups in the variety of semigroups given by an identity of the form $x^m = x^{m+n}$ *free Burnside semigroups of index*[10] *m and period n*. Relatively free groups in Burnside varieties $\mathcal{B}_n$ will be called *free Burnside groups of exponent n*.

It is interesting that so far there is no method which can prove that a free Burnside group is infinite without solving the word problem. In contrast, methods of proving the infiniteness of free Burnside semigroups (see [274]) do not require the solution of the word problem.

[10] Don't mix this index with the index discussed in the previous subsection. However completely regular semigroup varieties are of index 1 under both definitions, which is very convenient.

We have seen (Theorem 3.1) that it is very easy to prove the infiniteness of free Burnside semigroups of indexes ≥ 2. It turned out that the word problem in these free Burnside semigroups is very non-trivial and we still do not know the solution of the equational problem in Burnside semigroup varieties for some m and n.

Notice that the Pastijn-Trotter-Kaďourek-Polák Theorem (Theorem 3.19 above) reduces the equational problem for varieties given by identities of the form $x = x^n$ to the equational problem in the corresponding Burnside varieties of groups. So we do not have to worry about the case $m = 1$ here, in the semigroup corner of our survey.

The case when $m > 1$ is completely different. For example, the maximal subgroups of the free Burnside semigroups of index $m > 1$ are finite [77] (recall that in the case $m = 1$ these subgroups are just free Burnside groups of exponent n, which follows from results of [168]).

The decidability of the word problem in the free Burnside semigroups of index 2 and higher was first conjectured at least in 1977, perhaps much earlier. We personally heard it from L.N.Shevrin in 1977-1978. J.Brzozowski formulated this conjecture in the case of period 1 (i.e. $n = 1$) in 1969 (published in [60]). But until 1990 very little information about the free Burnside semigroups of index > 1 had been known. One can mention, for example, a paper of Brown [59] where he proved that any free Burnside semigroup of index 2 is a union of locally finite semigroups (this result answered a question by Shevrin [367]). I.Simon [377] obtained some results about the structure of $\mathcal{R}$-classes[11] in the free Burnside semigroups of period 1.

But in 1990–1991 significant progress was achieved almost simultaneously by A. de Luca and S.Varricchio [78] and J.McCammond [254]. They solved the equational problem in many Burnside semigroup varieties. The result of de Luca and Varricchio covered the cases with index $m \geq 5$ and period $n = 1$, and McCammond's result covered the cases with $m \geq 6, n \geq 1$. Notice that the only case which was covered by the de Luca-Varricchio paper and was not covered by McCammond's paper was $m = 5, n = 1$.

Proofs in [78] and [254] are based on different ideas. De Luca and Varricchio find a canonical word for every element of the free Burnside semigroup. In order to find this canonical word they obtain an (infinite) terminating Church-Rosser presentation of the free Burnside semigroup of index ≥ 5 and period 1 in the class of all semigroups (see Section 2.10 for the definition of terminating Church-Rosser presentations).

McCammond considers the set S_w of words which are equal to a given word w in this semigroup. He proves that this set is recognized by a finite non-deterministic automaton[12] A_w. The automaton A_w is constructed by induction on the so-called rank of w. It is interesting that, as was pointed out by J. Rhodes in a conversation with the second author of this survey, the canonical word of de Luca and Varricchio labels a

[11]Recall that two elements a and b are said to belong to the same $\mathcal{R}$-class of a semigroup S if $a = bc,\ b = ad$ for some elements c, d of S.

[12]That is a labeled graph with specified start and end vertices. An automaton *recognizes* a word w if w labels a path from the start vertex to an end vertex.

geodesic on the automaton of McCammond, and is called a "base" in McCammond's proof. McCammond proves more than 40 Lemmas by a simultaneous induction on this rank, while the proof in [78] proceeds in a more traditional manner.

Later A.do Lago [85] improved the method of de Luca and Varricchio, and solved the equational problem in Burnside semigroup varieties for every $m \geq 4,\ n \geq 1$. This result is stronger than McCammond's result. In fact do Lago found a kind of boundary for the applicability of the de Luca-Varricchio-do Lago method. He shows that it works fine for $m \geq 4, n \geq 1$. There is a strong hope that it works in cases $m = 2, n = 1$ and $m \geq 3, n \geq 1$. And it does not work, for example, in the case where $m = 2, n = 2$.

Independently and at the same time as do Lago, the solvability of the equational problem for $m = 4, n \geq 1$ was obtained by V.Guba [125], who is a well known specialist in Burnside problems in groups. Guba gave a talk about his proof at the Sverdlovsk Algebraic Systems seminar in January, 1992. He used the McCammond method. In fact, he showed that in order to get this result it is enough to slightly change the formulation of one of McCammond's Lemmas. Another nice consequence of this change is that instead of 40 lemmas proved by a simultaneous induction, one needs only 4.

Recently Guba found a further improvement of McCammond's method and solved the equational problem for arbitrary Burnside semigroup varieties of index ≥ 3 [127], [126].

The case $m = 3$ requires some new powerful combinatorial ideas. Unlike McCammond, Guba does not deal with automata A_w. His methods are purely one-dimensional. In fact, just as de Luca and Varricchio, Guba constructs a terminating Church-Rosser presentation for the relatively free semigroup in the corresponding Burnside semigroup variety.

Thus the solvability of the equational problem is not known only for $m = 2,\ n \geq 1$.

Problem 3.4 *Is the equational problem solvable in the Burnside semigroup variety of index 2?*

It is very unlikely that the answer is negative. The variety is too "natural" to have an undecidable equational problem. But let us not forget about the variety of modular lattices which is also very natural, but has an unsolvable equational problem (the celebrated result of Freese, [101]).

Notice that in all cases when the equational problem in Burnside semigroup varieties turned out to be decidable (see the papers cited above), the relatively free semigroups in these varieties were proved to be residually finite.

Problem 3.5 *Are the free Burnside semigroups of index 2 residually finite?*

3.3.5 The Equational Problem in Varieties of Inverse Semigroups

The variety of inverse semigroups $\mathcal{IS}$ is yet another subvariety of the variety of unary semigroups $\mathcal{U}$, which we considered in Section 3.3.3. The variety $\mathcal{IS}$ is defined by the following identities: $xx^{-1}yy^{-1} = yy^{-1}xx^{-1}, xx^{-1}x = x, (x^{-1})^{-1} = x$. The class of all groups is a subvariety of this variety. Varieties of inverse semigroups have been studied extensively during the last 20 years (see [295] for an introduction to the theory of varieties of inverse semigroups, and the papers [195], [194], [166], [167], [381], [247], [339]).

Traditionally inverse semigroups, discovered by Vagner and Preston in the fifties (see the history in [73] and [295]), were considered as ordinary semigroups, without explicitly mentioning the unary operation. They are von Neumann regular semigroups where idempotents commute. By the Vagner-Preston theorem [73] inverse semigroups are exactly semigroups of partial one-one transformations of a set, closed under taking inverse transformations. Then the operation $^{-1}$ is interpreted as the operation of taking the inverse of a partial one-one transformation. It turns out again, as in the case of completely regular semigroups, that it is more natural to consider inverse semigroups as algebras with two operations.

Inverse semigroups mysteriously appear in many investigations. For example, the fundamental groupoids of manifolds are in fact inverse semigroups. The Brandt monoid (2) is an inverse semigroup. Recently Margolis and Meakin [247] found a natural one-one correspondence between finite inverse semigroups and finitely generated subgroups of the free groups. In [246] a connection between identities of finite inverse semigroups and quasi-identities of finite semigroups was discovered. In [263], [264], it is shown that there exists a group-like correspondence between homomorphisms of the free semigroup and inverse subsemigroups of the so called inverse polycyclic semigroup which is in a sense an "inverse envelope" of the free semigroup. See also Theorem 3.46 by Ash below. There are lots of other examples.

A completely regular semigroup is connected with groups via its subgroups. Inverse semigroups and groups are also very closely connected but the connection is not so obvious. As we said, any inverse semigroup S is a semigroup of partial one-one transformations of a set. One can extend these transformations to some permutations of the set, and generate a group G by these transformations. We will call this group a *cover*[13] of S. So inverse semigroups are just restrictions of groups of permutations. Of course, an inverse semigroup can have different covers. If we fix a variety of groups $\mathcal{V}$ we can consider the class of inverse semigroups $\hat{\mathcal{V}}$ which have covers from $\mathcal{V}$ [243]. This class is a variety defined by all identities of the form $w = w^2$ where $w = 1$ is an identity of $\mathcal{V}$. Another way to describe this variety is the following: $\hat{\mathcal{V}}$ is the Mal'cev product of the variety of semilattices (commutative idempotent semigroups) and $\mathcal{V}$. Also this variety is the class of semigroups which divide semi-direct products

[13]Specialists say in this case that S has an e-unitary cover over G. It would take too much space to explain what "e-unitary" means, so we have coined another term.

of semilattices and groups from $\mathcal{V}$.

Varieties of the form $\hat{\mathcal{V}}$ form a skeleton in the lattice of varieties of inverse semigroups, just like varieties $\mathcal{CR}(\mathcal{V})$ in the lattice of varieties of completely regular semigroups. The variety of all inverse semigroups is just the "hat" of the variety of all groups.

The equational problem in the variety $\mathcal{IS}$ was solved by E.Kleiman, Munn, Schieblich, Schein (see [295]). Here we would like to present a nice solution of the word problem in relatively free semigroups in the varieties $\hat{\mathcal{V}}$ due to Margolis and Meakin [243].

Let $\mathcal{V}$ be a variety of groups, X a nonempty set and $F_X(\mathcal{V})$ the relatively free X-generated group in $\mathcal{V}$. Let $\Gamma_X(\mathcal{V})$ denote the Cayley graph of $F_X(\mathcal{V})$ relative to the set of generators X. Thus $\Gamma_X(\mathcal{V})$ has set $F_X(\mathcal{V})$ of vertices and has an edge labeled by $x \in X \cup X^{-1}$ from g to gx for each $g \in F_X(\mathcal{V})$. Define a semigroup $M_X(\mathcal{V})$ as the set of all pairs (Γ, g) where Γ is a finite connected subgraph of $\Gamma_X(\mathcal{V})$ containing 1 and g as vertices, $\Gamma \neq \{1\}$ with the multiplication

$$(\Gamma_1, g_1)(\Gamma_2, g_2) = (\Gamma_1 \cup g_1 \cdot \Gamma_2, g_1 g_2).$$

(Here $g_1 \cdot \Gamma_2$ denotes the natural action (left translation) of g_1 on Γ_2.)

Theorem 3.20 *(Margolis, Meakin, [243]). If $\mathcal{V}$ is any variety of groups and X is any non-empty set then $M_X(\mathcal{V})$ is the relatively free X-generated inverse semigroup in the variety $\hat{\mathcal{V}}$.*

The Munn description of the free inverse semigroups [295] is a particular case of Theorem 3.20 where $\mathcal{V}$ is the variety of all groups.

Theorem 3.20 shows that the equational problem is decidable in $\hat{\mathcal{V}}$ if and only if it is decidable in $\mathcal{V}$. This also follows from another description of the free inverse semigroups in $\hat{\mathcal{V}}$ obtained by Reilly and Trotter in [313] (see Lemma 3.5 in [313]).

There are several other recent results on the equational problem in inverse semigroup varieties. We would like to mention Stephen's results on Burnside varieties of inverse semigroups, which, of course, are defined by identities of the form $x^m = x^{m+n}$ [244], [381].

Theorem 3.21 *(Stephen, [381]). The equational problem is decidable in the Burnside variety of inverse semigroups of index m and period n for every pair (m, n) where $n \leq m$. Relatively free semigroups in these varieties are residually finite.*

This result is in contrast with results on the Burnside semigroup varieties which we discussed in Section 3.3.4. In the case of ordinary semigroups it is difficult to deal with small indexes (the decidability/undecidability of the equational problem for Burnside semigroup varieties of index 2 is still unknown) while the period did not play any significant role. Here, in the case of inverse semigroups the index is not important while the period must be small enough.

The method of proving Theorem 3.21 is in a sense similar to McCammond's method. For every word w in the free Burnside inverse semigroup $BI_{m,n}$ Stephen shows that the set of all words w' such that $w = w'e$ in $BI_{m,n}$ for some idempotent e is accepted by a finite deterministic automaton, the so called Cayley automaton of w.

Cayley automata, invented by Stephen, are very useful tools in studying the word problem in inverse semigroups. See, for example, [244], [381]. Meakin and the second author of this survey used these automata in [264] to prove that the word problem is undecidable in the variety $\hat{\mathcal{A}}$ where $\mathcal{A}$ is the variety of all Abelian groups (even though the equational problem in this variety is decidable by Theorem 3.20).

3.3.6 The Higman property

The semigroup analog of the Higman embedding theorem was proved by Murskii in 1967 [275]. Proper semigroup varieties which possess the Higman property and do not satisfy the condition $FP = FG$ (see Section 2.10) were unknown until the second author of this survey proved the following result which appears here for the first time.

Theorem 3.22 *(Sapir). The variety given by the identity $x^3 = 0$ possesses the Higman property, that is every semigroup from this variety which is given by a recursively enumerable set of defining relations is embeddable into a semigroup which is finitely presented in this variety.*

There are reasons to believe that the following conjecture of the second author of this survey may be proved with the help of the techniques employed in the proofs of Theorems 3.7, 3.8 and 3.22.

Conjecture 3.1 *(Sapir) Let $\mathcal{V}$ be a variety given by a system of identities $u_i = v_i$, $i \in I$ such that u_i and v_i do not contain nonempty free sets of letters. Then $\mathcal{V}$ satisfies the Higman property.*

If we allow u_i and v_i to contain free sets of letters then the method of the second author does not work. Nevertheless it is possible that every variety with locally finite periodic semigroups satisfies the Higman property. Currently we do not know any counterexamples.

Problem 3.6 *(Sapir) Is there any unavoidable word u with free letters such that the variety given by the identity $u = 0$ satisfies the Higman property? Is it true that the variety given by the identity $(xy)^2 = 0$ satisfies the Higman property?*

Notice that an analog of the Higman theorem for inverse semigroup varieties was obtained by Belyaev in [32]. No proper inverse semigroup varieties with this property are known (except for varieties with the property $FG = FP$). It is not known (Shevrin's problem) if the variety of all completely regular semigroups satisfies the Higman property.

3.4 The Club of The Word Problem

Permanent members: The word problem, the uniform word problem, the uniform word problem for finite semigroups, the residual finiteness of finitely presented semigroups, other algorithmic problems concerning finitely presented semigroups.

Undecided membership: The isomorphism problem.

Membership by association: The finiteness problem, the triviality problem.

3.4.1 Commutative Semigroups

The first non-trivial (that is non-locally finite) variety of semigroups with solvable word problem was found independently by A. I. Mal'cev [236] and by Ceitin and Emelichev [91]. It was the variety of all commutative semigroups.

Theorem 3.23 *(Mal'cev [236], Ceitin and Emelichev [91]). The variety of all commutative semigroups has a solvable word problem.*

Mal'cev showed that every finitely generated commutative semigroup is faithfully representable by matrices over a suitable field, and every finitely generated ring of matrices is residually finite. Therefore every finitely generated commutative semigroup is residually finite, and the McKinsey algorithm (see Connection 2.4 in the Introduction) gives the solution to the word problem.

As was pointed out by Emelichev in [91] the proof may be easily deduced from an old paper by Hermann [142] devoted to the membership problem for ideals in the ring of polynomials[14]. Indeed, let $S =< X|u_1 = v_1, \ldots, u_r = v_r >$ be any finitely presented commutative semigroup. Emelichev [91] proved that a relation $u = v$ holds in S if and only if the polynomial $p = u - v$ belongs to the ideal of the ring of polynomials $\mathbf{Q}[X]$ generated by polynomials $p_i = u_i - v_i$. This, in turn, is equivalent to the solvability of the following equation over $\mathbf{Q}[X]$:

$$p_1 f_1 + p_2 f_2 + \ldots + p_r f_r = p \tag{4}$$

with unknowns $f_1, \ldots, f_r$. Now we can use the following result from Hermann [142]:

Theorem 3.24 *(Hermann, [142]). Let $d = max\{deg(p_1), deg(p_2), \ldots, deg(p_r)\}$. If the equation (4) has a solution then there is a solution with $deg(f_i) \leq deg(p) + (rd)^{2^{|X|}}$.*

It is clear that if there is a bound for the degrees of unknown polynomials in (4) then the number of coefficients of these polynomials is also bounded. Then (4) is

[14]A proof of Hermann contained a small gap which has been fixed in [253].

equivalent to a finite system of linear equations over the field of rational numbers, which can be solved by, say, the Gauss elimination algorithm.

G. Bergman (see Section 10.3 of [34]) was probably the first to prove that every finitely presented commutative semigroup possesses a finite terminating Church-Rosser presentation which also implies the solvability of the word problem. R. Gilman [110], C. E. Peterson and M. E. Stickel [294] and by Ballantyne and Lankford [22] produced similar proofs independently.

A proof of Theorem 3.23 based on other ideas is presented below in Section 7.1.1.

Taiclin [389], [390] proved the following result which is much stronger than Theorem 3.23.

Theorem 3.25 *(Taiclin, [389], [390]). The elementary theory of every finitely presented commutative semigroup is decidable.*

A relatively easy proof of this theorem was published in [386]. The idea of the proof in [386] is the following. It is well known that the elementary theory of non-negative integers with addition (the Presburger theory) is decidable [305]. Then the elementary theory of any free commutative monoid (which is a direct power of the semigroup of non-negative integers) is decidable. The free commutative semigroup is equal to the free commutative monoid without the unit, so its elementary theory is also decidable. Every finitely presented commutative semigroup S is a factor of a finitely generated free commutative semigroup A over a finitely generated congruence σ. Taiclin proves that this congruence is elementary, that is there exists a first order formula $\theta(x, y)$ with two free variables such that

$$\theta(x, y) \text{ if and only if } (x, y) \in \sigma.$$

This implies Theorem 3.25 because now for every first order formula γ of S one can construct a first order formula $f(\gamma)$ of A such that

$$\gamma \text{ holds in } S \text{ if and only if } f(\gamma) \text{ holds in } A.$$

Indeed, in order to construct $f(\gamma)$, it is enough to replace all equality signs in γ by θ.

The same idea has been employed in a more general situation by Rozenblat and Zamjatin [333]. They considered varieties which satisfy a permutation identity, that is an identity of the form

$$x_1 x_2 \dots x_n = x_{\sigma(1)} x_{\sigma(2)} \dots x_{\sigma(n)} \tag{5}$$

where σ is a permutation.

Theorem 3.26 *(Rozenblat, Zamjatin, [333]). Let $\mathcal{V}$ be a finitely based variety satisfying a permutation identity. Then the elementary theory of every relatively free semigroup in $\mathcal{V}$ is decidable.*

An amazing thing about this theorem is that the converse statement also holds in the case of non-periodic varieties Rozenblat [332].

Theorem 3.27 *(Rozenblat, [332]). Let $\mathcal{V}$ be a semigroup variety where every relatively free semigroup has a decidable elementary theory. Then $\mathcal{V}$ satisfies either a permutation identity or an identity of the form $x_1x_2\dots x_n = u_1x_{n+1}u_2$ where $x_1,\dots,x_{n+1}$ are pairwise distinct letters, and u_1, u_2 are (possibly empty) words. In the last case $\mathcal{V}$ is a variety of finite index (see Section 3.3.3).*

3.4.2 The Description

The first example of a finitely based variety of semigroups, other than the variety of all semigroups, with an unsolvable word problem is, of course, the variety of Murskii [276]. This was the only known example until 1983 when an attack on the varieties with decidable word problem was initiated by I.Mel'nichuk. In particular, she proved [267] that the word problem is undecidable in any variety which contains a non-locally finite variety of semigroups given by identities of the form $u = 0$. Such varieties were been described by Bean, Ehrenfeucht, McNulty and Zimin (see Theorem 3.6 above). She also proved that the word problem is decidable in any finitely based variety which satisfies the permutation identity (5).

Then Mel'nichuk, and the authors of this survey [268] found a minimal variety with an undecidable word problem, a boundary between solvability and undecidability. It was the variety generated by the semigroup S_2 from Section 7.2.4. Semigroups S_1 and the dual semigroup $\overleftarrow{S_1}$ from Section 7.2.4 also appeared in [268]. Notice that generally speaking semigroups S_1 and S_2 depend on the Minsky machine used in the construction of these semigroups. But the varieties generated by these semigroups do not depend on the Minsky machine.

Finally, the second author of this survey [334], [335] proved that S_1 and $\overleftarrow{S_1}$ also generate boundaries between solvability and undecidability, proved that there are no more boundaries among non-periodic varieties of semigroups, every periodic semigroup in a non-periodic variety with decidable word problem must be locally finite, and every non-periodic variety with an undecidable word problem and locally finite periodic semigroups contains one of these three varieties. He also showed that a periodic variety with a solvable word problem and locally finite groups must be locally finite itself. This is everything that one can hope to get, because the problem of describing non-locally finite periodic varieties of groups with solvable word problem is hopeless. It also turns out that many other conditions for finitely presented semigroups in varieties are equivalent to the solvability of the word problem.

The results from [334] and [335] of the second author of this survey are summarized in the following three theorems.

Recall (see Section 2.10) that for every variety of semigroups $\mathcal{V}$ the set of all semigroups which are finitely presented inside $\mathcal{V}$ is denoted by $FP(\mathcal{V})$, the set of all (absolutely) finitely presented semigroups which belong to $\mathcal{V}$, is denoted by $FP \cap \mathcal{V}$.

We will need the following two subsemigroups of the Brandt monoid (2):

$$
\begin{aligned}
P &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \\
T &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.
\end{aligned} \tag{6}
$$

For every semigroup S the dual (anti-isomorphic) semigroup is denoted by $\overleftarrow{S}$, the semigroup S with an identity element adjoined is denoted by S^1.

Theorem 3.28 *(Sapir). Let $\mathcal{V}$ be a finitely based non-periodic variety. Then the following conditions are equivalent.*

1. *The word problem is decidable in any semigroup from $FP(\mathcal{V})$.*
2. *The elementary theory of any semigroup from $FP(\mathcal{V})$ is decidable.*
3. *The uniform word problem for $\mathcal{V}$ is decidable.*
4. *In every semigroup from $FP(\mathcal{V})$ the following "divisibility" problem is decidable:*

   For every two elements a and b decide if a is divisible by b.

5. *In every semigroup from $FP(\mathcal{V})$ the following "regularity" problem is decidable:*

   For every element a decide if a is regular.

6. *In every semigroup from $FP(\mathcal{V})$ the following "idempotent" problem is decidable:*

   For every element a decide if a is an idempotent.

7. *Every semigroup from $FP(\mathcal{V})$ is representable by matrices over a field.*
8. *Every nil-semigroup from $\mathcal{V}$ is locally finite and $\mathcal{V}$ does not contain $\overleftarrow{P} \times P^1$, $\overleftarrow{P}^1 \times P$, T.*
9. *Every nil-semigroup of $\mathcal{V}$ is locally finite and for some natural numbers k, m, n, p the variety $\mathcal{V}$ satisfies one of the following identities:*

$$x^n y (z^k t^k)^p z^m = x^m (t^k x^k)^p y z^n, \quad n = m + kp; \tag{7}$$

$$x y^n z = y^k x y^m z y^p, \quad n > m. \tag{8}$$

Theorem 3.29 *(Sapir). Let $\mathcal{V}$ be a finitely based periodic variety of semigroups where all groups are locally finite. Then each of the conditions 1, 2, 3, 4,5, 6, 8 from Theorem 3.28 is equivalent to the condition that $\mathcal{V}$ is locally finite.*

For $i = 1, \ldots, 8$ let i' be the condition i from Theorem 3.28 where the set $FP(\mathcal{V})$ is replaced by the set $FP \cap \mathcal{V}$.

Theorem 3.30 *(Sapir). Let $\mathcal{V}$ be a finitely based non-periodic variety of semigroups where all nil-semigroups are locally finite. Then each of the conditions 1, 2, 3, 4, 5, 6, 7 from Theorem 3.28 is equivalent to each of the conditions 1', 2', 3', 4', 5', 6', 7'. and to each of the following conditions*

9. *Every semigroup from $FP(\mathcal{V})$ is Hopfian.*
10. *Every semigroup from $FP \cap \mathcal{V}$ is Hopfian.*

Conditions 4, 5, 6 appear here for the first time. The fact that these conditions are equivalent to the others may be easily deduced from the results of Sapir [335].

Let us analyze these theorems. From Theorems 3.28, 3.29 it follows that in the class of non-periodic semigroup varieties where all nil-semigroups are locally finite every variety which has an undecidable (strongly undecidable) word problem must contain one of the three finite semigroups:

$$\overleftarrow{P} \times P^1, \ \overleftarrow{P}^1 \times P, \ T. \tag{9}$$

On the other hand, every non-periodic variety containing one of these three semigroups has an undecidable word problem. Therefore the following three varieties are the only minimal varieties with undecidable (strongly undecidable) word problem among non-periodic varieties with locally finite nil-semigroups:

$$\mathrm{var}(\overleftarrow{P} \times P^1 \times \mathbf{N}), \ \mathrm{var}(\overleftarrow{P}^1 \times P \times \mathbf{N}), \ \mathrm{var}(T \times \mathbf{N})$$

where $\mathbf{N}$ is the semigroup of natural numbers with respect to addition. In fact, these three varieties coincide with varieties generated by the semigroups S_1, $\overleftarrow{S}_1$, and S_2 respectively, presented in Section 7.2.4 below.

It is easy to see that each of these varieties is a join of a locally finite variety (generated by a finite semigroup) and the variety of all commutative semigroups. Both have solvable word problem. Therefore we have *examples of varieties with unsolvable word problem which are joins of varieties with solvable (even in polynomial time) word problem.*

If a finitely based variety does not contain any of the three semigroups (9) but still has an unsolvable word problem then either it contains a non-locally finite nil-semigroup or it is periodic and contains a "bad" group, an infinite finitely generated group of finite exponent (see Theorems 3.28, 3.30). Theorems 3.7 and 3.8 show that there are no minimal finitely based non-locally finite varieties of nil-semigroups, that is every non-locally finite finitely based variety of nil-semigroups contains a proper subvariety with the same properties. Thus in the first case the variety does not contain any minimal variety with unsolvable word problem. In the second case the situation is more complex. We do not know any minimal periodic variety of groups with unsolvable word problem. Moreover an analogy between nil-semigroups and periodic groups make the following two conjectures of the second author of the survey seem reasonable:

Conjecture 3.2 *Every finitely based periodic group variety with undecidable word problem contains a proper subvariety with an undecidable word problem.*

Conjecture 3.3 *Every non-locally finite finitely based variety of periodic groups has an undecidable word problem.*

Of course, we do not have any hope of proving these conjectures. So far we have only a few examples of periodic group varieties with undecidable word problem (see Theorems 6.15 and 6.16).

Thus, modulo groups, there are exactly three minimal varieties of semigroups with an undecidable word problem.

Now let us turn to the other properties mentioned in Theorems 3.28, 3.29, 3.30. These theorems show that the properties are equivalent in the class of finitely based non-periodic varieties with locally finite nil-semigroups.

If we leave this class then some of the equivalences become unknown. The most important connection which we would like to trace is the connection between weak and strong word problems.

Problem 3.7 *Is the decidability of the word problem in a variety of semigroups equivalent to the weak decidability of the word problem in this variety?*

And again the case of periodic varieties is crucial. Recall the following problem (see [384] Problem 3.61) which seems very difficult and interesting. The first part of this problem is well known, the second part was posed by Shevrin and the second author of this survey.

Problem 3.8 *a) Is there a finitely presented infinite periodic semigroup?*
b) Is there a finitely presented infinite nil-semigroup?

Since we do not know the answers to these questions it would be premature to even pose a question about describing periodic semigroup varieties with strongly undecidable word problem.

Properties 2 – 6 from Theorem 3.28 seem easier to handle.

In particular since every periodic semigroup of matrices over a field is locally finite (see [218], Chapter 10), Theorems 3.28, 3.29 imply a complete description of finitely based varieties of semigroups where every finitely presented semigroup is representable by matrices over a field.

Theorem 3.31 *(Sapir, [342]). For every finitely based variety $\mathcal{V}$ of semigroups the following conditions are equivalent.*

1. *Every semigroup from $FP(\mathcal{V})$ is representable by matrices over a field.*

2. *$\mathcal{V}$ either either is a locally finite variety or is non-periodic and satisfies conditions 1-9 of Theorem 3.28.*

Problem 3.9 *Describe semigroup varieties (not necessarily finitely based) where every finitely presented semigroup is residually finite.*

3.4.3 The Uniform Word Problem for Finite Semigroups

As we mentioned in the Introduction, in 1966 Yu. Gurevich [133] proved that the uniform word problem is undecidable in the class of all finite semigroups.

On the other hand, as follows from Connection 2.6, it is decidable in the finite trace of any variety where finitely presented semigroups are residually finite, and in any periodic variety where the restricted Burnside problem is solved positively. In [352] the second author of this survey proved that every variety $\mathcal{V}$ where the uniform word problem for finite semigroups is decidable is either non-periodic and belongs the first of these classes, or periodic and belongs to the second class. Since we know algorithmic descriptions of non-periodic varieties from the first class (Theorem 3.28) and periodic varieties from the second class (Theorem 3.13) we get the following theorem.

Theorem 3.32 *(Sapir, [352]). For any finitely based variety $\mathcal{V}$ the following conditions are equivalent:*

1. *The uniform word problem is decidable in $\mathcal{V}_{fin}$.*

2. *Every nil-semigroup from $\mathcal{V}$ is locally finite and either $\mathcal{V}$ is non-periodic and does not contain $\overleftarrow{P} \times P^1$, $\overleftarrow{P}^1 \times P$, T or $\mathcal{V}$ is periodic.*

As we have seen before, the condition 2 of this theorem may be algorithmically verified.

3.4.4 The Isomorphism Problem and Similar Problems

It is possible to prove that semigroups S_1, S_2 from Section 7.2.4 and the semigroup $S(M, \phi)$ from Section 7.2.5 are Hopfian. Therefore every variety with an unsolvable word problem, which is covered by Theorems 3.28, 3.29, 3.30, contains a Hopfian finitely presented semigroup with an unsolvable word problem. Thus Connection 2.1 applies and we can formulate the following statement proved recently by the second author of this survey. It appears here for the first time.

Theorem 3.33 *(Sapir). Let $\mathcal{V}$ be a finitely based variety of semigroups which satisfies one of the following conditions:*

1. *$\mathcal{V}$ contains a non-locally finite nil-semigroup;*

2. *$\mathcal{V}$ is non-periodic and contains one of the three semigroups (9).*

Then the isomorphism problem in $\mathcal{V}$ is undecidable.

If the following conjecture were correct then the isomorphism problem in semigroup varieties covered by Theorems 3.28, 3.29, 3.30 would be equivalent to the word problem.

Conjecture 3.4 *Let $\mathcal{V}$ be a finitely based variety of semigroups which is either non-periodic or periodic with locally finite groups. Then the solvability of the word problem is equivalent to the solvability of the isomorphism problem.*

Relatively detailed information about the structure of finitely presented semigroups in the varieties with decidable word problem is known: see Theorem 7.1 below. Theorem 7.1 shows that these semigroups are close to commutative semigroups. This must help to prove Conjecture 3.4 because, as usual, everything is fine with the variety of commutative semigroups. The following theorem was published in [388].

Theorem 3.34 *(Taiclin, [388]). The isomorphism problem in the variety of commutative semigroups is decidable.*

This theorem is in fact a result of the collective efforts of several authors. At first Taiclin [387] proved that with every pair of finitely presented commutative semigroups A, A' one can associate a pair of finite sequences of square matrices of the same finite size over the ring of integers: $U = \{M_1, \ldots, M_n\}$ and $U' = \{M_1', \ldots, M_n'\}$ such that the following statement holds.

Theorem 3.35 *(Taiclin, [387]). A is isomorphic to A' if and only if U and U' are conjugate, that is if and only if there exists an invertible matrix V over the integers such that $V^{-1}M_iV = M_i'$ for $i = 1, \ldots, n$.*

Remark. It is interesting that the conjugacy problem for sequences of square matrices and the isomorphism problem for commutative semigroups are in fact equivalent. Taiclin proves in [387] that for every two such sequences U and U' one can construct two finitely generated commutative semigroups (even completely regular commutative semigroups) A and A' such that U and U' are conjugate if and only if A and A' are isomorphic.

Later Grunewald and Segal [122], and Sarkisjan [357] came to the same problem through studying the isomorphism problem for nilpotent groups. By a coincidence (or was it more than a coincidence?) this isomorphism problem also reduced to the conjugacy problem for sequences of matrices over integers. Grunewald and Segal [122], and Sarkisjan [357] proved that the conjugacy problem is decidable which in turn implied the decidability of the isomorphism problem for commutative semigroups.

Perhaps one can generalize Taiclin's method and prove Conjecture 3.4. One can also try to employ the method of Pickel [296] (see the discussion before Connection 2.7 in the Introduction). Both approaches seem to be fruitful. But in order to apply Pickel's method one has to prove the following conjecture of the second author of this survey, which is interesting also by itself.

Conjecture 3.5 *(Sapir). For every finitely generated commutative semigroup A there exist only finitely many finitely generated commutative semigroups which are quasi-isomorphic to A (that is, which have the same finite homomorphic images as A).*

An analog of this conjecture is true for nilpotent groups [296]. And, as we mentioned before, commutative semigroups are closely related to nilpotent groups via sequences of matrices over integers. Thus one can expect that Conjecture 3.5 is also true.

By Connection 2.7, the triviality problem is decidable in any variety covered by Theorems 3.28, 3.29, 3.30. Since the triviality problem is related to the isomorphism problem (a semigroup is trivial if it is isomorphic to the 1-element semigroup), we have assumed that the triviality problem belongs to the word problem Club. But in fact we do not know any proper variety of semigroups where the triviality problem is undecidable. Same applies to the finiteness problem.

Problem 3.10 *(Sapir). Is there a proper semigroup variety where the triviality (finiteness) problem is undecidable?*

Thus it is possible that the Club of the triviality problem (finiteness problem) is even higher than the Burnside problem Club.

3.4.5 The Complexity of the Word and Uniform Word Problems

Let us formulate the word and uniform word problems in Computer Science terms from Section 2.8.

The word problem in an algebra $T =< X \mid R >$ from a variety $\mathcal{V}$.

- $B_D = \{(u, v) \mid u, v$ are words over $X\}$,
- $S_D = \{(u, v) \in B_D \mid u$ is equal to v modulo R and identities of $\mathcal{V}\}$.
- The size of the pair (u, v) is the sum of lengths of words u, v.

The uniform word problem in variety $\mathcal{V}$.

- $B_D = \{(u, v, R) \mid u, v$ are words over X, R is a set of relations $u_i = v_i\}$
- $S_D = \{(u, v, R) \in B_D \mid u$ is equal to v modulo R and identities of $\mathcal{V}\}$.
- The size of the triple (u, v, R) is the sum of lengths of words u, v, and all u_i, v_i such that $u_i = v_i \in R$.

As we mentioned in Section 2.8, the first variety where the complexity of the word and uniform word problems was investigated, was the variety of commutative semigroups.

E.Cardoza [68] noticed that the word problem in any fixed finitely generated commutative semigroup may be solved in linear time. This is an immediate corollary of the following three facts:

1. The result of Taiclin [386] that any congruence on a free finitely generated commutative semigroup A_n may be given by a first order Presburger formula in A_{2n} (see the proof of Theorem 3.25 above),

2. The result of Ginsburg and Spanier [111] that any subset $T \subseteq A_n$ definable by a Presburger formula in A_n is an effectively constructible finitely generated semilinear subset. This means that T is a finite union of "affine" sets of the following form $L(x, P) = \{x + n_1p_1 + \ldots + n_kp_k \mid n_i \in \mathbf{N}, p_i \in P\}$ for some element $x \in A_n$ and a some finite set $P \subset A_n$.

3. The result by Fischer, Meyer, and Rosenberg [99] that every semilinear subset of A_n can be recognized in linear time.

The uniform word problem for the variety of commutative semigroups is much harder. Mayr and Meyer [253] proved that it is exponential space complete (see also [385]).

The fact that there exists an exponential space algorithm solving this problem was announced earlier by Cardoza in [68]. It follows from Hermann's Theorem 3.24 (see Section 7.3).

The fact that every exponential space problem can be reduced to the uniform word problem for commutative semigroups, was proved by using a "bounded version" of Minsky machines with 3 tapes (see Section 7.3).

The time complexity of the uniform word problem for commutative semigroups is not known. It is certainly not polynomial, because polynomial time problems are solvable in polynomial space (see Connection 2.11). On the other hand by Connection 2.12 the time complexity is at most double-exponential. It is one of the famous open problems in Computer Science, whether or not every problem which can be solved in exponential space can be also solved in exponential time (this problem is only slightly less famous than its great P=NP brother). We would solve this problem if we could find an exponential time algorithm for the uniform word problem for commutative semigroups: this easily follows from the fact that the uniform word problem for commutative semigroups is exponential space complete. This shows that the possibility of finding an exponential time algorithm is very slim.

It is very interesting to find out the complexity of the word and the uniform word problems in the varieties where the decidability of these problems follows from Theorems 3.28, 3.29, 3.30. We think that the following conjecture of the second author of this survey may well be true.

Conjecture 3.6 *For every finitely based non-periodic semigroup variety $\mathcal{V}$ the following conditions are equivalent:*

1. *The word problem is decidable in $\mathcal{V}$;*
2. *The uniform word problem is decidable in $\mathcal{V}$;*
3. *The word problem in $\mathcal{V}$ may be solved in polynomial time;*
4. *The uniform word problem in $\mathcal{V}$ is exponential space complete.*

The equivalence of conditions 1 and 2 follows from Theorem 3.28.

3.5 The Club of Locally Residually Finite Varieties

Permanent members: "To be locally residually finite", "To have only a countable number of finitely generated semigroups", "To be locally representable by matrices over a field", "To be locally Hopfian", etc.

Temporary membership The Church-Rosser property.

3.5.1 Locally Residually Finite Varieties

Again everything started with the commutative semigroups. By the Redei Theorem [73] every finitely generated commutative semigroup is finitely presented. Thus the variety of commutative semigroups satisfies the property $FG = FP$. This, by the way, implies that there are only countably many finitely generated commutative semigroups. As we mentioned above, Mal'cev proved that this variety is locally residually finite and locally representable by matrices.

In the '70s Shevrin posed the problem of describing semigroup varieties which possess the property $FG = FP$ (see Problem 7 on page 3 of Bokut's survey [42]). Bokut' (Problem 1 on page 4 of [42]) posed a similar problem about locally residually finite semigroup varieties. These problems were stimulated, of course, by results of L'vov, Yu. Mal'cev, and Anan'in on varieties of associative algebras with similar properties (see Section 4).

Zimin [432] found a description of the finitely based non-periodic varieties which satisfy the property $FG = FP$. With every finitely based non-periodic semigroup variety $\mathcal{V}$ he associates a variety $\hat{\mathcal{V}}$. Then $\mathcal{V}$ satisfies the property $FG = FP$ if and only if $\hat{\mathcal{V}}$ is locally finite. The variety $\hat{\mathcal{V}}$ may be described in the following way (Zimin himself used another, more complicated, terminology). Let Σ be the finite basis of $\mathcal{V}$. Σ_1 be the subset of Σ which consists of all identities which do not hold in the semigroup $P\times \overleftarrow{P}$ where P is the same 3-element semigroup which figured in Theorem 3.28. Then $\hat{\mathcal{V}}$ is defined by all identities $u = 0, v = 0$ such that $u = v \in \Sigma_1$. Using Theorem 3.7 this description is easy to make algorithmic. A simpler algorithmic description was found by the second author of this survey in [337], [336]. It turned out that, at least in the finitely based non-periodic case, this property is equivalent to other properties mentioned above as the permanent members of the Club. Later the second author of this survey obtained complete descriptions of locally residually finite varieties and varieties which are locally representable by matrices [342]. Notice also that T.Nordahl [281] studied locally residually finite semigroup varieties which satisfy permutation identities (see identity (5) above).

The results of the second author of this survey are combined in the following two theorems (see [336]).

Theorem 3.36 *(Sapir, [337], [336]). For every semigroup variety $\mathcal{V}$ the following conditions are equivalent:*

1. *Every finitely generated semigroup from $\mathcal{V}$ is residually finite.*
2. *Every finitely generated semigroup from $\mathcal{V}$ is representable by matrices over a field.*
3. *Either $\mathcal{V}$ is a locally finite variety or the following three conditions hold:*
   - *(a) $\mathcal{V}$ is a non-periodic variety,*
   - *(b) All nil-semigroups from $\mathcal{V}$ are locally finite,*
   - *(c) $P\times \overleftarrow{P} \notin \mathcal{V}$.*
4. *Either $\mathcal{V}$ is a locally finite variety or the following three conditions hold:*
   - *(a) $\mathcal{V}$ is a non-periodic variety,*
   - *(b) All nil-semigroups from $\mathcal{V}$ are locally finite,*
   - *(c) $\mathcal{V}$ satisfies the identity*
   $$xy^n = y^k xy^m zy^p \tag{10}$$
   for some $k, m, n, p, n > m$.

Theorem 3.37 *(Sapir, [337], [336]). Let $\mathcal{V}$ be a semigroup variety which satisfies one of the following two conditions:*

1. $\mathcal{V}$ is non-periodic;

2. $\mathcal{V}$ is periodic and all groups in $\mathcal{V}$ are locally finite.

Then the following conditions are equivalent.

1. *Every finitely generated semigroup from $\mathcal{V}$ is residually finite.*
2. *Every finitely generated semigroup from $\mathcal{V}$ is representable by matrices over a field.*
3. *$FG(\mathcal{V}) = FP(\mathcal{V})$.*
4. *Every congruence on every finitely generated relatively free semigroup in $\mathcal{V}$ is finitely generated.*
5. *Every ideal in every finitely generated relatively free semigroup of $\mathcal{V}$ is finitely generated.*
6. *The set $FG(\mathcal{V})$ is countable.*

Remark 1. Notice that the identity (10) coincides with the identity (8) from Theorem 3.28. This, of course, reflects the fact that every locally residually finite variety of semigroups has a solvable word problem. The condition "to be locally residually finite" is strictly stronger than the condition "to have a solvable word problem". This is reflected by the fact that the "forbidden" semigroup $P\times \overleftarrow{P}$ is smaller than the "forbidden" semigroups from Theorem 3.28. It is a subsemigroup of the semigroups $\overleftarrow{P} \times P^1$ and $\overleftarrow{P}^1 \times P$, and it generates a smaller variety than $\mathrm{var}(T)$.

Remark 2. Condition 5 in Theorem 3.37 is published here for the first time.

Remark 3. There are several strong similarities between these theorems and results about locally residually finite associative and Lie algebras from Sections 4, 5. Let us mention two of these similarities:

- As in the associative and Lie algebra cases, local residually finiteness, local representability by matrices, and other similar conditions are equivalent.
- The identity (10) is a semigroup analog of L'vov's and Volichenko's identities (16), (22).

Remark 4. There is also a significant difference between Theorems 3.36, 3.37 and similar results in Lie and associative algebras. In the last two cases we have complete descriptions of varieties possessing the properties $FG = FP$ (locally weakly Noetherian varieties) and "to have only a countable number of finitely generated objects". In the case of semigroups we have descriptions modulo periodic varieties of groups. This difference roots in the differing situations of the Burnside properties in groups and associative (Lie) algebras (see Section 2.6).

3.5.2 Church-Rosser Presentations of Semigroups in Varieties

We have mentioned in Section 2.9 that there are many Church-Rosser semigroup varieties, although a complete description of such varieties is not yet known.

In order to present results which are currently known we need some definitions. Let F be a semigroup, $p, q, w \in F$. We say that the relation (p, q) is *applicable* to the element w if $w = spt$ for some elements $s, t \in F^1$. The result of the application of (p, q) to w is the element $sqt \in F$. Notice that if F is a free semigroup then this definition coincides with the definition used in Section 2.9. Notice also that the applicability problem discussed in Section 2.9 is, in the case of semigroups, the divisibility problem: "Given two elements $a, b \in F$, decide whether a is divisible by b in F" (we have discussed this problem before, see Theorems 3.28, 3.29). Thus with every set of pairs Σ from $F \times F$ we can associate a rewrite system $\Omega(F, \Sigma)$.

Of course, in order to find terminating Church-Rosser rewrite systems we shall use the Knuth-Bendix procedure (see Section 2.9). As we have mentioned in Section 2.9, before we are able to apply this procedure, we must meet some requirements.

First of all we shall assume that our semigroup F is ordered. This order must be stable with respect to multiplication. We shall assume that if s divides t in F then $s \leq t$. We shall also assume that for every element $s \in F$ there are only finitely many decompositions $s = s_1 s_2$ with $s_i \in F$. The latter assumption holds in any relatively free non-periodic semigroup. If these assumptions hold we shall call F *successfully ordered*. The absolutely free semigroups and the free commutative semigroups are successfully ordered by the ShortLex order.

We also need a definition of a critical pair. Some particular cases of our definition may be found in Bergman [34], Gilman [110] and Pedersen [292]). Let F be a finitely generated semigroup. An *overlap* of two relations $(p_1, q_1), (p_2, q_2) \in F \times F$ is a quadruple of elements $s_1, t_1, s_2, t_2 \in F$ such that $s_1 p_1 t_1 = s_2 p_2 t_2$ and $s_1 q_1 t_1 > s_2 q_2 t_2$. This overlap is called *critical* if there is no other overlap s_1', t_1', s_2', t_2' of the same relations such that:

1. $s_1' p_1 t_1' < s_1 p_1 t_1$;

2. for some elements $x, y \in S^1$

$$s_1 p_1 q_1 = x s_1' p_1 t_1' y, \; s_1 q_1 t_1 = x(s_1 q_1 t_1) y, \; s_1 q_1 t_1 > x(s_2' q_2 t_2') y.$$

The *critical pair* determined by such a critical overlap is the pair of elements $s_1 q_1 t_1$, $s_2 q_2 t_2$ of F.

Now let us consider the question of when a variety of semigroups is Church-Rosser (see the definition in Section 2.9). Periodic finitely based varieties of semigroups with solvable word problem are either locally finite (hence Church-Rosser, see Section 2.9) or contain "bad" groups (see Theorems 3.28, 3.29). Therefore we can restrict ourselves to non-periodic varieties.

It turns out that if $\mathcal{V}$ is a locally residually finite variety of semigroups and all free semigroups in $\mathcal{V}$ are successfully ordered then we always have a finite number of critical pairs and the Knuth-Bendix procedure always terminates. So we have the following result.

Theorem 3.38 *(Sapir, [338]). Let $\mathcal{V}$ be a finitely based variety which either is non-periodic or contains no infinite finitely generated groups of finite exponent. Suppose $\mathcal{V}$ is locally residually finite and that the free semigroups in $\mathcal{V}$ are successfully ordered. Then the Knuth-Bendix procedure applied to any finite $\mathcal{V}$-presentation of a semigroup $S \in \mathcal{V}$ halts, producing a Church-Rosser terminating presentation of S. Therefore $\mathcal{V}$ is a Church-Rosser variety (the free semigroups of $\mathcal{V}$ play the role of pseudo-free semigroups in the definition of Church-Rosser varieties).*

This theorem generalizes the main results in [292] and [110].

Not every locally residually finite variety has orderable free semigroups. The following example is due to the second author of this survey. Let us take the variety $\mathcal{V}$ given by the following identities:

1. $xyxzxyxtxyxzxyx = x^2yzxyxtxyxzxyx$,
2. $xy^8 = y^8x$,
3. $x^2y^2z^3t^3z^3x^2y^2 = y^2x^2z^6t^3x^2y^2$,
4. $x^2y^2z^6t^3x^2y^2 = y^2x^2z^3t^3z^3x^2y^2$.

The first two identities make $\mathcal{V}$ locally residually finite (see Theorem 3.37). Now take $X = \{x, y, z, t\}$ and suppose that $F = F_X(\mathcal{V})$ admits a complete stable order $<$. It is easy to see that words $u = z^3t^3z^3$ and $v = z^6t^3$ represent different elements in F. Thus either $u < v$ or $v < u$. Suppose that $u < v$. Since $<$ is a stable order, we must have that

$$x^2y^2ux^2y^2 \leq x^2y^2vx^2y^2,\ y^2x^2ux^2y^2 \leq y^2x^2vx^2y^2. \tag{11}$$

But identities 3 and 4 yield

$$x^2y^2ux^2y^2 = y^2x^2vx^2y^2,\ x^2y^2vx^2y^2 = y^2x^2ux^2y^2.$$

This and (11) imply:

$$y^2x^2vx^2y^2 \leq y^2x^2ux^2y^2 \leq y^2x^2vx^2y^2.$$

Hence

$$y^2x^2vx^2y^2 = y^2x^2ux^2y^2$$

But it is easy to see that this identity does not follow from identities 1 – 4. This contradiction shows that the free semigroups from $\mathcal{V}$ are not orderable.

Nevertheless there are enough locally residually finite varieties with orderable free semigroups.

Theorem 3.39 *(Sapir, [338]). Every non-periodic variety $\mathcal{V}(n, W)$ of semigroups given by two identities $Z_n = W$ and $x^{2n}y = x^n y x^n$ has orderable free semigroups. Therefore every non-periodic finitely based locally residually finite variety is Church-Rosser (one can take free semigroups in the corresponding variety $\mathcal{V}(n, W)$ as pseudo-free semigroups for $\mathcal{V}$).*

This theorem shows yet again that locally residually finite varieties of semigroups inherit many important properties of the variety of commutative semigroups. We have mentioned above (Section 2.9) that rewrite systems of relations in free commutative semigroups are exactly the Petri nets. Thus rewrite systems of relations in free semigroups of locally residually finite varieties of semigroups are non-commutative analogs of Petri nets.

3.6 The Club of Residually Finite Varieties

Permanent members: "To have solvable elementary theory", "To have solvable elementary theory of the finite trace", "To be residually finite", "To be residually small", "To have finitely many quasi-varieties", "To have at most countably many subquasi-varieties".

3.6.1 The Elementary Theory

A description of locally finite group varieties with decidable elementary theories was obtained by Ershov [93]. He proved that a locally finite variety of groups has a decidable elementary theory if and only if it is Abelian.

Based on this result, Zamjatin found a complete list of locally finite semigroup varieties with decidable elementary theories [424]. Then Zamjatin proved, solving a Tarski problem, that every group variety with a decidable elementary theory is Abelian [422]. This result and results from [424] imply that every semigroup variety with a decidable elementary theory is locally finite. Thus the main theorem of [424] gives in fact the list of all semigroup varieties with decidable elementary theories. In order to formulate this theorem we need some notation:

- $\mathcal{L}$ is the variety of left-zero semigroups (=var$\{xy = x\}$),
- $\mathcal{R}$ is the variety of right-zero semigroups (=var$\{xy = y\}$),
- $\mathcal{N}$ is the variety of semigroups with zero product (=var$\{xy = 0\}$),
- $\mathcal{I}$ is the variety of all commutative idempotent semigroups (=var$\{xy = yx,\ x^2 = x\}$).
- L is the two element left zero semigroup,
- R is the two element right semigroup,

- N is the two element semigroup with zero product,

- I is the two-element commutative idempotent semigroup,

- M_n is the Rees matrix semigroup over the cyclic group of order n with 2×2-sandwich matrix $P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, that is the set of triples $\{(i, g, j) \mid i, j \in \{1,\ 2\}, g \in \mathbf{Z}_n\}$ with the following multiplication $(i, g, j)(k, h, \ell) = (i, g + P(k, j) + h, \ell)$ where "+" is the operation in $\mathbf{Z}_n$.

- If $\mathcal{V}_1$ and $\mathcal{V}_2$ are two varieties of semigroups then $\mathcal{V}_1 \times \mathcal{V}_2$ denotes the class of all direct products $A \times B$ where $A \in \mathcal{V}_1$, $B \in \mathcal{V}_2$; $\mathcal{V}_1 + \mathcal{V}_2$ denotes the joins of these varieties in the lattice of all semigroup varieties.

Theorem 3.40 *(Zamjatin, [424], [422]). A variety of semigroups has a decidable elementary theory if and only if it coincides with one of the following varieties: $\mathcal{G}$, $\mathcal{G} \times \mathcal{L}$, $\mathcal{G} \times \mathcal{R}$, $\mathcal{G} \times \mathcal{L} \times \mathcal{R}$, $\mathcal{N}$, $\mathcal{L} + \mathcal{N}$, $\mathcal{R} + \mathcal{N}$, where $\mathcal{G}$ is an Abelian group variety of finite exponent.*

From the proof of this result, it is easy to deduce a characterization of varieties with decidable elementary theories in terms of "forbidden" semigroups. As far as we know, this result is published here for the first time. In this theorem and later *wr* denotes the wreath product.

Theorem 3.41 *A variety of semigroups has a decidable elementary theory if and only if it is locally finite and does not contain the following finite semigroups*

- $\mathbf{Z}_p wr \mathbf{Z}_q$ *where p, q are distinct primes,*

- *The nilpotent non-Abelian groups of order p^3 where p is a prime,*

- *The semigroup I,*

- *The semigroup M_n for every $n > 1$,*

- *The four element commutative 3-nilpotent semigroup $\{a, b, c, 0\}$ where $ab = c = ba$, all other products are equal to 0,*

- *The semigroup $L \times R \times N$,*

- *The semigroup $\mathbf{Z}_n \times N$ where $\mathbf{Z}_n$ is a non-trivial cyclic group.*

While the list of forbidden semigroups in this theorem is infinite, it contains only finitely many semigroups of any given period. Given a finite set of identities, which are known to define a periodic variety of semigroups $\mathcal{V}$, it is easy to find out the period of $\mathcal{V}$. Then we have to check only a finite portion of this list.

Theorem 3.41 gives an algorithmic description of these varieties in the class of locally finite varieties. The problem of a complete algorithmic description of these varieties is equivalent to the problem of finding an algorithm, which, given a finite number of group identities, decides if the corresponding variety is Abelian of finite exponent. The last problem seems to be hopeless at the present time.

Notice that the results of Zamjatin may be deduced from general results of McKenzie and Valeriote [260] on locally finite varieties of arbitrary universal algebras with decidable elementary theory.

In [421] Zamjatin considered elementary theories of finite traces of semigroup varieties.

Theorem 3.42 *(Zamjatin, [421]). Let $\mathcal{V}$ be a semigroup variety. Then the elementary theory of $\mathcal{V}_{fin}$ is decidable if and only if $\mathcal{V}$ coincides with one of the following varieties: $\mathcal{G}$, $\mathcal{G} \times \mathcal{L}$, $\mathcal{G} \times \mathcal{R}$, $\mathcal{G} \times \mathcal{L} \times \mathcal{R}$, $\mathcal{N}$, $\mathcal{L} + \mathcal{N}$, $\mathcal{R} + \mathcal{N}$, where $\mathcal{G}$ is a periodic variety of groups where all finite groups are Abelian.*

Thus it turned out that for every variety $\mathcal{V}$ such that the elementary theory of $\mathcal{V}_{\text{fin}}$ is decidable there exists another variety $\mathcal{U}$ such that $\mathcal{V}_{\text{fin}} = \mathcal{U}_{\text{fin}}$ and the elementary theory of $\mathcal{U}$ is decidable. If $\mathcal{V}$ is locally finite then one can take $\mathcal{U} = \mathcal{V}$. Notice that in the case of rings, for example, the situation is quite different. From Zamjatin's Theorems 4.27 and 4.28, it follows that, say, the variety generated by the ring $\mathbf{Z}/4\mathbf{Z}$ (integers modulo 4) has an undecidable elementary theory, but its finite trace has a decidable elementary theory.

3.6.2 Residually Finite Varieties

Residually finite varieties, that is varieties consisting of residually finite algebras, are exactly varieties where every subdirectly irreducible algebra is finite. Subdirectly irreducible algebras play very important role in the theory of varieties because by the Birkhoff theorem every variety is generated by its subdirectly irreducible members. This is why residually finite varieties of algebras have attracted a robust interest during the last 25 years.

In 1969 Ol'shanskii [289] described all residually finite varieties of groups, i.e. varieties which consist of residually finite groups.

Theorem 3.43 *(Ol'shanskii, [289]). A group variety is residually finite if and only if it is generated by a finite group with Abelian Sylow subgroups.*

This theorem stimulated all further investigations of residually finite varieties of semigroups, rings, Lie rings, and universal algebras in general.

As far as semigroups are concerned, descriptions of residually finite varieties have been found in some particular cases by Gerhard [107] and Mamikonian [241]. A complete description of residually finite varieties of semigroups has been found by

Golubov and the second author of this survey [113], [114]. A full proof appeared in [112]. Independently but later other descriptions were found by Kublanovskii [210] and R.McKenzie [259]. We present here an algorithmic description of these varieties in terms of "forbidden" semigroups, which can be deduced from descriptions in Sapir [349], Golubov and Sapir [112], Sapir [350] and is published here for the first time. Notice that the three element semigroup P and its mirror image $\overleftarrow{P}$ appear in this description once again (it appeared in the descriptions of varieties with solvable word problem, and locally residually finite varieties). We shall also meet these semigroups a number of times later. Perhaps there exists some mysterious connection between residual finiteness and these semigroups.

Theorem 3.44 *(Sapir). A semigroup variety $\mathcal{V}$ is residually finite if and only if it is locally finite and does not contain the following semigroups:*

- $(\mathbf{Z}_p wr \mathbf{Z}_q) wr (\underbrace{\mathbf{Z}_r \times \mathbf{Z}_r \times \ldots \times \mathbf{Z}_r}_{n})$ *where p, q, r are distinct primes, $n = 1, 2, \ldots$,*
- *The non-Abelian groups of order p^3 where p is a prime,*
- M_n, $n > 1$,
- $P \times (\mathbf{Z}_p wr \mathbf{Z_q})$ *where p, q are distinct primes,*
- $\overleftarrow{P} \times (\mathbf{Z}_p wr \mathbf{Z_q})$ *where p, q are distinct primes,*
- *The semigroups $P \times L$ and $\overleftarrow{P} \times R$,*
- *The three element semigroup L^1,*
- *The four element commutative 3-nilpotent semigroup $\{a, b, c, 0\}$ where $ab = c = ba$, all other products are equal to 0,*
- *The four element semigroup $W_1 = \{a, b, c, d\}$ with the following multiplication table:*

	a	b	c	d
a	a	a	a	a
b	d	d	a	d
c	c	c	c	c
d	d	d	d	d

  and the semigroup $\overleftarrow{W}_1$,

- *The seven element semigroup $W_2 = \{a, b, c, d, e, f, 0\}$ with the following multiplication table:*

	0	a	b	c	d	e	f
0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	0
b	0	0	0	0	0	0	0
c	0	a	b	c	d	a	b
d	0	a	b	c	d	b	a
e	0	0	0	0	0	0	0
f	0	0	0	0	0	0	0

  and the semigroup $\overleftarrow{W}_2$.

Again this list of semigroups is "almost finite". Given a finite set of identities, we have to check only a finite part of this list.

This theorem and Theorem 3.41 show that every variety with decidable elementary theory is residually finite. The converse is not true even for periodic group varieties. But still these two descriptions have much in common.

As a corollary of Theorem 3.44 one can deduce that every residually finite variety is of index 2 (in the sense of the Sapir-Sukhanov hierarchy from Section 3.3.3).

Golubov and Sapir [114], Kublanovskii [209], and Savina [359], [358], [361], [360] (see also [368]) found algorithmic descriptions of varieties of semigroups with the following properties:

- Every semigroup is residually finite with respect to membership in a subsemigroup;

- Every semigroup is residually finite with respect to membership in a (left, right) ideal;

- Every semigroup is residually finite with respect to the regularity (for every non-regular element there is a homomorphism onto a finite semigroup such that the image of this element is not regular);

- Every semigroup has recognizable ideals (that is every ideal is a union of classes of a congruence of finite index).

All these properties turned out to be "almost" equivalent to residual finiteness, but, for example, in the case of residual finiteness with respect to membership in an ideal, we do not have restrictions on the group subvariety (because every group is residually finite with respect to membership in ideals).

Another property which turned out to be equivalent to residual finiteness in all known cases, is the property "to be residually small". A variety is called *residually small* if the orders of its subdirectly irreducible algebras are bounded by some cardinal (recall that in residually finite varieties these orders are finite). This property is very closely connected to many nice properties of varieties (see [396]), so it is very popular among specialists in universal algebra.

Residually small varieties of semigroups have been studied by McKenzie [259], [256]. He showed that the problem of describing these varieties would be reduced to the analogous problem for periodic group varieties, if a certain group-theoretic problem had a positive solution. Then the second author of this survey and Shevrin [354] solved this problem positively, which completed the description of residually small semigroup varieties modulo periodic group varieties. Every locally finite residually small group variety is residually finite (this follows from Olshanskii's results [289]), so, in the case of locally finite varieties of semigroups, the properties "to be residually finite" and "to be residually small" are equivalent. Examples of non-locally finite residually small varieties of periodic groups are not known, and it seems very unlikely that such examples exist. Thus there is a high probability that these two properties are equivalent in the class of all semigroup varieties.

3.6.3 Subquasi-Varieties

A quasi-variety is a class of algebras defined by quasi-identities.

By a result of Mal'cev every two distinct finite subdirectly irreducible algebras generate distinct quasi-varieties. Thus a locally finite variety of algebras with finitely many quasi-varieties must be residually finite. By a result of the second author of this survey [351], for groups and associative rings the converse statement is also true. But this is not the case for semigroups.

A complete description of such varieties appeared in [351]. Here we present an algorithmic description which can be deduced from [351] but has not been published before. The main characters of this description are the familiar semigroups L, R, N, I, P.

Theorem 3.45 *(Sapir, [351]). A locally finite variety of semigroups contains only finitely many quasi-varieties if and only if it is residually finite and does not contain the following semigroups:*

- *The semigroup $I \times \mathbf{Z}_{p^6}$ where p is a prime,*
- *Semigroups $P \times \mathbf{Z}_{p^3}$ and $\overleftarrow{P} \times \mathbf{Z}_{p^3}$, where p is a prime,*
- *The semigroup $N \times (\mathbf{Z}_p wr \mathbf{Z}_q)$ where p, q are distinct primes,*
- *The semigroup $I \times (\mathbf{Z}_p wr \mathbf{Z}_q) \times (\mathbf{Z}_r wr \mathbf{Z}_p)$ where p, q, r are distinct primes,*
- *The semigroup $I \times (\mathbf{Z}_p wr \mathbf{Z}_q) \times Z_{p^2}$.*

The proof of this theorem is very non-trivial. It involves, in particular, a careful study of finite Abelian groups with one or two distinguished subgroups. It turned out that the properties of such systems depend very much on whether the exponent of the group is free of 6-th powers (resp. 3-d powers) or not.

3.7 The Membership Problem For Pseudovarieties of Finite Semigroups

It would take too much space to review all known algorithmic results about the membership problem for pseudovarieties of finite semigroups. Fortunately there are good surveys, published, in particular, in this journal, devoted exclusively to this subject (see, for example, [141], [242]). This is actually a whole theory where semigroups are mixed with automata, languages, profinite groups, etc. John Rhodes calls this devil mixture "Global Semigroup Theory". We will briefly discuss only four interesting topics of this theory.

A pseudovariety may be generated by a known class of semigroups (say, by one finite semigroup). It may appear as a result of a construction (the Mal'cev product, the join, etc.). It may be given by identities or so called pseudoidentities. Finally it may appear accidentally: an important class of finite semigroups turns out to be a pseudovariety. Thus we will consider four cases of the membership problem and give some examples of the solutions in each of these cases. We will also consider (when possible) the computational complexity issues. For every pseudovariety $\mathcal{P}$ the instance of the membership problem is the Cayley table of a finite semigroup.

3.7.1 Pseudovarieties Generated by Important Classes of Semigroups

If a semigroup S is a homomorphic image of a subsemigroup of a semigroup T then we will say that S *divides* T.

If $\mathcal{C}$ is a class of finite semigroups then pvar$\mathcal{C}$ is the minimal pseudovariety containing $\mathcal{C}$: that is the set of all semigroups which divide finite direct products of semigroups from $\mathcal{C}$. If $\mathcal{C}$ is closed under finite direct products then pvar$\mathcal{C}$ is just the set of semigroups which divide semigroups from $\mathcal{C}$. The following result, obtained by C.Ash [14], answered a question by Pin.

Theorem 3.46 *(Ash, [14]). The pseudovariety generated by the class of all finite inverse semigroups coincides with the class of all finite semigroups where idempotents commute.*

Thus this pseudovariety has decidable and even polynomial time membership problem. As a corollary, one can conclude that every finite semigroup with commuting idempotents divides a semidirect product of a finite semilattice (commutative idempotent semigroup) and a finite group. Indeed, as we have pointed out in Section 3.3.5 every inverse semigroup divides such a semidirect product.

Using Ash's method Birget, Margolis, and Rhodes [39] proved the following generalization of Theorem 3.46.

Theorem 3.47 *(Birget, Margolis, Rhodes, [39]). The pseudovariety generated by all finite regular semigroups whose idempotents form subsemigroups (such semigroups* *are called orthodox), coincides with the set of all finite semigroups whose idempotents form subsemigroups.*

Every regular semigroup, whose idempotents form a subsemigroup, divides a semi-direct product of an idempotent semigroup and a group [39]. Thus every finite semigroup whose idempotents form a subsemigroup divides such a semi-direct product.

One of the most important classes of finite semigroups is the class of 0-simple finite semigroups. Recall that a semigroup is called 0-simple if it does not have ideals except itself and possibly $\{0\}$. Every finite semigroup may be obtained from 0-simple semigroups by a sequence of ideal extensions. Thus the role of finite 0-simple semigroups in the theory of semigroups is similar to the role of finite simple groups in the theory of groups. Every group is, of course, a 0-simple semigroup. The classic theorem of Sushkevich [73] shows that finite 0-simple semigroups have the following structure. Let G be a finite group, let L and R be two finite sets and let P be an $R \times L$-matrix over the group G with 0 adjoint such that every row and every column of P contains a non-zero element. Let $M^0(G; L, R, P)$ be the set $L \times G \times R \cup \{0\}$ with the following binary operation:

$$(\ell, g, r)(\ell', g', r') = \begin{cases} (\ell, gP_{r\ell'}g', r') & \text{if } P_{r\ell'} \neq 0; \\ 0 & \text{if } P_{r\ell'} = 0. \end{cases}$$

Then $M^0(G; L, R, P)$ is a finite 0-simple semigroup and every finite 0-simple semigroup is isomorphic to $M^0(G; L, R, P)$ for some G, L, R, P. If $L = R$ and P is the identity matrix then $M^0(G; L, R, P)$ is called the *Brandt semigroup over the group G*. It is denoted by $B_L(G)$. Brandt semigroups are precisely the 0-simple inverse semigroups. As one can see the structure of finite 0-simple semigroups and finite 0-simple inverse semigroups is extremely clear. Thus the following result of Kublanovsky was very unexpected.

Theorem 3.48 *(Kublanovsky, 1994). The set of all subsemigroups of finite 0-simple semigroups is not recursive. The set of finite semigroups in the quasi-variety generated by the class of finite 0-simple semigroups is not recursive also.*

Kublanovsky uses the unsolvability of the uniform word problem for finite groups (Slobodskoii [378]) and Connection 2.2. Then T.Hall, Kublanovsky and Sapir proved the following stronger result.

Theorem 3.49 *(T.Hall, Kublanovsky, Sapir, [140]). For every pseudovariety of finite groups $\mathcal{V}$ the following conditions are equivalent.*

1. *The uniform word problem in $\mathcal{V}$ is solvable.*

2. *The set of subsemigroups of finite 0-simple semigroups $M^0(G; L, R, P)$ with $G \in \mathcal{V}$ is recursive.*

3. *The set of finite subsemigroups of Brandt semigroups $B_L(G)$ with $G \in \mathcal{V}$ is recursive.*

4. *The set of finite nilpotent of degree 4 subsemigroups of finite 0-simple semigroups $M^0(G; L, R, P)$ with $G \in \mathcal{V}$ is recursive.*

5. *The set of finite nilpotent of degree 3 subsemigroups of finite Brandt semigroups $B_L(G)$ with $G \in \mathcal{V}$ is recursive.*

6. *The set of finite nilpotent of degree 4 semigroups in the quasi-variety generated by finite 0-simple semigroups $M^0(G; L, R, P)$ with $G \in \mathcal{V}$ is recursive.*

7. *The set of finite nilpotent of degree 3 semigroups in the quasi-variety generated by finite Brandt semigroups $B_L(G)$ with $G \in \mathcal{V}$ is recursive.*

Remark. The degrees 3 and 4 in Conditions 4 - 7 are impossible to make smaller (see [140]).

This result is in constrast with the following result from [140].

Theorem 3.50 *(Kublanovsky, Margolis, Sapir, Trotter, [140]). Let $\mathcal{V}$ be a decidable pseudovariety of groups. Then the pseudovariety $\mathcal{CS}^0(\mathcal{V})$ generated by finite 0-simple semigroups $M^0(G; L, R, P)$ with $G \in \mathcal{V}$ is decidable and the pseudovariety $\mathcal{B}(\mathcal{V})$ generated by all Brandt semigroups $B_L(G)$ with $G \in \mathcal{V}$ is decidable. These two pseudovarieties are finitely based provided $\mathcal{V}$ is finitely based. In particular the membership problem for the pseudovariety generated by all finite 0-simple semigroups and the membership problem for the pseudovariety generated by all finite Brandt semigroups are decidable in polynomial time.*

An important step in the proof of Theorem 3.50 is the representation of the pseudovariety $\mathcal{CS}^0(\mathcal{V})$ as a star-product $(\mathcal{I} + \mathcal{V}) * \mathcal{R}$ where as above $\mathcal{I}$ is the variety of semilattices and $\mathcal{R}$ is the variety of right zero semigroups. The star-products of pseudovarieties are defined in the next section.

3.7.2 Operations on Pseudovarieties

We have already met some of important operations on pseudovarieties. Given two pseudovarieties $\mathcal{B}$ and $\mathcal{C}$ of finite semigroups one can consider:

- The join $\mathcal{B} + \mathcal{C}$, that is the set of all finite semigroups which divide direct products $B \times C$ where $B \in \mathcal{B}$, $C \in \mathcal{C}$;

- The star-product $\mathcal{B} \star \mathcal{C}$, that is the set of all finite semigroups which divide semi-direct products $B\lambda C$ where $B \in \mathcal{B}$, $C \in \mathcal{C}$;

- The Mal'cev product $\mathcal{B}m\mathcal{C}$, that is the set of all finite semigroups which divide semigroups which have homomorphisms onto semigroups from $\mathcal{C}$ such that each pre-image of an idempotent belongs to $\mathcal{B}$.

There is a natural connection between finite semigroups and automata via transition semigroups. Then the join of varieties corresponds to the parallel connection of automata, the star-product corresponds to the sequential connection of automata. The Mal'cev product does not correspond to anything natural, but it coincides with the star-product in many important cases (like the case of group pseudo-varieties, and cases considered in the previous subsection).

We have seen that the join of two pseudovarieties with decidable membership problem may have undecidable membership problem (see Theorem 3.18). Rhodes conjectured that the same is true for Mal'cev and star-products [141]. But very often joins or the star-product are decidable. We refer again to Theorems 3.46 and 3.47, 3.50.

The star-product is especially important because of the following variant of the celebrated Krohn-Rhodes theorem.

Theorem 3.51 *Let $\mathcal{A}$ be the pseudovariety of all aperiodic finite semigroups (that is semigroups with trivial subgroups). Let $\mathcal{G}$ be the pseudovariety of all finite groups. Let us define a pseudovariety $\mathcal{R}_n$ as follows:*

$$\mathcal{R}_0 = \mathcal{A}, \ldots, \mathcal{R}_{n+1} = \mathcal{A} \star \mathcal{G} \star \mathcal{R}_n.$$

Then every finite semigroup belongs to $\mathcal{R}_n$ for some n.

The minimal number n such that $S \in \mathcal{R}_n$ is called the *group complexity* of S. The problem of recognizing the group complexity, that is the membership problem for the pseudovariety $\mathcal{R}_n$ is the famous group complexity problem, posed by Rhodes. The answer in general is not known even for $n = 1$. But the group complexity problem for inverse and completely regular semigroups is known to be decidable [207]. This means that, given a completely regular or inverse semigroup S one can effectively decide what the group complexity of S is. It is interesting that methods used by Krohn, Rhodes, and Tilson in [207], in particular, the Rhodes expansion, are similar to methods used later by L.Polák and others in their study of completely regular varieties of semigroups.

The latest achievement in the group complexity problem was made by C.Ash [15]. He proved the so-called Rhodes type II conjecture, one of the main consequences of which is the following.

Theorem 3.52 *(Ash, [15]). If the membership problem is decidable for a pseudovariety $\mathcal{V}$ then it is decidable for the pseudovariety $\mathcal{V}m\mathcal{G}$.*

In fact the type II conjecture provides an algorithm which solves the membership problem in $\mathcal{V}m\mathcal{G}$. With every finite semigroup one can associate a subsemigroup $D(S)$ which is called the *type II subsemigroup*. It is constructed inductively as follows:

1. All idempotents belong to $D(S)$,

2. If $a \in D(S)$, and for some $b, c \in S$ $b = bcb$ then $bac, cab \in D(S)$.

It is obvious that given S one can easily (in polynomial time) construct $D(S)$.

One of the formulations of the type II conjecture is the following: *A semigroup S belongs to $\mathcal{V}m\mathcal{G}$ if and only if $D(S) \in \mathcal{V}$.*

This theorem implies Theorems 3.46 and 3.47 above (since the variety of semilattices and the variety of all idempotent semigroups obviously have decidable membership problems). It has many other corollaries as well (see surveys [141] and [242]).

In the proof of Theorem 3.52 Ash heavily uses the theory of inverse semigroups, in particular, covers defined above in Section 3.3.5, and the Ramsey-type technique developed by him when he was proving Theorem 3.46.

Later Ribes and Zalesskii [320] found a "profinite" proof of Theorem 3.52. A connection between the type II conjecture and questions on profinite groups has been discovered by Pin, Reutenauer and Margolis [298], [245]. They proved that the following conjecture is equivalent to the type II conjecture: *Let $H_1, H_2, \ldots, H_n$ be finitely generated subgroups of a free group F. Then the product $H_1H_2\cdots H_n$ is closed in the profinite topology of F.* Recall that the basis of the profinite topology on F is the set of all subgroups of F of finite index. For $n = 1$ this conjecture is exactly the famous result of M.Hall. Ribes and Zalesskii proved this conjecture using the theory of profinite groups acting on profinite graphs, developed earlier by Gildenhuys, Ribes, and Zalesskii.

It is amazing that the theory of finite semigroups is so closely connected with the theory of profinite groups. It would be very interesting to compare the semigroup theoretic proof of Ash and the topological proof of Ribes and Zalesskii, and discover new relationships between these two areas.

3.7.3 Identities and Pseudo-Identities

When the membership problem is decidable for a pseudovariety $\mathcal{V}$, the computational complexity of this problem becomes very important. It would be very interesting, in particular, to characterize pseudovarieties which have polynomial time membership problem.

The first step in this direction was made by J.Almeida [10]. It is easy to see that if a pseudovariety $\mathcal{V}$ is given by a finite number of identities, that is if it is a finite trace of a finitely based variety, then the membership problem for $\mathcal{V}$ is decidable in polynomial time. J.Almeida conjectured that the converse statement is also true. In fact his conjecture is more general. By a theorem of Reiterman [314] every pseudovariety of semigroups may be given by so called pseudoidentities, which are basically pairs of elements of the free pro-$\mathcal{V}$ semigroup. Almeida [10] conjectured that *a pseudovariety has a polynomial membership problem if and only if it has a finite basis of pseudoidentities, each of which may be computed in polynomial time.* The second author of this survey constructed the following counterexample to the conjecture of Almeida. This example appears here for the first time. The mysterious three-element semigroups P and $\overleftarrow{P}$ play here a role again.

Theorem 3.53 *(Sapir). Let $\mathcal{N}$ be the 2-nilpotent group variety of exponent 4, and for every $n \geq 1$ let w_n be the word $x_1^3y_1^3x_1y_1 \cdots x_n^3y_n^3x_ny_n$. Let $\mathcal{S}$ be the intersection of the variety var$(P\times \overleftarrow{P}) + \mathcal{N}$ and the variety given by all identities of the form $zw_nz = (zw_nz)^2$. Then the following conditions hold:*

1. *$\mathcal{S}$ is generated by a finite semigroup,*
2. *$\mathcal{S}$ does not have a finite basis of pseudo-identities,*
3. *The membership problem for $\mathcal{S}_{\mathrm{fin}}$ is polynomial.*

The proof is based on the following facts:

- The variety $\mathcal{S}$ is generated by a finite semigroup and is not finitely based. This is proved in [344].
- If a finite semigroup has a finite basis of identities in the class of finite semigroups then it has a finite basis of identities in the class of all semigroups. This is proved in [348].
- If a finite semigroup has a finite basis of pseudo-identities then it has a finite basis of identities in the class of all finite semigroups. This is proved in [10].
- The membership problem for $\mathcal{S}_{\mathrm{fin}}$ is polynomial. This is a new result.

In fact we do not know any pseudovariety generated by a finite semigroup which has a non-polynomial time membership problem.

Problem 3.11 *Is there a pseudovariety generated by a finite semigroup whose membership problem is not polynomial time? Is the membership problem for the pseudovariety generated by the Brandt monoid (2) polynomial time?*

The fact that we do not know the answer to such questions is very annoying. This means that we still know very little about finite semigroups.

Another counterexample to Almeida's conjecture has been found independently by Volkov [413]. He proved, in particular, that the Mal'cev product of the pseudovariety of all aperiodic finite semigroups and the pseudovariety of all finite groups is not finitely based. The fact that the membership problem for this pseudovariety is decidable in polynomial time follows from Ash's Theorem 3.52. Notice that this pseudovariety is not generated by a finite semigroup, and is not a finite trace of any variety. In the case of inverse semigroups a counterexample to Almeida's conjecture has been found by Margolis (unpublished). He proved that the membership problem for the Mal'cev product of the pseudovariety of semilattices and the pseudovariety of Abelian groups of exponent 2 is solvable in polynomial time. The fact that this pseudovariety is not finitely based follows from results of E.Kleiman [194].

While Almeida's conjecture turned out to be incorrect, it seems to point in the right direction. It is quite possible that the property "the membership problem is solvable in polynomial time" may be expressed by a simple second order formula. Connection 2.13 and similar connections discussed in Section 2.8 also hints at such a possibility.

3.7.4 Inherently Non-Finitely Based Finite Semigroups

One of the most famous algorithmic problems about finite universal algebras is the following Tarski problem.

Problem 3.12 *(Tarski, [392]). Is the set of finite algebras, possessing a finite basis of identities, recursive?*

This problem was reduced to the case of groupoids (algebras with one binary operation) by R.McKenzie [257]. It was proved by McNulty and Shallon [262] that there are in a sense a "few" finitely based non-associative groupoids. This made associative groupoids, that is semigroups, an important case for the Tarski problem. In fact, McNulty and Shallon proved that there are "few" non-associative groupoids which are not inherently non-finitely based. We discussed this concept a little in Section 3.3.1. Recall that a locally finite variety of algebras (resp. a finite algebra) is called *inherently non-finitely based* if every locally finite variety, containing it, is not finitely based. If a variety (resp. an algebra) is not inherently non-finitely based then we shall call it *weakly finitely based.* McNulty and Shallon asked if there exists an inherently non-finitely based semigroup. This question has been answered by the second author of this survey (see Theorem 3.11 above): the six-element Brandt monoid (2) is inherently non-finitely based.

Then the following "weak" version of the Tarski problem arose:

Is the set of weakly finitely based finite semigroups decidable?

This problem also has been solved by the second author of this survey [341]. The answer is "Yes". The following theorems combine results from papers [346, 341]. Before we formulate this theorem, let us recall that if e is an idempotent of a semigroup S then the maximal subgroup of S containing e is denoted by S_e (e is the identity element in this subgroup); if G is a finite group then the upper hypercenter $\Gamma(G)$ is the last term of the upper central series of G.

Theorem 3.54 *(Sapir, [346], [341]). A locally finite variety of semigroups is weakly finitely based if and only if it satisfies a nontrivial identity of the form $Z_n = W$, where Z_n is the Zimin word, and n is a natural number.*

Theorem 3.55 *(Sapir, [346], [341]). For every finite semigroup S, the following conditions are equivalent:*

1. *S is weakly finitely based,*

2. *S satisfies a non-trivial identity of the form $Z_n = W$.*

3. *S satisfies a non-trivial identity of the form $Z_n = W$, where $n \leq |S|^2$, and W is obtained from Z_n by replacing some letters x_i by x_i^{p+1} where p is the period of the semigroup S,*

4. *For every idempotent $e = e^2 \in S$ the semigroup $T = eSe$ satisfies the following condition:*

   > *For every element $a \in T$ and for every idempotent $f \in TaT$ elements faf and $fa^{p+1}f$ belong to the same coset of the maximal subgroup T_f of T with respect to the upper hypercenter $\Gamma(T_f)$.*

This theorem has many corollaries. Let us mention only some of them (see also [341]).

Theorem 3.56 *(Sapir, [341]). Let Z_ω be the infinite (in both directions) word which is a limit of Zimin words Z_n. Let $S(Z_\omega)$ be the semigroup corresponding to Z_ω (more precisely, to the symbolic dynamical system generated by Z_ω, see Section 3.3.1). Then the following three conditions hold:*

1. *The variety generated by $S(Z_\omega)$ is the only minimal inherently non-finitely based non-group variety of semigroups.*

2. *Every inherently non-finitely based semigroup variety contains either $S(Z_\omega)$ or an inherently non-finitely based variety of groups.*

3. *The equational theory of $S(Z_\omega)$ is decidable.*

Remark. It is not known if there exist inherently non-finitely based varieties of groups. A celebrated result of Zelmanov [426] shows that there **are no** inherently non-finitely based group varieties of prime exponent. So there is a big probability that there are no inherently non-finitely based group varieties at all. If this turns out to be the case then $\mathrm{var}(S(Z_\omega))$ will be the *absolutely unique* minimal inherently non-finitely based semigroup variety.

The following corollaries are about finite semigroups.

Theorem 3.57 *(Sapir, [341]). A. If all subgroups of a finite semigroup S are nilpotent then S is weakly finitely based if and only if the Brandt monoid B_2^1 does not divide S.*

B. There exists an inherently non-finitely based finite semigroup S such that B_2^1 does not divide S.

Theorem 3.58 *(Sapir, [341]). The set of all weakly finitely based finite semigroups is a pseudovariety. This pseudovariety has a polynomial time membership problem.*

The second statement of Theorem 3.58 has not been published before. It follows from condition 4 of Theorem 3.55.

It is interesting that if we consider B_2^1 as an inverse semigroup, that is as an algebra with two operations: binary (multiplication) and unary (which in this case coincides with the operation of taking the transpose of a matrix), then it is no longer inherently non-finitely based. Moreover the following theorem of the second author of this survey holds [339].

Theorem 3.59 *(Sapir, [339]). Every finite inverse semigroup, considered as an algebra with two operations, is weakly finitely based.*

This theorem follows from the following two facts proved in [339]. Let Z_n' be the Zimin word Z_n without the rightmost letter. Then:

- Every finite inverse semigroup satisfies the identity
$$Z_n' = Z_n' x_1 x_1^{-1} \tag{12}$$
for some n;
- An inverse semigroup S is locally finite if it satisfies the identity (12) and all subgroups of S are locally finite.

Remark. Just recently, after the text of this section was written, Ralph McKenzie solved the Tarski problem (Problem 3.12) in the negative. More precisely, for every Turing machine T, calculating a partially recursive function $f(n)$, and every number n, McKenzie constructs a finite algebra $A(T,n)$ such that if $f(n)$ is defined then $A(T,n)$ is finitely based, and if $f(n)$ is not defined then $A(T,n)$ is inherently non-finitely based. Thus the following two sets of finite algebras are not recursive:

- The set of finitely based finite algebras;
- The set of inherently non-finitely based finite algebras.

This outstanding result sheds a new light on the questions discussed in this section. For example, since the set of inherently non-finitely based finite semigroups is recursive, finite semigroups behave better algorithmically than general finite algebras. And - who knows - perhaps the set of finitely based finite semigroups is also recursive?

4 Associative Algebras

4.1 Basic Definitions

We will use the terminology from Jacobson [157], Rowen [331], Kemer [171], Kemer [172].

We will mainly consider associative algebras with or without unit over a field, say K, although some of the results hold for arbitrary associative rings. *If in the formulation of a result we do not specify that we consider algebras with (without) unit then the result holds in both cases.*

We would like the ground field K not to be the main cause for the undecidability of algorithmic problems. In particular we would like to apply the McKinsey algorithm in order to solve the word problem in residually finite algebras. This means that we need to be able to count finite dimensional K-algebras. Thus we will always assume that K is countable and recursive. This means that there exists an effective enumeration of elements of K and the basic field operations of K are recursive functions.

Let S be a semigroup, let K be a field. Then the semigroup algebra KS is the algebra of formal sums $\sum_i \alpha_i s_i$ where $\alpha_i \in K,\ s_i \in S$ with the natural distributive multiplication. If S contains 0 then we will always identify 0 of S with 0 of K The algebra KS may be considered as an algebra generated by the direct product $K \times S$ subject to the relations:

$$(\alpha_1 + \alpha_2, s) = (\alpha_1, s) + (\alpha_2, s),\ (\alpha_1, s_1)(\alpha_2, s_2) = (\alpha_1\alpha_2, s_1 s_2),$$
$$\alpha(\alpha_1, s_1) = (\alpha\alpha_1, s_1),\ (\alpha, 0) = (0, 0).$$

The *free associative algebra* over a set X is the semigroup algebra KX^+.

A tensor product $A \otimes B$ of two K-algebras A and B is the algebra generated by $A \times B$ subject to the relations:

$$(a_1 + a_2, b) = (a_1, b) + (a_2, b),\ (a, b_1 + b_2) = (a, b_1) + (a, b_2),$$
$$(a_1, b_1)(a_2, b_2) = (a_1 a_2, b_1 b_2),\ \alpha(a, b) = (\alpha a, b) = (a, \alpha b).$$

If A is a finite dimensional algebra with a basis $a_1, \ldots, a_n$ and structure constants $\alpha_{i,j,k}$ (that is $a_i a_j = \sum_k \alpha_{i,j,k} a_k$), then the m-generated *algebra of generic elements* of A is generated by formal sums

$$\sum_i X_i^{(j)} a_i,\ j = 1, \ldots, m, \tag{13}$$

where $X_i^{(j)}$ are commuting unknowns, subject to the following relations

$$\sum_i X_i^{(p)} a_i \sum_i X_i^{(q)} a_i = \sum_{i,j} X_i^{(p)} X_j^{(q)} a_i a_j = \sum_{i,j,k} \alpha_{i,j,k} X_i^{(p)} X_j^{(q)} a_k.$$

In other words this is the subalgebra of the tensor product $K[X_1^{(1)}, \ldots, X_n^{(m)}] \otimes A$ generated by elements (13).

An *identity* in the case of associative algebras is simply a polynomial $\alpha_1 u_1 + \ldots + \alpha_n u_n$ where u_i is a word, α_i is an element of K. One says that this identity *holds* in an algebra R if this polynomial is identically 0 on this algebra. An algebra satisfying a non-trivial identity with GCD of coefficients equal to 1 is called a *PI-algebra.*

We shall denote the ring commutator $xy - yx$ by $[x, y]$, and write $[[x, y], z]$ as $[x, y, z]$.

A polynomial p is called *homogeneous* if every letter occurs the same number of times in each monomial of p.

An algebra (without a unit) is called *nilpotent of step k* if it satisfies the identity $x_1 x_2 \ldots x_{k+1}$. Notice that traditionally such associative algebras are called nilpotent of step $k+1$. We break the tradition for the sake of uniformity with the cases of Lie algebras and groups. The variety of all nilpotent algebras of step k is denoted by $\mathcal{N}_k$. The variety of commutative algebras is denoted by $\mathcal{A}$.

We denote by $\mathcal{W}_1\mathcal{W}_2$ the Mal'cev product of the varieties $\mathcal{W}_1$ and $\mathcal{W}_2$. For example, $\mathcal{N}_k\mathcal{A}$ is the variety of algebras with k-step nilpotent derived subalgebra. If $\mathcal{V}$ is a variety of algebras then the class of all algebras A which have a central ideal I such that $A/I \in \mathcal{V}$ is a variety denoted by $Z\mathcal{V}$. For example, $Z\mathcal{N}_k\mathcal{A}$ is the variety of center-by-$\mathcal{N}_k\mathcal{A}$ associative algebras.

A K-algebra is called *residually finite* if it has enough K-homomorphisms into finite dimensional K-algebras to separate every two distinct elements.

For every algebra K the algebra of all $n \times n-$matrices over K is denoted by $M_n(K)$.

Similar notation and definitions will be used in the group and Lie algebra cases in the next sections.

4.2 Overview

Algorithmic problems in varieties of associative algebras have a much better neighborhood than algorithmic problems in varieties of other systems considered in this survey.

Indeed, while in the case of groups and semigroups we struggle with periodic non-locally finite varieties, in the case of associative algebras we have the Kaplansky-Shirshov Theorem [161], [371] which states that every associative algebra with a non-trivial identity whose one-generated subalgebras are finite dimensional is locally finite dimensional.

While in the cases of semigroups and groups there are lots of varieties without finite bases of identities, in the case of associative algebras over fields of characteristic 0 we have the Kemer Theorem [170], which states that every variety is finitely based. In the case of positive characteristic, every variety of algebras over an infinite field, that does not contain the algebra of all 2×2-matrices, is finitely based (Krasil'nikov,

[204]). There is a strong hope that every variety of associative algebras over an infinite field is finitely based, and nobody knows any counterexamples even in the case of finite fields.

While in the cases of semigroups, groups and Lie algebras relatively free systems in varieties may have awful structure, by another of Kemer's theorems [173] every relatively free associative algebra over a field of characteristic 0 is representable by matrices over a direct power of the so-called Grassmann algebra, and if a variety is generated by a finitely generated algebra then its free algebras are in some sense almost finite dimensional. More precisely these are algebras of generic elements of some finite dimensional algebras.

Arbitrary algebras in varieties of associative algebras have nice properties too. By the theorem of Razmyslov, Kemer, and Braun [56] every finitely generated associative algebra A satisfying a non-trivial identity has a nilpotent ideal $J(A)$ (the Jacobson radical) such that $A/J(A)$ is a subdirect product of matrix algebras.

All these features make associative algebras a very nice place to work with algorithmic problems.

For more information about the theory of varieties of associative rings and algebras we refer the reader to the book of Rowen [331]. The current state of this theory is presented in a recent survey by Kemer [172].

Of course, sometimes associative algebras are harder to work with than, say, semigroups. For example, the simple tools like Minsky machines do not suffice to get deep enough in the lattice of varieties of associative algebras. Thus, this lattice is "deeper" (or would it be better to say "more viscous"?) than the lattice of varieties of semigroups. Also some of the properties which are equivalent in the case of semigroups turn out to be distinct in the case of associative algebras. It is actually interesting to see how the specifics of associative algebras influence the behavior of properties which belong to the same Club in the semigroup case.

4.3 The Identity Problem and Related Problems

Let $p = 0$ be an identity of associative algebras. The polynomial p can be represented as a sum of homogeneous polynomials $p = p_1 + \ldots + p_n$. It is well-known and easy to prove (see [172]) that in the case of associative algebras over an infinite field the identity $p = 0$ is equivalent to the system of identities $p_1 = 0, \ldots, p_n = 0$. Now let F be a finitely generated relatively free algebra of a variety of algebras over an infinite field. Then for every n, elements from F of degree $\geq n$ form an ideal. The intersection of all these ideals is 0. Each of them has a finite co-dimension. Thus every relatively free associative algebra over an infinite field is residually finite. Therefore it has a decidable word problem, and by Connection 2.5 we have the following result.

Theorem 4.1 *The equational theory is decidable in every variety defined by homogeneous identities. The identity problem is decidable in the variety of all associative algebras over an infinite field, and in all its subvarieties.*

Thus the equational problem in every variety of associative algebras over an infinite field is decidable.

Notice that a similar argument has been used to prove the decidability of the equational problem in non-periodic varieties of semigroups (see Theorem 3.15). Free semigroups in these varieties have a similar sequence of ideals of finite index (see Section 3.3.2).

As we mentioned before, residual finiteness provides very slow algorithms. In the case of characteristic 0 there exist much faster algorithms solving the equational problem.

The following result, noticed by the second author of this survey, is a direct consequence of Kemer's results [170], [173]. It is published here for the first time.

Theorem 4.2 *Let K be a field of characteristic 0 such that sums and products of elements of K are computable in polynomial time (say, the field of rational numbers). Then every variety of associative algebras over K has NP decidable equational theory.*

In order to show how this follows from Kemer's results let us recall some definitions. An algebra A is called a *superalgebra* if A has two distinguished subspaces A_0 and A_1 satisfying the following conditions:

$$A = A_0 \oplus A_1,\ A_0^2, A_1^2 \subseteq A_0, \quad A_0A_1, A_1A_0 \subseteq A_1.$$

The pair (A_0, A_1) is called the *grading* of A. Every associative algebra A may be considered as a superalgebra if one takes $A_0 = A, A_1 = 0$. This grading is called trivial. The most famous example of a superalgebra with a non-trivial grading is the *Grassmann algebra*. It is generated by an infinite set $\{e_1, e_2, \ldots\}$ with defining relations $e_ie_j = -e_je_i,\ i, j = 1, 2, \ldots$. The grading of G is defined as follows: G_0 is the subspace of G generated by the words of even length in the generators, and G_1 is the subspace generated by the words of odd length. For every superalgebra A with grading (A_0, A_1) the subalgebra $A_0 \otimes G_0 + A_1 \otimes G_1$ of the tensor product of A and G is called the *Grassmann hull* of the superalgebra A.

The following two outstanding results were proved by Kemer.

Theorem 4.3 *(Kemer, [170]). Every variety of associative algebras over a field of characteristic 0 is finitely based.*

Theorem 4.4 *(Kemer, [173]). Every non-trivial variety of algebras is generated by the Grassmann hull of some finite dimensional superalgebra. If a variety does not contain the Grassmann algebra (in particular, if it is generated by a finitely generated algebra) then it is generated by some finite dimensional algebra.*

Now take any variety of associative algebras over a field of characteristic 0. By Kemer's finite basis theorem (Theorem 4.3) it can be given by a finite number of identities. Kemer's proof of Theorem 4.4 shows that, given a finite system of identities Σ, one can effectively construct a finite dimensional superalgebra A such that the Grassmann hull of A generates the variety defined by Σ. It is clear that, given this algebra A, it takes only linear space and non-deterministic polynomial time to check if an identity $p = 0$ holds in the Grassmann hull of A (a similar fact for finite dimensional algebras was noted after Problem 2.4 in Section 2.8).

Problem 4.1 *(Sapir). Is there a variety of associative algebras over Q (or any other field of characteristic 0 where the addition and the multiplication are computable in polynomial time) such that its equational theory is not decidable in polynomial time?*

The residual finiteness of relatively free algebras over an infinite field implies the following result which also appears here for the first time. For every variety $\mathcal{V}$ of algebras $\mathcal{V}_{\text{fin}}$ denotes the class of all finite dimensional algebras of $\mathcal{V}$.

Theorem 4.5 *The identity problem for $\mathcal{V}_{fin}$ (the Rhodes problem) is decidable for every variety $\mathcal{V}$ of associative algebras over a field of characteristic 0.*

Problem 4.2 *Do analogs of Theorems 4.1 and 4.5 hold for algebras over finite fields?*

Notice also that the equational problem for varieties of associative rings (algebras over $\mathbf{Z}$) was formulated first by Mal'cev in 1966 (see [200], Problem 2.40b). This problem is still open.

4.4 The Word Problem

It is very easy to construct a finitely presented algebra over an arbitrary field which has an undecidable word problem: just take the semigroup algebra of a finitely presented semigroup S with an undecidable word problem. In particular S can be one of the semigroups S_1, S_2, $\overleftarrow{S}_2$ which generate minimal varieties of semigroups with undecidable word problem (see Section 7.2.4 below). The second author can show (unpublished) that varieties generated by KS_1, KS_2, $K\overleftarrow{S}_2$ coincide and do not depend on the corresponding Minsky machine. The same is true if we add units to S_1, S_2, $\overleftarrow{S}_2$ and consider varieties of unitary algebras generated by semigroup algebras of these monoids. Thus let S be S_1 or S_1^1. Then the algebra KS has the following three properties (see [268]):

1. It is finitely presented,
2. It has an undecidable word problem,
3. It has an ideal I, nilpotent of class 3, such that KS/I is commutative.

4. I^3 is in the center of KS.

The last two properties mean that KS belongs to the variety $\mathcal{N}_3\mathcal{A}\cap Z\mathcal{N}_2\mathcal{A}$. hence it satisfies the identities

$$[x_1,y_1][x_2,y_2][x_3,y_3][x_4,y_4]=0,\ [[x_1,y_1][x_2,y_2][x_3,y_3],z]=0$$

Recently the second author showed (unpublished) that KS belongs to the variety generated by the algebra $M_3(K_1)$ of 3×3 matrices over any infinite extension K_1 of K (if K is infinite then we can take $K_1=K$). Thus we have the following result.

Theorem 4.6 *(Sapir). The word problem is strongly undecidable in the variety $varM_3(K_1)\cap\mathcal{N}_3\mathcal{A}\cap Z\mathcal{N}_2\mathcal{A}$. Here $K_1=K$ if K is an infinite field or $\mathbf{Z}$ and K_1 is any infinite extension of K if K is a finite field.*

In the case of prime characteristic $varKS$ is the smallest known variety with undecidable word problem. We do not know, for example if the word problem is decidable in the variety generated by $M_2(K_1)$.

It is interesting to note that one can find a nice-looking algebra V which generates the variety $varKS$. This algebra is given by generators $\{A,B,q,a,b\}$ and defining relations

$$xA=Bx=q^2=qa=bq=0.$$

where x is an arbitrary generator.

The second author can show that in the case of positive characteristic every homomorphic image of any subalgebra of V generates either the same variety as V or a variety with decidable word problem.

Problem 4.3 *(Sapir). Is it true that the variety generated by the algebra V is a (the) minimal variety of associative algebras with undecidable word problem in the case of positive characteristic?*

In the case of characteristic 0 we can go deeper.

Let U be the associative algebra or associative unitary algebra over a field of characteristic 0, given by generators $\{A,B,q,a\}$ and defining relations

$$xA=Bx=q^2=qaq=[q,a,a]=0$$

where x is an arbitrary generator. It is easy to see that U is a homomorphic image of the subalgebra of V generated by $A,q,B,a+b$. Algebra U has more identities than V. For example, U satisfies the identity $[x_1,y_1][x_2,y_2,z][x_3,y_3]=0$ (see the next theorem) and V does not satisfy this identity (take $x_1=A,\ y_1=a,\ x_2=q,\ y_2=z=b,\ x_3=B,\ y_3=b.$).

The following theorem holds.

Theorem 4.7 *(Sapir, [343].) The word problem is strongly undecidable in any variety containing U. In the unitary case the word problem is decidable in every subvariety of the variety var$U$ and this variety has the following basis of identities.*

1. $[x_1, x_2][x_3, x_4][x_5, x_6][x_7, x_8] = 0$;
2. $[[x_1, x_2][x_3, x_4][x_5, x_6], x_7] = 0$;
3. $[x_1, x_2][x_3, x_4, x_5][x_6, x_7] = 0$;
4. $[[x_1, x_2, x_3][x_4, x_5, x_6], x_7] = 0$;
5. $[x_1, x_2, x_3][x_4, x_5, x_6] - [x_1, x_2, x_6][x_4, x_5, x_3] - [x_1, x_2][x_3, x_6][x_4, x_5]$
$+[x_1, x_2][x_4, x_5][x_3, x_6] - [x_3, x_6][x_1, x_2][x_4, x_5] = 0$;
6. $[x_1, x_2][x_3, x_4][x_5, x_6] + [x_5, x_6][x_3, x_4][x_1, x_2] + [x_1, x_6][x_3, x_4][x_2, x_5]$
$+[x_2, x_5][x_3, x_4][x_1, x_6] + [x_2, x_6][x_3, x_4][x_5, x_1] + [x_5, x_1][x_3, x_4][x_2, x_6] = 0$;

The decidability/undecidability parts of this theorem are consequences of the following three results.

Theorem 4.8 *(Sapir, [343]). The variety generated by U contains an absolutely finitely presented associative algebra with an undecidable word problem.*

Theorem 4.9 *(Sapir, [343]). In the unitary case every proper subvariety of var$U$ satisfies an identity of the form $[x_1, x_2][x_3, x_4][x_5, \dots, x_{n+5}] = 0$.*

Theorem 4.10 *(Sapir, [343]). In the unitary case every variety of associative algebras satisfying an identity of the form*

$$[x_1, \dots, x_n][x_{n+1}, x_{n+2}][x_{n+3}, \dots, x_{n+m}] = 0$$

has a decidable word problem.

It is possible that appropriate analogs of Theorems 4.9, 4.10 hold in the case of algebras without units. Then in this case we would have an analog of Theorem 4.7.

Theorem 4.8 is obtained by applying the method of interpreting systems of differential equations. This method will be discussed later in Section 7.6.

In the proof of Theorem 4.9 and 4.10 some methods of Latyshev [222] are employed. These methods are especially designed to work with non-matrix varieties, that is varieties which do not contain the algebra of 2×2 matrices. It is easy to see that the variety generated by the algebra U is non-matrix.

We will discuss the methods of proving the solvability of the word problem employed in the proof of Theorem 4.10 later in Section 7.1.

It is quite possible that the variety $varU$ is the only minimal variety with undecidable word problem (characteristic 0). Notice that the lattice of varieties of associative algebras over a field of characteristic 0 satisfies the descending chain condition (by Kemer's finite basis Theorem 4.3). Therefore every variety with undecidable word problem contains a minimal variety with this property. Recall that in the semigroup case there are varieties with undecidable word problem which do not contain any minimal variety with this property (for example, the variety defined by the identity $x^2 = 0$).

Problem 4.4 *(Sapir). Is it true that varU is the only minimal variety of associative algebras over a field of characteristic 0 with undecidable word problem?*

4.5 The Isomorphism Problem

The isomorphism problem for associative algebras is much harder than for semigroups. Even for finite dimensional algebras the solvability of the isomorphism problem is not at all clear. Let A and B be two n-dimensional K-algebras. Let $X = \{x_1, \ldots, x_n\}$ be a basis of A and $Y = \{y_1, \ldots, y_n\}$ be a basis of B. Let

$$x_i x_j = \sum_k \alpha_{i,j,k} x_k,$$

$$y_i y_j = \sum_k \beta_{i,j,k} y_k$$

where $\alpha_{i,j,k}$ and $\beta_{i,j,k}$ are (structure constants) from K. Algebras A and B are isomorphic if and only if there exists a basis $Z = \{z_1, \ldots, z_n\}$ in A such that

$$z_i z_j = \sum_k \beta_{i,j,k} z_k. \tag{14}$$

The basis Z is determined by the invertible transition matrix M from X to Z. A standard linear algebra argument gives us that (14) holds if and only if some system of equations in n^2 unknown entries of M has a solution. Therefore if K has a decidable elementary theory in the signature including all constants then the isomorphism problem for finite dimensional algebras over this field is decidable. In fact we need only an algorithm to solve systems of polynomial equations over this field. Every algebraically closed field has a decidable elementary theory with constants (see [395]). If the elementary theory of K is not decidable (for example, if K is the field of rational numbers $\mathbf{Q}$ or the ring of integers $\mathbf{Z}$), then the question becomes much more delicate. We do not know any example of a recursive field for which the isomorphism problem for finite dimensional algebras is undecidable. If $K = \mathbf{Q}$ or $\mathbf{Z}$ then the decidability of the isomorphism problem for finite dimensional algebras has been proved by Sarkisian [356] and Grunewald and Segal [122].

If we assume that the isomorphism problem for finite dimensional K-algebras is decidable then the isomorphism problem is decidable in every locally finite dimensional variety, for example, in any nilpotent variety.

We are not aware of any results about the computational complexity of the isomorphism problem for finite dimensional algebras. This problem may be computationally very hard even for finite algebras over finite fields. Some important characteristics of a finite algebra can be found in polynomial time. In particular, Lajos Rónyai [329] proved that the radical of a finite algebra over a finite field can be found in polynomial time. If the algebra is semisimple then its decomposition into a sum of simple algebras (the Wedderburn decomposition) can be found in polynomial time provided the Turing machine can use an oracle to factor polynomials over finite fields.

The variety of commutative algebras is the minimal non-locally finite dimensional variety. The situation with this variety is very complicated. In fact, almost nothing is known about the isomorphism problem for finitely generated (= finitely presented) commutative algebras. There are no examples of recursive fields for which this isomorphism problem is undecidable, and there are no examples of fields for which this problem is known to be decidable. We do not know the answer even for finite fields like $\mathbf{Z}_2$. The isomorphism problem for commutative associative algebras is clearly important by itself. The isomorphism problems in some important varieties of groups, for example, the variety of metabelian groups, reduces to the isomorphism problem for commutative K-algebras for some K (see Section 6.7). It would be interesting to apply the Pickel method (see Connection 2.7) to finitely generated commutative algebras.

Not much is known about varieties which are higher than the variety of commutative algebras. We can prove only that isomorphism is undecidable in any variety where we can prove the undecidability of the word problem. This is so because we can always interpret Minsky machines or systems of differential equations in such a way that the resulting associative algebra is Hopfian. Then the undecidability of the isomorphism problem follows from Connection 2.1. Moreover for each of these varieties $\mathcal{V}$ there exists an associative algebra L that is finitely presented in $\mathcal{V}$ and such that the problem of whether an algebra from $FP(\mathcal{V})$ is isomorphic to L is undecidable.

4.6 Varieties Where Finitely Presented Algebras are Residually Finite

The problem of describing varieties of associative algebras where every finitely presented algebra is residually finite has been posed by Bokut' in [42].

In contrast with the semigroup case, not every finitely presented associative algebra in a variety with a decidable word problem is residually finite. For example, let us take a semigroup $Y = < A, a, b, q >$ with defining relations

$$xA = 0,\ ab = ba = 0,\ qa = 0,\ Ab = 0,\ aqb = q,\ Aqb = 0, \tag{15}$$

where x is any generator. It is easy to see that this semigroup is not residually finite. Indeed, elements Aq and 0 cannot be separated by a homomorphism into a finite semigroup: every such homomorphism must identify two elements Aa^n and Aa^m for $n < m$, then it will identify $Aa^n qb^m = Aqb^{m-n} = 0$ and $Aa^m qb^m = Aq$.

From results of Mal'cev [237] it follows that for every commutative ring K and every semigroup S if the semigroup algebra KS is residually finite (as a K-algebra) then S is residually finite.

Therefore the semigroup algebra KY is not residually finite for every commutative ring K. On the other hand it is easy to see that the algebra $KY^{\#}$ (the algebra KY with unit adjoint) satisfies the identity $[x_1, y_1][x_2, y_2][x_3, y_3] = 0$ and so by Theorem 4.10 it belongs to a variety of associative algebras with solvable word problem (in the case of characteristic 0).

The algebra KY (without unit) satisfies the identity $x[y_1, z_1][y_2, z_2] = 0$. Thus we have the following result announced by Kublanovskii in [212] but, as far as we know, never published with a proof before.

Theorem 4.11 *(Kublanovsky [212]). If every finitely presented algebra in a variety of associative algebras is residually finite then this variety cannot contain varieties given by the identity $x[y_1, z_1][y_2, z_2] = 0$ or $[y_1, z_1][y_2, z_2]x = 0$.*

In the same paper [209] Kublanovskii announced the following result, which also has not been published with a proof. The second author of this survey can prove this result, so we decided to include it in the survey.

Theorem 4.12 *(Kublanovsky [212]). Let $\mathcal{V}$ be a variety of algebras given by finitely many homogeneous identities, which satisfies a permutational identity of the form*

$$x_1 x_2 \ldots x_n = x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)}.$$

Then every finitely presented algebra in $\mathcal{V}$ is residually finite and representable by matrices over an extension of K.

Note that the condition that $\mathcal{V}$ be given by homogeneous identities holds automatically if the ground field is infinite.

This result shows that at least in the case of algebras without unit the class of varieties where every finitely presented algebra is residually finite does not coincide with the class of locally residually finite varieties. Indeed, by this theorem every finitely presented algebra in the variety given by the identity $xyzt = xzyt$ is residually finite, but this variety is not locally residually finite by Theorem 4.17 below.

In the case of algebras with units, we do not know if these two classes are different. To prove that they are different in the case of characteristic 0 one has to consider the variety given by identities $[x_1, y_1][x_2, y_2][x_3, y_3] = 0$, $[[x_1, y_1][x_2, y_2], z] = 0$, $S_4 = 0$

(here S_4 is a standard identity – see Section 4.1) because this is the minimal non-locally residually finite variety by Theorem 4.18.

One can see that we don't know much about varieties where finitely presented algebras are residually finite.

Nevertheless our intuition tells us that the description of such varieties, at least in the case of characteristic 0, is not very far ahead. In particular we can formulate the following conjectures.

Conjecture 4.1 *(Sapir) Let K be a field of characteristic 0. The variety $varKY$ (resp. $varKY^{\#}$) is a minimal variety containing finitely presented non-residually finite algebras.*

Conjecture 4.2 *(Sapir) Let again K be a field of characteristic 0. Then the only minimal varieties containing finitely presented non-residually finite algebras are $varKY$, $varK\overleftarrow{Y}$, $varKZ$, $varK\overleftarrow{Z}$ (resp. $varKY^{\#}$, $varK\overleftarrow{Y}^{\#}$, $varKZ^{\#}$, $varK\overleftarrow{Z}^{\#}$ in the case of algebras with unit) where Z is a semigroup given by generators A, a, b, q and defining relations*

$$xq = 0,\ Aa = 0,\ ab = ba,\ qab = q,\ qAb = 0$$

(here x is any generator).

4.7 Locally Residually Finite Varieties. Non-Commutative Commutative Algebra

As we saw in the two previous sections, properties which formed the Club of the Word Problem in the semigroup case are no longer equivalent in the case of associative algebras. In this section, we shall show that the properties which formed the Club of Locally Residually Finite Varieties are much more strongly bonded with each other.

The following theorems are a result of the work of A.Mal'cev, Latyshev, L'vov, Anan'in, Yu.Mal'cev, Nechaev, Tonov, Kublanovskii, Drensky, Kemer, Sapir (in time order). Together they give a description of varieties which satisfy the properties from the Club of locally residually finite varieties. This description depends on the ground field (ring) K and on the existence of a unit. As far as K is concerned there are four possible situations:

(Char 0) K is a field of characteristic 0;

(Inf) K is an infinite field of positive characteristic;

(Fin) K is a finite field;

(Z) $K = \mathbf{Z}$.

Thus there are a total of eight possible cases.

Theorem 4.13 *Let $\mathcal{V}$ be a variety of associative algebras. In all eight cases the following conditions are equivalent*

1. *$\mathcal{V}$ is locally residually finite;*
2. *$FG(\mathcal{V}) = FP(\mathcal{V})$;*
3. *Every finitely generated algebra from $\mathcal{V}$ is representable by endomorphisms of a finitely generated module over a field if K is a field or over a ring of polynomials if $K = \mathbf{Z}$.*
4. *$\mathcal{V}$ satisfies an identity of the form*

$$xy^n z = \sum_{n_i<n} u_i x y^{n_i} z v_i \tag{16}$$

where u_i and v_i are some words.

If K is countable then these conditions are equivalent to the following condition.

5. *The set $FG(\mathcal{V})$ is countable.*

Theorem 4.14 *If K is a field then condition 3 from Theorem 4.13 is equivalent to the following condition:*

3'. *Every finitely generated algebra of $\mathcal{V}$ is representable by matrices over an extension of K.*

Theorem 4.15 *If K is an infinite field then condition 4 of Theorem 4.13 is equivalent to each of the following conditions (for some $n > 1$):*

4'. *$\mathcal{V}$ satisfies an identity of the form*

$$[x, y, y, \ldots, y]z^n[t, u, u, \ldots, u] \tag{17}$$

4". *Every finitely generated algebra of $\mathcal{V}$ satisfies an identity of the form*

$$[x_1, \ldots, x_n]y_1 \ldots y_n[z_1, \ldots, z_n] = 0.$$

Theorem 4.16 *In the case of algebras without unit and in the case of characteristic 0, the conditions of Theorem 4.13 are equivalent to the following condition:*

6. *Every finitely generated algebra in $\mathcal{V}$ is Hopfian.*

Theorem 4.17 *In the case of algebras without unit, the conditions of Theorem 4.13 are equivalent to each of the following conditions:*

7. $\mathcal{V}$ does not contain the variety defined by the following identity:

$$xyzt - xzyt = 0.$$

7'. $\mathcal{V}$ does not contain the algebra $K[x] \times K(P \times \overleftarrow{P})$ where P is the same semigroup which figures in Theorem 3.28 and 3.36. Here $K[x]$ is the algebra of polynomials with one variable.

Theorem 4.18 *In the case of characteristic 0 and algebras with unit, the conditions of Theorem 4.13 are equivalent to each of the following conditions:*

8. $\mathcal{V}$ does not contain the variety given by the following three identities

$$[x_1, y_1][x_2, y_2][x_3, y_3] = 0, \ \ [[x_1, y_1][x_2, y_2], z] = 0, \ \ S_4 = 0.$$

8'. $\mathcal{V}$ does not contain the following algebra of matrices

$$\begin{pmatrix} K[x]/(x^3) & xK[x]/(x^3) \\ xK[x]/(x^3) & K[x]/(x^3) \end{pmatrix}.$$

where $K[x]$ is the ring of polynomials with one variable, (x^3) is the ideal generated by the polynomial x^3.

As we mentioned above these theorems are a result of the work of many people. In 1943 A.I.Mal'cev [237] proved that every finitely generated commutative algebra is representable by matrices over a field. In 1958 [236] he proved that every finitely generated algebra of matrices is residually finite. He also proved that every finitely generated commutative ring is residually finite. This was the beginning of the investigation of algebras representable by matrices and residually finite algebras. One should mention also the problem of Kaplansky [162] of whether every finitely generated PI-algebra is representable by matrices. After an easy example of a non-representable PI-algebra was found, there arose the natural problem of describing representable PI-algebras in terms of identities. Among partial results related to this problem we can mention the result of Lewin [224] which states that every (not only finitely generated) algebra satisfying the identity $[x, y][z, t] = 0$ is representable by matrices over a commutative ring. From this result and from the result of Mal'cev [237], it follows that every finitely generated algebra satisfying the identity $[x, y][z, t] = 0$ is representable by matrices over a field. We should also mention an unpublished result of V.T.Markov which states that every finitely generated algebra over an infinite field, which satisfies the Engel identity

$$[x, y, y, \ldots, y] = 0 \tag{18}$$

is representable by matrices over a field.

In 1966 Latyshev [221] studied what was at first glance a completely different question: how to generalize the Hilbert finite basis theorem to PI-algebras. He described varieties of algebras of characteristic 0 with unit where every finitely generated algebra is Noetherian. Recall that an associative algebra is called *(left) Noetherian* if it possesses the descending chain condition for left ideals. It turned out that these are precisely the varieties satisfying the Engel identity (18). We will consider locally Noetherian varieties in the next section. In 1969 L'vov [226] described the class of varieties of algebras of characteristic 0 where every finitely generated algebra is *weak Noetherian* (has the ascending chain condition for two-sided ideals). It is easy to see that this condition is equivalent to the condition $FG(\mathcal{V}) = FP(\mathcal{V})$. He proved that a variety satisfies this condition if and only if it satisfies an identity of the form (16). These identities were later called the *L'vov identities*. In 1976 L'vov [225] announced a similar description of locally weak Noetherian varieties of algebras over an arbitrary commutative Noetherian ring (in particular, the ring of integers, any field, the ring of polynomials over any field or over the ring of integers, etc.). L'vov never published a proof of this result though.

Yu.Mal'cev [239] proved that in the case of algebras without unit over a field of characteristic 0 there is only one minimal non-weak Noetherian variety, namely the variety given by the identity $xyzt = xzyt$. So he proved that condition 7 is equivalent to condition 2. Then he used an example of a non-Hopfian semigroup W constructed by Zel'manov, to show that this minimal variety contains the non-Hopfian semigroup algebra KW. He showed this only in the non-unitary case of characteristic 0. This and L'vov's result mentioned in the previous paragraph imply the statement of Theorem 4.16 in the non-unitary case of characteristic 0. But it is easy to see that KW satisfies the identity $xyzt = xzyt$ regardless of the characteristic. This gives the non-unitary part of Theorem 4.16. The second author of this survey can prove that the semigroup algebra KW^1 (which is also non-Hopfian) belongs to the variety of unitary algebras from condition 8 (Theorem 4.18). This and Theorem 4.18 imply the unitary part of Theorem 4.16.

It was Anan'in who discovered strong connections between locally representable varieties, locally residually finite varieties, and locally weak Noetherian varieties[15]. He proved the equivalence of conditions 1, 2, 3', 4, 4', 4" of Theorems 4.13, 4.14 and 4.15 in the case of an infinite field (of any characteristic).

Then Yu.Mal'cev and Nechaev [240] added condition 5 in the case of algebras without unit over a field of characteristic 0.

Then Kublanovskii [211] proved that conditions 1, 2, 3, 4 are equivalent in the case of algebras without unit over arbitrary fields or **Z**. His proof can be easily rewritten to the case of algebras with unit. In [212], [208] he proved that conditions 1, 2, 3, 4 are equivalent to the condition 7 for algebras without unit. Obviously this is not true for algebras with units. From this, it follows almost immediately that condition 5 ("there are countably many finitely generated algebras") is equivalent to conditions

[15] Anan'in [11] mentions that L'vov discovered the equivalence of conditions 1 and 2 independently.

1, 2, 3, 4 in the case of arbitrary countable or finite field or $\mathbf{Z}$. Condition 7' was noticed by the second author of this survey. Semigroups P and $\overleftarrow{P}$ are the 3-element semigroups from (6) which appear in Theorems 3.28 and 3.36. The similarity between this condition and condition 3 of Theorem 3.36 is very interesting.

Tonov [400] found condition 8 of Theorem 4.18. Condition 8' of this theorem was found by Drensky [86] and independently by Kemer during his visit to Lincoln (Nebraska) in October, 1993 (by request of the second author of this survey). The algebra from condition 8' appeared earlier in a paper by Drensky [88] as R_3 (see page 214 of this paper). Algebras $R_k = \begin{pmatrix} K[x]/(x^k) & xK[x]/(x^k) \\ xK[x]/(x^k) & K[x]/(x^k) \end{pmatrix}$ play an exceptional role in Drensky's description of subvarieties of the variety generated by the algebra of 2×2-matrices.

Notice that Bergman [35] proved that the ring of endomorphisms of the group $\mathbf{Z}_p \times \mathbf{Z}_{p^2}$ is not representable by matrices over a field. This is a finite ring, so it generates a locally residually finite (even locally finite) variety of rings. Thus condition 3 of the Theorem 4.13 is not equivalent to condition 3' in the case of rings.

Thus we know almost everything about connections between the properties which belong to the Club of locally residually finite varieties. The only major thing which we don't know is the description of minimal non-locally residually finite varieties in the case of algebras with units over fields of positive characteristic and over $\mathbf{Z}$.

Problem 4.5 *Find all minimal non-locally residually finite varieties of associative algebras with units over a field of positive characteristic and minimal non-locally finite varieties of rings with units.*

Theorems 4.13 - 4.18 show that the L'vov identity (16) (or, equivalently, the identity (17) in the case of characteristic 0) is an appropriate generalization of commutativity if one wants to have an analog of the Hilbert finite basis theorem. There were several attempts to generalize in a similar way other classical results of commutative algebra. For example in 1984 Kharchenko [174] showed that generalizations of E.Noether's famous theorem about invariants also lead to L'vov's identity (16).

Recall some definitions. We shall consider algebras with units over a field K of characteristic 0. Let $V_m = \mathrm{span}\{x_1, \ldots, x_m\}$ be an m-dimensional vector space, $m > 1$. Consider the natural action of the group $GL(m, K) = GL(V_m)$ on V_m. Now let $\mathcal{V}$ be a variety of K-algebras, and let $F = < x_1, \ldots, x_m >$ be the m-generated free algebra in this variety. Then any linear map $g : V_m \to V_m$ may be extended (uniquely) to an endomorphism of F, and this endomorphism will be an automorphism if and only if g belongs to $GL(m, K)$. Thus the action of $GL(m, K)$ on V_m may be extended to an action of this group on F. For every subgroup $G \leq GL(M, K)$ we can define the *subalgebra of invariants* of this group:

$$F^G = \{p \mid g(p) = p \text{ for every } g \in G\}.$$

For example, if $\mathcal{V}$ is the variety of commutative algebras then F is the algebra of polynomials in m variables, and F^G is the algebra of invariant polynomials. The classical Noether theorem states that in this case for every finite group G the algebra of invariants is always finitely generated. If we take $\mathcal{V}$ to be the variety of all algebras then the analog of this result is not true [82], [174]. Kharchenko [174] found a description of varieties where an analog of the Noether theorem holds. Recently Drensky [86] strengthened Kharchenko's theorem. As a result we have the following statement.

Theorem 4.19 *(Kharchenko, [174], Drensky, [86]). Let $\mathcal{V}$ be a variety of associative algebras with units over a field of characteristic 0. Then conditions 1-4 of Theorem 4.13 are equivalent to each of the following conditions:*

9. *For every finite group $G < GL(m, K)$ the algebra F^G is finitely generated.*

9'. *There exists an element g from $GL(m, K)$ such that g has a finite order, at least two eigenvalues of g have different orders, and the algebra $F^{<g>}$ is finitely generated.*

Condition 9 is due to Kharchenko, condition 9' is due to Drensky. For a survey of results on non-commutative invariant theory we refer the reader to Formanek [100].

4.8 The Higman Property

In [33] V.Belyaev proved the following analog of Higman's theorem.

Theorem 4.20 *(Belyaev, [33]). Let K be either a finitely generated commutative unitary ring or a field which is a finite extension of its simple subfield. Then every K-algebra given by a recursively enumerable set of defining relations is embeddable into a finitely presented K-algebra.*

This result answered a problem by Bokut' from [84] (Problem 1.22).

It is interesting that among the relations of this finitely presented algebra only one relation has the form $a + b = c$ (a, b, c are generators), others are equalities of words of generators. Thus this algebra is a factor algebra of a semigroup algebra over the ideal generated by $a + b - c$. In the proof, Belyaev encodes the addition and multiplication of an arbitrary associative algebra A in a semigroup. Then he uses the Mursky analog of the Higman Theorem [275] to embed this semigroup into a finitely presented semigroup S. Then he proves that algebra A is embeddable into $KS/Id(a + b - c)$.

Theorem 4.20 answered a question of Bokut' from [84]. Every locally weak Noetherian variety (see a description of such varieties in Section 4.7) satisfies the Higman property because every finitely generated algebra there is finitely presented. The second author of this survey conjectures that there are no other varieties of associative algebras over a field which satisfy this property.

4.9 Finitely Presented Relatively Free Algebras

The question of when a relatively free universal algebra is finitely presented in some bigger variety is very natural, and it seems strange that this question has been studied intensively only in the case of associative algebras (at least as far as we are aware).

Let $\mathcal{M}_n$ be the variety generated by the algebra of $n \times n$-matrices over an infinite field. Procesi [306] and Lewin [224] raised the question of whether the relatively free algebras in $\mathcal{M}_n$ are finitely presented. The answer to this question was given by V.Markov in [251]. In fact Markov described all varieties of associative algebras with units where relatively free algebras are finitely presented. More precisely, Markov proved the following three theorems. In each of these theorems we consider algebras with units over an infinite field.

For every variety $\mathcal{V}$ let $md(\mathcal{V})$ be the minimal number among all $deg_x(f)$ such that $f = 0$ is a nontrivial identity of $\mathcal{V}$. Here $deg_x(f)$ is the degree of x in the polynomial f. If $\mathcal{V}$ is the variety of all associative algebras then we let $md(\mathcal{V})$ be infinity.

Theorem 4.21 *(V. T. Markov, [251]). Let $\mathcal{V}$ and $\mathcal{V}'$ be two varieties of associative algebras with units. If $1 < md(\mathcal{V}) < md(\mathcal{V}')$ then the relatively free k-generated algebra in $\mathcal{V}$ is not finitely presented in $\mathcal{V}'$ for every $k > 1$.*

Theorem 4.22 *(V. T. Markov, [251]). For any $n \geq 2$ the relatively free algebra in $\mathcal{M}_n$ with more than one generator is not finitely presented in the variety $\mathcal{M}_{n+1}$, furthermore it is not absolutely finitely presented.*

Theorem 4.23 *(V. T. Markov, [251]). For every variety of associative algebras with units $\mathcal{V}$ the following conditions are equivalent.*

1. *For every $k > 1$ the relatively free k-generated algebra from $\mathcal{V}$ is finitely presented.*
2. *The relatively free algebra in $\mathcal{V}$ with two generators is finitely presented.*
3. *$\mathcal{V}$ is locally Noetherian (every finitely generated algebra possesses the descending chain condition for left ideals).*
4. *$\mathcal{V}$ satisfies an Engel identity $[x, y, ..., y] = 0$.*
5. *$\mathcal{V}$ satisfies the "left L'vov identity"*

$$xy^n + \sum_i \alpha_i y^i x y^{n-i}$$

   for some n.

6. *$md(\mathcal{V}) = 1$.*

Theorem 4.22 follows from Theorem 4.21 because $md(\mathcal{M}_n) = n$ (see [251]). Markov mentions that Theorem 4.23 was independently proved by L'vov. The equivalence of conditions 3, 4, and 5 in Theorem 4.23 was proved by Latyshev in [221] in the case of characteristic 0. Markov noticed that Latyshev's prove does not depend on the characteristic.

4.10 Gröbner Bases

In this section we follow the paper [338] of the second author of this survey.

We consider only algebras over an infinite field. The situation with rings and algebras over finite fields is similar but technically more complicated.

As we mentioned in Section 2.9, in the case of associative algebras over a field the Church-Rosser presentations correspond to so-called Gröbner bases. There exists a big literature devoted to Gröbner bases (see, for example, the book by T. Becker and V. Weispfenning [30] or the survey by Ufnarovsky [404]).

First of all let us give the precise definition of a Gröbner basis. Our definition will be more general than that in [30], [404], or other sources which we are aware of. This definition is similar to what we had in the semigroup case (see Section 3.5.2). Let $F =< X >$ be a finitely generated algebra. Let $Sgp(F)$ be the multiplicative semigroup generated by X in F. Algebra F is spanned by $Sgp(F)$. Let $Y \subseteq Sgp(F)$ be a basis of the vector space F. We shall call elements of Y monomials and elements from F polynomials. Let $<$ be a complete order on Y. Then every polynomial f in F has a leading monomial which will be denoted by $\bar{f}$. We shall call F *ordered* if there exists a basis $Y \subseteq Sgp(F)$, and a complete order $<$ on Y such that:

1. if F contains a unit 1 then 1 is the minimal element in Y;

2. $>$ satisfies the descending chain condition;

3. $>$ is *stable*, that is if $u, v \in Y$ and $u \geq v$ then for every $s, t \in Y$

$$\overline{sut} \geq \overline{svt}.$$

Let us assume that F is ordered and that an ordered basis Y of monomials is fixed. We shall call F *effectively ordered* if

- The set Y has a recursive enumeration;

- There exists an algorithm which, given an element from $Sgp(F)$, produces the decomposition of this element into a linear combination of elements from Y;

- The order $<$ is recursive.

Every relation $(u, v) \in F \times F$ is equivalent to the relation $(u - v, 0)$, thus we can restrict ourselves to relations of the form $(f, 0)$. We also will suppose that f is *monic*, that is the leading coefficient of f is 1. Instead of the pair $(f, 0)$ we shall simply write f.

Let f be a (monic) relation. We say that f *is applicable* to the polynomial g if $\bar{g} = \overline{tfs}$ for some $t,\ s \in Y$. In this case we can replace g by $g - \alpha tfs$, where α is the coefficient of the leading monomial of g. This "kills" the leading monomial of g. The polynomial $g - \alpha tfs$ is called the *result of an application of f to g*. Since the order $<$ is stable, the leading monomial of $g - \alpha tfs$ is smaller than the leading monomial of g. It is easy to see that this definition of an application of a relation is more restrictive than the general definition given in Section 2.9.

Thus with every basis Σ of the ideal I we can associate a rewrite system $\Omega(F, \Sigma)$ with elements of F as objects and applications of basic elements as elementary transformations. The connected components of this rewrite system are the cosets modulo the ideal I generated by Σ. Since the order on monomials satisfies the descending chain condition, this rewrite system is always terminating. It is easy to verify that this rewrite system satisfies the Church-Rosser property if and only if for every element g of I there exists an element f_j from the basis which is applicable to g. If this condition holds, we will call the basis $\{f_1, \ldots, f_n\}$ a *Gröbner basis* of the ideal I.

The term "Gröbner basis" of an ideal in a free commutative algebra was introduced by Buchberger in 1965 [61] but the concept had been introduced a year earlier by Hironaka [145]. He used the term "standard basis". Notice that if F is a free commutative algebra then $Sgp(S)$ (the set of all commutative words) is a basis of the vector space F. Both Hironaka and Buchberger used the ShortLex order on commutative words (see Section 3.5.2) They proved that every ideal of a finitely generated free commutative algebra has a Gröbner basis. (see [62], [63], [30]). Thus they proved, in our terms, that the variety of commutative algebras is a Church-Rosser variety and one can take the free algebras in this variety as the pseudo-free algebras in the definition of a Church-Rosser variety (see the definition in Section 2.9).

In the non-commutative case, Gröbner bases were introduced (under different names) independently by many authors including Shirshov [372] and Bergman [34] (see Ufnarovsky's survey [404]).

Gröbner bases of ideals in free commutative algebras play an extremely important role in computational commutative algebra. They are used in solving non-linear systems of polynomial equations, in solving the word problem in commutative algebras, in computing syzygies, in solving the membership problem for finitely generated subalgebras of commutative algebras, and so on. For applications of Gröbner bases see, for example, [30], [62], [63], [321], [404].

Of course, Gröbner bases would be practically useless if we did not have algorithms to find them. Hironaka's proof was non-constructive, but Buchberger [61] found an algorithm which produces a Gröbner basis in every ideal of a free commutative algebra. This algorithm may be viewed as the Knuth-Bendix procedure (see Section

2.9) applied to a finitely generated congruence on the free commutative algebras. Other variants of the Knuth-Bendix procedure have been developed in [372], [34], [12], [273], [220], [160], [404] (see also references in [30]).

The difference between these variants lies in their differing definitions of a critical pair (see the discussion at the end of Section 2.9). Here we present what appears to be a generalization of all these definitions. Notice that our definition of a critical pair in the associative algebra case is almost the same as in the semigroup case (see Section 3.5.2).

An *overlap* of two relations $f_1, f_2 \in F$ is a quadruple of monomials $s_1, t_1, s_2, t_2 \in F$ such that

$$\overline{s_1 f_1 t_1} = \overline{s_2 f_2 t_2}$$

and

$$\overline{s_1 f_1 t_1 - \overline{s_1 f_1 t_1}} > \overline{s_2 f_2 t_2 - \overline{s_2 f_2 t_2}}$$

This overlap is called *critical* if and there is no other overlap s_1', t_1', s_2', t_2' of the same relations such that

1. $\overline{s_1 f_1 t_1} > \overline{s_1' f_1 t_1'}$,
2. There exist monomials x, y possibly empty such that $\overline{s_1 f_1 t_1} = x\overline{s_1' f_1 t_1'}y$,

$$\overline{s_1 f_1 t_1 - \overline{s_1 f_1 t_1}} = x\overline{s_1' f_1 t_1' - \overline{s_1' f_1 t_1'}}y > x\overline{s_1 f_2 t_1 - \overline{s_1 f_2 t_1}}y.$$

The *critical pair* determined by such a critical overlap is the pair of elements $s_1(f_1 - \overline{f_1})t_1, s_2(f_2 - \overline{f_2})t_2$ of F.

In many cases one can prove that for every two relations f_1, f_2 there are only finitely many critical pairs, and that the Knuth-Bendix procedure halts and so every ideal has a finite Gröbner basis (see Apel and Lassner [12], Mora [273], Latyshev [220] and Kandri-Rody and Weispfenning [160]). The following algebras, which arise naturally in mathematics and physics, have this property:

- free commutative algebras;
- enveloping algebras of finite dimensional Lie algebras (see the definition in Section 5);
- the Weyl algebras $W_n =< p_i, q_i \mid [p_i, q_i] = 1, [p_i, q_j] = 0, 1 \leq i \neq j \leq n >$.

More generally, the so called solvable algebras [160], [206] satisfy these conditions. By definition, an associative algebra F is called *solvable* if F is a factor-algebra of the free algebra KX^+ over an ideal I generated by polynomials of the type $[x_i, x_j] + p_{ij}$ such that the following conditions hold:

1. p_{ij} is a linear combination of commutative words in X;

2. $\overline{p_{ij}} < x_j x_i$ in the ShortLex order on commutative monomials;

3. I does not contain any non-zero linear combination of commutative words.

These conditions imply that the commutative words in X form a basis of the vector space F. The ShortLex order on this basis makes F an ordered algebra. Notice that there exists a strong similarity between solvable algebras and polycyclic groups (see Section 6). Perhaps it would be better to call these algebras "polycyclic" instead of "solvable".

The following analog of Theorem 3.38 holds.

Theorem 4.24 *(Sapir, [338]). Let $\mathcal{V}$ be a locally residually finite variety of associative algebras. Suppose that the free finitely generated algebras of $\mathcal{V}$ are ordered. Then the Knuth-Bendix procedure applied to any finite $\mathcal{V}$-presentation of an algebra $A \in \mathcal{V}$ halts, producing a Church-Rosser terminating presentation of S. Therefore $\mathcal{V}$ is a Church-Rosser variety (the free algebras of $\mathcal{V}$ play the role of pseudo-free algebras in the definition of Church-Rosser varieties).*

Conjecture 4.3 *(Sapir). For every n every finitely generated free algebra in the variety given by the identity*

$$[x_1, \ldots, x_n] y_1 y_2 \cdots y_n [z_1, \ldots, z_n] = 0 \tag{19}$$

is effectively orderable.

If this conjecture is true then, as in the case of semigroups, every locally residually finite variety of associative algebras is Church-Rosser. One would be able to take the free algebras in the varieties given by identities (19) as pseudo-free algebras. Indeed, by Theorem 4.15 every finitely generated algebra in a locally residually finite variety satisfies this identity, and every variety which satisfies this identity is locally residually finite. As in the case of semigroups one also has to use the fact that locally residually finite varieties are locally weak Noetherian.

So far the biggest known class of Church-Rosser varieties consists of varieties which satisfy the Lie nilpotency identity $[x_1, x_2, \ldots, x_n] = 0$. The following theorem is proved by Latyshev [220] (Latyshev used different terminology).

Theorem 4.25 *(Latyshev, [220]). Every variety of unitary associative algebras, which satisfies the identity $[x_1, \ldots, x_n] = 0$ for some n is Church-Rosser. One can take the enveloping algebra of a free nilpotent Lie algebra with m generators as a rank m pseudo-free algebra for this variety.*

Problem 4.6 *(Sapir). Describe Church-Rosser varieties of associative algebras.*

4.11 Residually Finite Varieties

The description of residually finite varieties of associative rings turned out to be similar to the description of residually finite varieties of groups, but nicer since rings do not have Burnside complications. It also turned out that the property of being residually finite is equivalent to the property of having finitely many subquasi-varieties (similar to the group case, but in a sharp contrast with the semigroup case).

Let $\mathcal{M}_0$ be the variety of all rings with zero multiplication, for every prime $p \geq 2$ let $\mathcal{M}_p$ be the variety defined by the following identities:

$$xyz = xy + yx = px = 0,$$

for every prime q let $\mathcal{M}'_q$ be the variety defined by the following identities:

$$xyz = xy - yx = qx = 0.$$

Theorem 4.26 *For every variety of associative rings $\mathcal{V}$ the following conditions are equivalent.*

1. *Every ring in $\mathcal{V}$ is residually finite.*
2. *$\mathcal{V}$ contains only finitely many subquasi-varieties.*
3. *$\mathcal{V}$ contains no more than countably many subquasi-varieties.*
4. *$\mathcal{V}$ is generated by a finite ring R such that every nilpotent subring of R is a ring with zero multiplication.*
5. *$\mathcal{V}$ does not contain the varieties $\mathcal{M}_0$, $\mathcal{M}_p$, $\mathcal{M}_q$ for every prime $p \geq 3$ and q.*
6. *$\mathcal{V}$ satisfies an identity $nx = 0$ for some n (here x is a variable, n is a natural number) and an identity $xy + \sum_i u_i = 0$ where lengths of words u_i are greater than 2, x and y are variables.*
7. *$\mathcal{V}$ satisfies an identity $nx = 0$ for some n and does not contain finite rings given by the following presentation $< a, b \mid a^2 = b^2 = ab + ba = pa = pb = 0 >$, $< a \mid qa = a^3 = 0 >$ where p is any prime and q is any odd prime.*

Given a finite set of identities Δ which defines the variety $\mathcal{V}$, it is easy to check whether $\mathcal{V}$ satisfies an identity $nx = 0$ for some n: the set Δ must contain an identity of the form $mx + \sum_i u_i = 0$ where none of the words u_i is equal to x. After this n is found, the list of "forbidden" rings to be checked becomes finite because the prime parameters p and q participating in the presentations of these rings must divide n. Therefore condition 7 of Theorem 4.26 gives an algorithmic description of residually finite varieties of rings. A similar description holds in the case of algebras over an arbitrary field.

Theorem 4.26 has a strange history. In 1977 Volkov proved the equivalence of conditions 1 and 3. The proof employed ideas from Ol'shanskii's paper on residually finite groups [289] and L'vov's results from [227]. He wrote a paper containing the proof of this result. The second author of this survey personally saw the paper because he was a co-author of it. His contribution was a description of varieties of rings where every ring is residually finite with respect to subrings (for every element and every subring which does not contain the element there exists a homomorphism onto a finite ring which separates the element and the subring). Then Volkov told L'vov about this result and it turned out that L'vov had proved it three years earlier, in 1974. He was a graduate student then and mentioned this result in his graduate student annual report. Then, in 1978, L'vov gave a talk about residually finite varieties of rings at the "Algebra and Logic" seminar in Novosibirsk (see [228]), but he never published the paper. Of course, Volkov did not publish his proof either. L'vov also mentioned that Belkin independently found a similar proof. This is quite possible because Belkin's paper [31] contains similar results. But Belkin's proof too has never been published. Meanwhile in 1979 R.McKenzie, who did not know about L'vov, Volkov, and Belkin, found his proof and published it in [255] and then in [258]. The equivalence of conditions 1, 2, 4 was obtained by the second author of this survey in [349] and [350]. The equivalence of conditions 4 and 5 follows immediately from the results of L'vov's paper [227]. The conditions 6 and 7 are taken from Volkov's paper [414].

Notice that the equivalence of conditions 1 and 3 holds in the case when $\mathcal{V}$ is an arbitrary congruence-modular variety of universal algebras that is generated by a finite algebra [102]. Notice also that in the case of rings and arbitrary congruence-modular varieties the property "to be residually finite" is equivalent to the condition "to be residually small" (this condition was introduced in Section 3.6.2), see [258], [102].

4.12 Varieties with Decidable Elementary Theory

A complete description of varieties of associative K-algebras where K is either a field with a decidable elementary theory or $\mathbf{Z}$ was found by Zamjatin in [423]. It turned out that if K is an infinite field then a variety has a decidable elementary theory if and only if it is a variety with zero products. In the case where K is a finite field or $\mathbf{Z}$ his description may be formulated in the following way.

Theorem 4.27 *(Zamjatin, [423]). Let $\mathcal{V}$ be a variety of associative K-algebras where K is a finite field or $\mathbf{Z}$. Then the following conditions are equivalent:*

1. *The elementary theory of $\mathcal{V}$ is decidable.*

2. *There exists a variety with zero product $\mathcal{U}$ and a variety $\mathcal{W}$ generated by finitely many finite fields such that every algebra in $\mathcal{V}$ is a direct product of an algebra from $\mathcal{U}$ and an algebra from $\mathcal{W}$.*

3. *$\mathcal{V}$ is residually finite and does not contain the following two semigroup algebras: $\mathbf{Z}_pP$, $\mathbf{Z}_p \overleftarrow{P}$ (if $K = \mathbf{Z}$), or KP, $K \overleftarrow{P}$ (if K is a finite field). Here P is the 3-element semigroup from (6), $\overleftarrow{P}$ is the dual (anti-isomorphic) semigroup, and p is a prime number.*

Condition 3 did not appear in [423] and has been added by the second author of this survey. This condition can be extracted from the proof of Theorem 1 from [423]. Recall that by virtue of Theorem 4.26 the class of residually finite varieties has an algorithmic description.

Zamjatin [423] also described varieties of associative algebras whose finite traces have decidable elementary theories. Again only the case where $K = \mathbf{Z}$ or K is a finite field is non-trivial. In order to formulate Zamjatin's result we need some notation.

First we introduce rings of types 1–5. By a ring of type 1 we mean a finite field; a ring of type 2, the ring KP where K is a finite field and P is our 3-element semigroup from (6); a ring of type 3, the ring $K \overleftarrow{P}$ where K is a finite field; a ring of type 4, the ring of all matrices of the form $\begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix}$ over a finite field in which $a_2 = a_1^{p^k}$ where p is the characteristic of the field and k is a fixed natural number; a ring of type 5, the Galois ring of characteristic p^2 for some prime p. (A Galois ring of characteristic p^2 is a ring isomorphic to $\mathbf{Z}[x]/(p^2, f(x))$ where $\mathbf{Z}[x]$ is the ring of polynomials in the variable x and $(p^2, f(x))$ is the ideal generated by p^2 and a polynomial $f(x)$ with leading coefficient unity which is irreducible modulo p.) We also need the 4-element semigroup $E = \{e, f, c, 0\}$ in which $e^2 = e, f^2 = f, ef = c$ and all other products are equal to 0.

Theorem 4.28 *(Zamjatin, [423]). Let $\mathcal{V}$ be a variety of associative K-algebras where K is a finite field or $\mathbf{Z}$. Then the following conditions are equivalent:*

1. *The elementary theory of $\mathcal{V}_{fin}$ is decidable.*
2. *$\mathcal{V}$ is generated by a finite set of rings of types 1–5.*
3. *$\mathcal{V}$ is residually finite and does not contain the semigroup algebra $\mathbf{Z}_pE$ (if $K = \mathbf{Z}$) or KE (if K is a finite field).*

Condition 3 gives an algorithmic description. It has been extracted from the proof of Theorem 2 of [423] by the second author of this survey.

Remark. Varieties with decidable elementary theories have been intensively studied in the general case of universal algebras. Burris and McKenzie [64] described all locally finite congruence-modular varieties of universal algebras which have decidable elementary theory. Theorem 4.27 can be easily deduced from their description, although it should be mentioned that Burris and McKenzie used Zamjatin's ideas.

McKenzie and Valeriote [260] considered elementary theories of varieties without the assumption of congruence modularity. They reduced the problem of describing decidable locally finite varieties to two special cases: varieties of modules over a finite ring and the so-called discriminator varieties. A discriminator variety is a variety of algebras that can be thought of as Boolean algebras with extra operations. Thus starting with varieties of rings the problem of describing varieties with decidable elementary theories went to general algebras and then returned to rings again but in a different form. Congruence-modular finitely generated varieties of universal algebras with decidable finite traces were considered by Idziak [149]. He also reduced the problem of describing such varieties to some varieties of modules over a finite ring. Theorem 4.28 can be deduced from Idziak's results but this requires some work.

5 Lie Algebras

5.1 Basic Definitions

For notation, definitions and basic facts on Lie algebras and their varieties see Jacobson [156], Bakhturin [18], Kostrikin [201].

A linear algebra L over a commutative unitary ring K is called a *Lie algebra* if L satisfies the following two identities (the product of x and y is denoted by xy):

$$xy + yx = 0,$$

$$(xy)z + (yz)x + (zx)y = 0.$$

The second identity is called the *Jacoby law*.

Since the Lie algebra product is non-associative, we need to agree on how to read the word xyz. We follow the left faction of the Lie algebra community and read this word as $(xy)z$. The word xy^n reads as $(\ldots(xy)y\ldots)$. Note that in the main reference book on varieties of Lie algebras, [18], xyz stands for $x(yz)$. Thus it belongs to the wrong (right) faction of the community.

Let L be a Lie algebra. A linear function $\phi : A \to A$ is called a *derivation* if $\phi(xy) = \phi(x)y + x\phi(y)$. Let x be an element of L. Then *ad* x denotes the function $y \to yx$, which is called an *internal derivation*. The Jacoby law and $xy + yx = 0$ imply that *ad* x is indeed a derivation.

The *derived subalgebra* A' of an algebra A is the subalgebra generated by all products xy, $x, y \in A$. The *centre* $Z(A)$ of A is the ideal of all elements $x \in A$ such that $xy = 0$ for every $y \in A$.

As in the case of associative algebras we assume that K is either a field or the ring of integers. In the latter case L is called a *Lie ring*. If K is a field then we assume that it is recursive.

There is a natural connection between Lie algebras and associative algebras. If A is an associative algebra, then the commutator $[x, y] = xy - yx$ may be considered as a new operation. This operation together with the addition and the multiplication by elements of K make A a Lie algebra. Let L be a Lie algebra with product $\circ$ which is a free K-module (this holds automatically if K is a field). Let E be a basis of L. Consider the associative algebra A generated by the set E subject to defining relations $e_1e_2 - e_2e_1 = e_1 \circ e_2$, $e_1, e_2 \in E$. This algebra is called the *enveloping algebra* of A. Let us fix a complete order on E. By the famous Poincare-Birkhoff-Witt Theorem [18], products $e_1e_2 \cdots e_n$ where $e_1 \geq e_2 \geq \ldots \geq e_n$ form a basis of A. In particular if the basis E is recursive and the order $<$ is recursive then A has a recursive basis.

We denote the variety of Abelian Lie algebras by $\mathcal{A}$ (this variety is defined by the identity $xy = 0$). By $\mathcal{N}_k$ we denote the variety of nilpotent Lie algebras of step $\leq k$. This variety is defined by the following identity:

$$x_1 \cdots x_{k+1} = 0.$$

The Engel variety of step k is defined by the (Engel) identity $yxx \cdots x = 0$ (x repeats k times). This identity can be rewritten in the form: $(ad\ x)^k = 0$.

As in the general case we denote by $\mathcal{UV}$ the Mal'cev product of the varieties $\mathcal{U}$ and $\mathcal{V}$. For example, $\mathcal{N}_k\mathcal{A}$ is the variety of algebras with k-step nilpotent derived subalgebra. The n-th power of the variety of Abelian algebras $\mathcal{A}^n$ is the variety of all solvable Lie algebras of step $\leq n$. For every variety $\mathcal{V}$, $Z\mathcal{V}$ denotes the variety of centre-by-$\mathcal{V}$ algebras. An algebra A belongs to $Z\mathcal{V}$ if and only if $A/Z(A) \in \mathcal{V}$. For example, $Z\mathcal{N}_k\mathcal{A}$ is the variety of centre-by-$\mathcal{N}_k\mathcal{A}$ Lie algebras.

Let $F(X)$ be the free Lie algebra over $X = \{x_1, \ldots, x_n, \ldots\}$. A *poly-degree* of a monomial $v(x_1, \ldots, x_m) \in F(X)$ is an m-tuple $\alpha = (\alpha_1, \ldots, \alpha_m)$, where α_i is the number of occurrences of x_i in $v(x_1, \ldots, x_m)$. Let F_α be the subspace of $F(X)$ spanned by monomials of polydegree α. An element in $F(X)$ is called *homogeneous* if it belongs to F_α for some α. Every element in $F(X)$ can be uniquely written as a sum of homogeneous elements. A variety $\mathcal{W}$ is called *homogeneous* if it may be given by homogeneous identities. This is equivalent to the following property: if $v = 0$ is an identity in the variety $\mathcal{W}$ and $v = \sum_\alpha v_\alpha$, where v_α is homogeneous then for any α the identity $v_\alpha = 0$ also holds in $\mathcal{W}$.

5.2 Overview

The neighborhood of algorithmic problems in the case of Lie algebras is not as good as in the case of associative algebras, but not as bad as in the group case. This puts Lie algebras in the position between associative algebras and groups.

Let us briefly review what is known about the finite basis problem and the Burnside-type problems in Lie algebras.

It is still unknown whether there exists a non-finitely based variety of Lie algebras over a field of characteristic zero.

The varieties $\mathcal{N}_k\mathcal{A}$ over a field of characteristic zero are hereditary finitely based (Krasilnikov [205]). This is the biggest known class of hereditary finitely based varieties (characteristic = 0). Notice that these varieties play a significant role in the theory of varieties of Lie algebras. Indeed, every solvable finite dimensional Lie algebra over a field of characteristic zero has a nilpotent derived subalgebra [156], hence it belongs to some variety $\mathcal{N}_k\mathcal{A}$.

Il'tyakov [151] proved that every finite dimensional Lie algebra over a field of characteristic 0 is finitely based.

Vaughan-Lee and Drensky showed that if the characteristic p is not 0, then there exist non-finitely based varieties even inside $Z\mathcal{N}_{p-1}\mathcal{A}$ (see [408], [409], [87]). They also proved that even a finite dimensional Lie algebra over an infinite field of positive characteristic can be non-finitely based. On the other hand Bakhturin and Olshanskii showed that every finite Lie ring is finitely based [20].

The role of the Engel identity $(ad\ x)^k = 0$ in the theory of Lie algebras is similar to the role of the identity $x^n = 0$ in the theory of semigroups and associative algebras. Every nilpotent Lie algebra satisfies this identity. The question of when the Engel identity implies local nilpotency is the Burnside-type problem for Lie algebras. It is closely related to the restricted Burnside problem for groups [201].

It was proved by Kostrikin [202], [203] that every finitely generated Lie algebra of characteristic p is nilpotent provided it satisfies the Engel identity $(ad\ x)^k = 0,\ k < p$ (k is arbitrary in the case of characteristic zero).

The following result of Zelmanov is a generalization of Kostrikin's results, it summarizes the investigations of Lie algebras with Engel identity: A finitely generated Lie ring which satisfies an Engel identity is nilpotent (see Zelmanov [428], [427], Vaughan-Lee and Zelmanov [410]). This is an analog of the Kaplansky-Shirshov theorem for associative algebras (see Section 4.2).

For Lie algebras of characteristic 0 the following analog of the Dubnov-Ivanov-Nagata-Higman theorem has been proved by Zelmanov [425]: a Lie algebra over a field of characteristic zero which satisfies an Engel identity is nilpotent. In the case of positive characteristic p there exist simple examples of non-nilpotent Lie algebras satisfying the Engel identity $(ad\ x)^{p+1} = 0$ (see Cohn [74]). Razmyslov [312] constructed a (much more difficult) example of a similar Lie algebra with a "smaller" identity $(ad\ x)^{p-2} = 0$. The existence of such an example implies several important results about varieties of groups (see [312]).

5.3 The Identity Problem and Related Problems

The question of whether the identity problem is solvable for all varieties of Lie rings was posed in 1966 by Mal'cev ([200], problem 2.40). It is still unsolved. Notice that the analogous problem for groups was solved in the negative by Kleiman (see a sketch of his proof in Section 7.7.3). Unfortunately all attempts to find a similar proof for Lie rings have failed. The main difficulty is to find a Lie ring analog of the main personage of Kleiman's proof, the group $C(A, N, M)$.

There are positive solutions of the equational problem in some important varieties.

If a variety of Lie algebras can be given by homogeneous identities then, as in the case of semigroups and associative algebras (Sections 3.3.2, 4.3), every relatively free algebra in this variety is residually finite. This can be proved using Theorem 4.2.2 from [18] in the same manner as Theorems 3.15 and 4.1 above. Every variety of Lie algebras over an infinite field may be given by homogeneous identities ([18], Theorem 4.2.4). Thus we have the following theorem, which is similar to Theorem 4.1 in Section 4.3).

Theorem 5.1 *The equational theory is decidable in every variety of Lie algebras defined by homogeneous identities. The identity problem is decidable in the variety of all Lie algebras over an infinite field.*

In particular, the equational problem is solvable in the product of nilpotent varieties $\mathcal{N}_{c_1}\mathcal{N}_{c_2}\cdots\mathcal{N}_{c_n}$ for arbitrary natural numbers $c_1, \ldots, c_n$.

Using the proof of Krasilnikov's result that the varieties $\mathcal{N}_c\mathcal{A}$ over a field of characteristic 0 are hereditary finitely based [205], we can prove that the identity problem is solvable in any subvariety of the variety $\mathcal{N}_c\mathcal{A}$.

By Connection 2.9 the Rhodes problem is solvable in any subvariety of the variety $\mathcal{N}_c\mathcal{A}$.

Problem 5.1 *Is the Rhodes problem solvable in the variety of all Lie algebras?*

5.4 The Word Problem

A.I. Shirshov was one of the first to study the word problem in Lie algebras [370], [372]. He developed the so-called composition method (this was one of the first implementations of what was later called the Knuth-Bendix procedure, see Section 2.9) and used it in the proof of the following theorem

Theorem 5.2 *(Shirshov, [370], [372]). The word problem is solvable in any Lie algebra given by one defining relation.*

In fact Shirshov proved that the Knuth-Bendix procedure applied to any one relator presentation of a Lie algebra produces a finite terminating Church-Rosser presentation of this algebra.

It is interesting that Theorem 5.2 cannot be proved by using the residual finiteness argument. Agalakov [7] proved that the Lie algebra $< x, y \mid yxx - yx + x = 0 >$ over a field of characteristic 0 is not residually finite. It is hard to believe, but over any field of positive characteristic the algebra given by the same relation is residually finite [7].

We have already mentioned that the word problem is decidable in relatively free algebras of products of nilpotent varieties $\mathcal{N}_{c_1}\mathcal{N}_{c_2}\cdots\mathcal{N}_{c_n}$. Talapov [391] proved a more general result that is similar to Theorem 5.2.

Theorem 5.3 *(Talapov, [391]). The word problem is solvable in Lie algebras given by 1 defining relation in the variety* $\mathcal{N}_{c_1} \dots \mathcal{N}_{c_k}$ *for arbitrary natural numbers* $c_1, \dots, c_k$.

The result was originally formulated for Lie algebras over a field which is a finitely generated over its simple subfield, but the proof can be easily adapted to the case of an arbitrary recursive field.

The word problem for Lie algebras was first formulated by Shirshov (see [84], Problem 1.154). The first example of a finitely presented Lie algebra (over an arbitrary field) with undecidable word problem was implicitly constructed by L.A. Bokut' [44]. Bokut' used a Lie algebra analog of the technique used by Higman and Valiev in their proofs of the embedding theorem for groups [144], [406]. In [46] G.P.Kukin presented the following simple construction of a Lie algebra with an undecidable word problem. Let S be a semigroup given by generators x_i and defining relations $u_j = v_j$. The Lie algebra L_S over a ring K is generated by elements a, x_i, y_i. The defining relations of L_S are the following:

$$x_i y_i = 0,$$

$$a x_i = a y_i,$$

$$< a u_j > = < a v_j >$$

for every i and j. Here the notation $< u >$ means that we consider the word u as a Lie algebra word. It is easy to see that two words u and v over X are equal in S if and only if the Lie algebra words $< au >$ and $< av >$ are equal in L_S. Thus if S has an undecidable word problem then so does L_S. Algebra L_S may be viewed as a Lie algebra analog of the semigroup algebra KS.

The first attempt to construct a Lie algebra that is finitely presented in a nontrivial variety, namely $\mathcal{N}_2\mathcal{A}$, and has an undecidable word problem was made by Kukin in [213]. Unfortunately, there is a serious gap in the method used there (see Section 7.5). An attempt [217] to construct an absolutely finitely presented Lie algebra with an undecidable word problem and a non-trivial identity contained a similar gap. A detailed discussion of this example may be found in Baumslag, Gildenhuys and Strebel [28].

Nevertheless the variety $\mathcal{N}_2\mathcal{A}$ from Kukin [213] was so small that it hinted to the possibility of describing all varieties of Lie algebras with decidable word problem. This problem has been mentioned in [42] and [66].

The first correct examples of varieties with strongly undecidable word problem appeared in Kharlampovich [177], Mel'nichuk, Sapir, Kharlampovich [268] and Baumslag, Gildenhuys, Strebel [28].

Theorem 5.4 *(Kharlampovich, [177]). The word problem is strongly undecidable in the variety of Lie algebras* $Z\mathcal{N}_3\mathcal{A} \cap \mathcal{A}^3$ *over a field of characteristic* $\neq 2$.

Constructions in all these papers are based on an interpretation in Lie algebras of a 2-tape Minsky machine with an undecidable halting problem, see section 7.2.1. The methods of the works mentioned above are described in Sections 7.2.6 and 7.2.7.

If the main field has characteristic 2, these methods do not work quite as successfully, and we do not know whether this variety has a strongly undecidable word problem.

Notice that the variety $Z\mathcal{N}_3\mathcal{A} \cap \mathcal{A}^3$ is the Lie algebra analog of the variety of associative algebras from Theorem 4.6. Thus Theorem 5.4 is a Lie algebra analog of Theorem 4.6.

As far as the (ordinary) word problem is concerned, we have the following result.

Theorem 5.5 *(Kharlampovich, [187]). The word problem is undecidable in any variety of Lie algebras (over an arbitrary field) containing $Z\mathcal{N}_2\mathcal{A}$.*

For a field of characteristic 0 the result of Theorem 5.4 has been improved.

Theorem 5.6 *(Kharlampovich, [176]). The word problem is strongly undecidable in the variety $Z\mathcal{N}_2\mathcal{A}$ of Lie algebras over a field of characteristic* 0.

The Minsky machine technique is insufficient in this case (see Section 7.6.5). The proof of the theorem is based upon Sapir and Kharlampovich's result [353] on the algorithmic unsolvability of systems of linear differential equations over a ring of polynomials (see Section 7.6.3). In the case of positive characteristic our interpretation of differential equations does not work, so we cannot drop the restriction on the characteristic in this theorem.

It is interesting to compare Theorem 5.6 with Theorem 4.7 for associative algebras: in the case of associative algebras the variety $Z\mathcal{N}_2\mathcal{A}$ also has a strongly undecidable word problem. We shall see in Section 6.5.1 that a similar result holds for groups.

The variety $Z\mathcal{N}_2\mathcal{A}$ is not a minimal variety with undecidable word problem. But the following theorem shows that it is very close to the boundary between decidability and undecidability.

Theorem 5.7 *(Kharlampovich). If $\mathcal{V} \subseteq \mathcal{N}_2\mathcal{A}$ then the word problem for $\mathcal{V}$ is decidable.*

The idea of the proof is described in section 7.1.

Theorem 5.7 generalized several results obtained earlier.

It has been known since 1954 that finitely generated metabelian Lie algebras are residually finite (Hall [137]). Hence the word problem is decidable in the variety $\mathcal{A}^2$ and in the subvarieties of this variety.

Umirbaev [405] proved the solvability of the word problem in the variety of centre-by-metabelian Lie algebras $Z\mathcal{A}^2$. His proof employs the same ideas as Romanovskii's earlier proof of the similar result for groups (see Section 6).

It is still unknown if there are Lie algebras finitely presented in $Z\mathcal{A}^2$ which are not residually finite, but it is known that this variety is not locally residually finite (see Theorem 5.26). Notice that the result of Umirbaev does not give any information about subvarieties of the variety $Z\mathcal{A}^2$. As we have noted in the Introduction, the solvability of the word problem is not hereditary for subvarieties.

Thus the gap between varieties with decidable word problem and those with undecidable word problem is very narrow, so in spite of the mistakes in Kukin's papers [213], [217], the problem of describing varieties with decidable word problem, inspired by these papers, seems doable.

Our intuition tells us that the following two conjectures "must" be true.

Conjecture 5.1 *Every variety of solvable Lie algebras over a field of characteristic zero with decidable word problem is contained in the variety $\mathcal{N}_c^2$ for some c.*

Conjecture 5.2 *A subvariety of $\mathcal{N}_c^2$ has decidable word problem if and only if its intersection with $\mathcal{N}_3\mathcal{A}$ has decidable word problem, that is the variety $\mathcal{N}_3\mathcal{A}$ is an indicator variety with respect to the decidability of the word problem (see the definition in the Section 2.7).*

These two conjectures justify the following problem.

Problem 5.2 *(Kharlampovich). Describe all subvarieties of the variety $\mathcal{N}_3\mathcal{A}$ with hereditary solvable word problem.*

An important step towards the solution of this problem is the investigation of subvarieties of the variety $Z\mathcal{N}_2\mathcal{A}$. The first author of this survey and D. Gildenhuys investigated subvarieties of this variety over a field of characteristic 0. The bad news is that even inside this variety the property of having an undecidable word problem is not hereditary for subvarieties. More precisely the following surprising results were obtained.

Theorem 5.8 *(Kharlampovich, [184]) There exists an infinite chain of varieties of Lie algebras ($char. = 0$) inside $Z\mathcal{N}_2\mathcal{A}$ in which varieties with decidable and undecidable word problem alternate.*

This theorem is the consequence of the following two theorems.

Theorem 5.9 *(Kharlampovich, [184]). The word problem is decidable in the varieties $\mathcal{N}_2\mathcal{N}_k \cap Z\mathcal{N}_2\mathcal{A}$.*

Let $\mathcal{M}_k$ be the variety defined in $Z\mathcal{N}_2\mathcal{A}$ by the identity

$$(x_1 \dots x_{k+2})(y_1 \dots y_{k+2})(z_1 \dots z_k) = 1.$$

Theorem 5.10 *(Kharlampovich, [184]). The word problem is undecidable in the varieties $\mathcal{M}_k$.*

There are obvious inclusions

$$\cdots \subseteq \mathcal{M}_k \subseteq Z\mathcal{N}_2\mathcal{A} \cap \mathcal{N}_2\mathcal{N}_{k+1} \subseteq \mathcal{M}_{k+2} \subseteq \cdots.$$

In order to study subvarieties of $Z\mathcal{N}_2\mathcal{A}$ one needs a special system of concepts that has been developed by the first author of this survey. We need some of these concepts now.

Let F be the free Lie algebra of infinite rank with generators $X = \{x_1, x_2, \dots\}$ in the variety $Z\mathcal{N}_2\mathcal{A}$. The letter ℓ, with or without subscripts, will denote a product $x_i x_j$ of two distinct generators. The generators appearing in different ℓ-symbols are assumed to be distinct from each other, and from those represented by other letters.

Let $x \in F$ and u, v elements of F such that $u \equiv v \pmod{F'}$; i.e. $u - v \in F'$. Then we have $(\ell_1 u)\ell_2 x = (\ell_1 v)\ell_2 x$, $\ell_1(\ell_2 u)x = \ell_1(\ell_2 v)x$, $\ell_1\ell_2(\ell_3 u) = \ell_1\ell_2(\ell_3 v)$. Therefore it makes sense to consider the elements $(\ell_1 u)\ell_2 x$, $\ell_1(\ell_2 u)x$, $\ell_1\ell_2(\ell_3 u)$, where u belongs to the enveloping algebra $U(F/F')$ of F/F'. Since F/F' is an Abelian algebra which is spanned by x_i, by the Poincare-Birkhoff-Witt theorem the enveloping algebra $U(F/F')$ is isomorphic to the ring of polynomials $K[x_1, \dots, x_n, \dots]$, where K is the ground field. Let $\check{x}_1, \dots, \check{x}_n, \dots$ be some new unknowns. Then the mapping $x_i \mapsto \check{x}_i$ can be extended to an isomorphism: $K[x_1, \dots, x_n, \dots] \to K[\check{x}_1, \dots, \check{x}_n, \dots]$ We denote

$$\ell_1\ell_2\ell_3 \circ u\check{v} = (\ell_1 u)(\ell_2 v)\ell_3,$$

where $u, v, w \in K[x_1, \dots, x_n, \dots]$, and $\check{v}$ is the image of v under the indicated isomorphism. We denote also

$$\ell_1\ell_2 x \circ u\check{v} = (\ell_1 u)(\ell_2 v)x.$$

Let $P(\alpha, x)$ be the polynomial containing only one occurrence of every factor of the polynomial

$$(x + (1+\alpha)\check{x})(\alpha x + (1+\alpha)\check{x})((1+\alpha)x + \check{x})((1+\alpha)x + \alpha\check{x})$$

(polynomials which differ by the scalar coefficient are considered the same). For example, $P(-1, x) = x\check{x}$.

A polynomial $R(\alpha, x_1, x_2, x_3, x_4)$ is to be obtained from $P(\alpha, x)$ by introducing a different variable for each factor. For example,

$$R(-1, x_1, x_2, x_3, x_4) = x_1\check{x}_2.$$

The identities of the form $\ell_1\ell_2\ell_3 \circ f(x_1, \check{x}_1, \dots, x_n, \check{x}_n) = 0$ we call *enveloping identities.*

Let A_3 be the alternating group on 3 elements. Let $\mathcal{W}_\alpha$ be the variety defined inside $Z\mathcal{N}_2\mathcal{A}$ by the identities

$\ell_1\ell_2 z \circ R(\alpha, x_1, x_2, x_3, x_4) R(\alpha, x_5, x_6, x_7, x_8) = 0,$
$\sum_{\sigma\in A_3}(\ell x_{\sigma(1)})(x_{\sigma(2)}x_{\sigma(3)})z \circ R(\alpha, x_1, x_2, x_3, x_4) = 0,$
$\sum_{\sigma,\delta\in A_3}(x_{\sigma(1)}x_{\sigma(2)}y_{\delta(1)})(y_{\delta(2)}y_{\delta(3)}x_{\sigma(3)})z = 0.$

Theorem 5.11 *(Kharlampovich, [183],[178]). Let K be a field of characteristic 0. For every $\alpha \in K, \alpha \neq 0, -1$ there exists a finitely presented Lie algebra with undecidable word problem that belongs to the variety $\mathcal{W}_\alpha$.*

This theorem is stronger than Theorem 5.6.

Theorem 5.12 *(Kharlampovich, [178]). If K is a field of algebraic numbers, $\alpha \neq 0, -1$, and $|\alpha|$ and $|\alpha + 1|$ are multiplicatively independent (that is products of integer powers of $|\alpha|$ and $|\alpha+1|$ are equal to 1 only if the powers are equal to 1) then the variety $\mathcal{W}_\alpha$ is a minimal variety with a strongly undecidable word problem. For every such α and β the intersection $\mathcal{W}_\alpha \cap \mathcal{W}_\beta$ has hereditarily decidable word problem.*

The varieties from Theorem 5.12 are candidates for being minimal with undecidable word problem, but we still can not prove that every proper subvariety of them has not only weakly decidable, but decidable word problem. Since by the theorem of Krasilnikov [205] the lattice of subvarieties of $\mathcal{N}_3\mathcal{A}$ satisfies the descending chain condition, each variety $\mathcal{W}_\alpha$ contains a minimal finitely based variety with undecidable word problem. Thus there exist infinitely many minimal varieties of Lie algebras over any number field. This is in a sharp contrast with the cases of semigroups (three minimal varieties) and associative algebras (most probably just one minimal variety).

An explicit example of a minimal variety with undecidable word problem will be given in Theorem 5.13.

The restriction on the field to be a number field is necessary here, because the solution of the word problem in the proper subvarieties of the variety $\mathcal{W}_\alpha$ is connected with the solution of some systems of exponential Diophantine equations. In Section 7.6.1 the connections between the word problem and some number theoretic questions will be discussed in more detail. In general, the answer to the following question is required in order to describe all subvarieties of the variety $Z\mathcal{N}_2\mathcal{A}$ with decidable word problem.

Problem 5.3 *(Kharlampovich). Does there exist an effective bound for the integral solutions of the exponential Diophantine equation*

$$\sum_{i=1}^{n} \lambda_i(x)\alpha_i^x = 0, \tag{20}$$

where the $\lambda_i(x)$ are polynomials in x with algebraic coefficients, and the α_i are algebraic numbers with the property that α_i/α_j $(i \neq j)$ are not m-th roots of unity for any m?

There is a well known theorem by Scolem [379] which says that if the $\lambda_i(x)$ in (20) are constants then this equation has only finitely many integer solutions. The proof of Scolem's theorem works in the case when the $\lambda_i(x)$ are arbitrary polynomials. Thus the equation (20) always has only finitely many solutions. The problem is to find an effective bound for these solutions. For $n = 1$ an effective bound is obvious. For $n = 2$ the effective bound follows from a theorem of Baker [16]. Mignotte, Shorey and Tijdeman [269] have obtained an effective bound in the case when $n = 3$ and all $\lambda_i(x)$ are constants. This is all that is currently known about Problem 5.3.

In [182], there is a reference to an unpublished positive solution of this problem by S.V.Kotov. The solution turned out to be incorrect. Fortunately it is not used in the proof of any of the results in [182].

The following theorem gives the only known explicit examples of a minimal variety of Lie algebras over a field of characteristic 0 with undecidable word problem. As we mentioned above the set of such varieties is infinite.

Theorem 5.13 *(Kharlampovich and Gildenhuys, [190]). Let $\mathcal{W}$ be the subvariety of the variety $Z\mathcal{N}_2\mathcal{A}$ defined in $Z\mathcal{N}_2\mathcal{A}$ by the identities*

$$(x_1x_2x_3)(x_4x_5x_6)x_7 = 0,$$
$$\textstyle\sum_{\sigma\in A_3}(y_1y_2)(t_{\sigma(1)}t_{\sigma(2)}z_1\dots z_{2h+1})t_{\sigma(3)} = 0, h \geq 1,$$
$$\textstyle\sum_{\sigma\in A_3}(y_1y_2x_{\sigma(1)})(x_{\sigma(2)}x_{\sigma(3)})z = 0.$$

Then $\mathcal{W}$ has an undecidable word problem and all its proper subvarieties have decidable word problem.

There exist varieties containing the variety $\mathcal{W}$ and having solvable word problem, for example, the varieties $Z\mathcal{N}_2\mathcal{A} \cap \mathcal{N}_2\mathcal{N}_{k+1}$, see Theorem 5.9. Thus we currently do not know any variety $\mathcal{V}$ of Lie algebras such that every proper subvariety of $\mathcal{V}$ has solvable word problem and every variety containing $\mathcal{V}$ has an undecidable word problem. In the case of associative algebras and semigroups such varieties exist (see Theorems 3.28, 4.7).

Yet another interval in the lattice of varieties which consists of varieties with undecidable word problem is provided by the following theorem.

Theorem 5.14 *(Kharlampovich and Gildenhuys, [191]). In the case of characteristic 0 the interval between $Z\mathcal{A}\mathcal{N}_2 \cap Z\mathcal{N}_2\mathcal{A}$ and $Z\mathcal{N}_2\mathcal{A}$ consists of varieties with undecidable word problem.*

This theorem implies, in particular, that the varieties $Z(\mathcal{A}\mathcal{N}_2 \cap \mathcal{N}_2\mathcal{A})$ and $Z\mathcal{A}\mathcal{N}_2$ have undecidable word problem. This result is in contrast with the fact that the word problem is decidable in the variety $\mathcal{A}\mathcal{N}_c$ for any c. Moreover the latter varieties are locally residually finite by a result of Hall [137].

Another approach consists of trying to find "strong" identities which hold in varieties with decidable word problem. This would give, in some sense, upper bounds for varieties with solvable word problem. The following theorems provide some nice identities.

Theorem 5.15 *(Kharlampovich and Gildenhuys, [188]). Over a field of characteristic* 0 *every proper subvariety of variety* $Z\mathcal{N}_2\mathcal{A}$ *satisfies an enveloping identity of the form*

$$\ell_1\ell_2\ell_3 \circ (x_1\check{x}_2 - x_2\check{x}_1)^n = 0.$$

Theorem 5.16 *(Kharlampovich and Gildenhuys, [190]). If a variety* $\mathcal{V}$ *of Lie algebras over a field of characteristic 0 has hereditary decidable word problem then* $\mathcal{V}$ *satisfies for some* n *the identity*

$$\sum_{1\le i+j\le n} \alpha_{ij} x_1 x^{n+1-i-j}(tx^i)tx^j = 0, \tag{21}$$

with $\alpha_{1,0} \neq 0$

Theorem 5.17 *(Kharlampovich, Gildenhuys, [188]). If a variety of Lie algebras, over the field of algebraic numbers, is contained in* $Z\mathcal{N}_2\mathcal{A}$ *and has solvable word problem, then it admits for some integer* $n \geq 0$ *the identity*

$$\ell_1\ell_2\ell_3 \circ x_1^n \check{x}_1^n \bar{x}_1^n (x_1\check{x}_2 - x_2\check{x}_1) = 0.$$

5.5 The Generalized Freiheitssatz

Free algebras in varieties usually behave better than arbitrary algebras. Thus it is important to find conditions under which a finitely presented algebra is close to a free algebra or contains a "big" free subalgebra. The classical example is the Freiheitssatz proved by Magnus for one related groups (see [229]): in every group with n generators and one relation one can effectively find a free subgroup freely generated by $n-1$ generators. Magnus used this theorem to prove the solvability of the word problem in groups with one defining relation. A generalization of this theorem for $n+m$-generated groups with m relations was conjectured by Lyndon in "Kourovskaya Tetrad" [200] and proved by Romanovskii [326] (see Section 6.6). A similar result for semigroups was proved earlier by L. Shneerson [376]. He also proved the following characterization of free Lie (associative) algebras which is conceptually close to the Freiheitssatz.

Theorem 5.18 *(Shneerson, [375]). Let* F *be a Lie (or associative) algebra given by* $n+k$ *generators* $a_1, \ldots, a_{n+k}$ *and* n *relations. If* F *can be generated by* k *generators* $c_1, \ldots, c_k$ *then* F *is free and* $c_1, \ldots, c_k$ *are its free generators.*

Let us give precise formulations of the Freiheitssatz and the Generalized Freiheitssatz in varieties.

Let $\mathcal{V}$ be a variety of universal algebras. We say that the generalized Freiheitssatz holds in a variety $\mathcal{V}$, if for any algebra $L \in \mathcal{V}$ given inside $\mathcal{V}$ by $n+m$ generators $x_1, \ldots, x_{n+m}$ and m relations there exist n generators $x_{i_1}, \ldots, x_{i_n}$ which freely generate a $\mathcal{V}$-free subalgebra of L. If in this definition we fix $m = 1$ then we shall say that the (ordinary) Freiheitssatz holds in $\mathcal{W}$. We say that the effective Freiheitssatz holds in $\mathcal{W}$ if those free generators may be found effectively.

Theorem 5.19 *The effective Freiheitssatz holds in the following varieties:*

1. *(Shirshov [372]). The variety of all Lie algebras.*
2. *(Talapov [391]). The Mal'cev product $\mathcal{N}_{c_1} \ldots \mathcal{N}_{c_k}$ (for arbitrary $c_1, \ldots, c_k$), where $\mathcal{N}_c$ is the variety of nilpotent Lie algebras of step c.*

As a consequence of this theorem we have the effective Freiheitssatz for solvable and nilpotent varieties.

The following theorem answered a question by Bokut' [42].

Theorem 5.20 *(Kharlampovich, [177]) The Generalized Freiheitssatz holds in the varieties $\mathcal{A}^n$ and in the variety of all Lie algebras.*

Notice that unlike Theorem 5.19, this theorem is not effective. It does not say how to find the free n-generated subalgebra which figures in the generalized Freiheitssatz. We do not know if the effective generalized Freiheitssatz holds in the variety of all Lie algebras or in the variety $\mathcal{A}^n$.

Romanovskii told the first author that the following result may be proved in the same way as its group analog in [327].

Theorem 5.21 *(Romanovskii) The Generalized Freiheitssatz holds in the variety $\mathcal{N}_c$ of Lie algebras for every c.*

Again, it is not known whether the effective generalized Freiheitssatz holds in $\mathcal{N}_c$ for every c.

5.6 The Isomorphism Problem

The isomorphism problem for Lie algebras is as hard as it is for associative algebras. The solvability of the isomorphism problem for finite dimensional Lie algebras over an algebraically closed field can be proved by the same argument as for associative algebras (see Section 4.5). Recall that the decidability of the isomorphism problem for finite dimensional algebras over $\mathbf{Q}$ and $\mathbf{Z}$ was proved by Sarkisian [356] and Grunewald and Segal [122] (see Section 4.5).

As a corollary we have the decidability of the isomorphism problem for finitely generated nilpotent Lie $\mathbf{Q}$-algebras and finitely generated nilpotent Lie rings.

In the proof of Theorem 5.4 one can choose a Minsky machine in such a way that the corresponding Lie algebra is Hopfian. Hence by Connection 2.1 we have the following two results.

Theorem 5.22 *If $\mathcal{V} \supseteq Z\mathcal{N}_3\mathcal{A} \cap \mathcal{A}^3$ then the isomorphism problem for absolutely finitely presented algebras in $\mathcal{V}$ is undecidable. There exists a finitely presented Lie algebra $L \in \mathcal{V}$ such that the problem of whether an algebra from $FP \cap \mathcal{V}$ is isomorphic to L is undecidable.*

Theorem 5.23 *If $\mathcal{V} \supseteq Z\mathcal{N}_2\mathcal{A}$ then the isomorphism problem for relatively finitely presented algebras in $\mathcal{V}$ is undecidable. There exists a Lie algebra L that is finitely presented in $\mathcal{V}$ and such that the problem of whether a Lie algebra from $FP(\mathcal{V})$ is isomorphic to L is undecidable.*

The original proof of Theorem 5.22 in Kharlampovich [177] and Baumslag, Gildenhuys and Strebel [28] is similar to Kirkinskii and Remeslennikov's proof of the unsolvability of the isomorphism problem for the group variety $\mathcal{A}^7$ [193].

The construction used for the proofs of the results about the unsolvability of the word problem in subvarieties of the variety $Z\mathcal{N}_2\mathcal{A}$ is based on the interpretation in Lie algebras of a system of linear differential equations. This system can be chosen in such a way that the Lie algebra with undecidable word problem will be Hopfian.

Hence the following theorem holds.

Theorem 5.24 *If a variety $\mathcal{W}$ of Lie algebras over a field of characteristic 0 contains one of the varieties $\mathcal{W}_\alpha, \alpha \neq 0, -1$, then the isomorphism problem for finitely presented algebras which belong to $\mathcal{W}$ is undecidable. There exists a finitely presented Lie algebra $L \in \mathcal{W}$ such that the problem of whether a Lie algebra from $FP \cap \mathcal{W}$ is isomorphic to L is undecidable.*

5.7 The Uniform Word Problem For Finite Traces

The uniform word problem for finite traces of varieties of Lie rings has been raised by Bokut' (see [84], Problem 2.24). It was investigated in Kharlampovich [179], [181]. Let $\mathcal{N}$ be the class of all nilpotent rings and $\mathcal{F}$ be the class of all finite Lie rings. Let $\mathcal{X}$ be an arbitrary class of Lie rings such that $Z\mathcal{N}_2\mathcal{A} \subseteq \mathcal{X}$.

Theorem 5.25 *(Kharlampovich [179], [181]). The universal problem is undecidable for the following classes of Lie rings: $\mathcal{F}$; $\mathcal{N}$; $\mathcal{F} \cap \mathcal{N}$; $\mathcal{F} \cap \mathcal{X}$; $\mathcal{N} \cap \mathcal{X}$; $\mathcal{N} \cap \mathcal{X} \cap \mathcal{F}$.*

If one considers finite dimensional Lie algebras over a field instead of finite Lie rings then the same result is true.

In general, like in the semigroup case (see Section 3.4.3), we can prove the undecidability of the uniform word problem for finite trace of some variety of Lie rings provided we can prove the undecidability of the word problem in this variety using the technique of Minsky machines. The following natural conjecture arises

Conjecture 5.3 *The uniform word problem for the finite trace of a variety of Lie rings is decidable if and only if the word problem for this variety is decidable.*

5.8 The Higman Property

As we have already mentioned, Bokut' [44] used a Lie algebra analog of the Higman and Valiev technique employed in the proofs of the embedding theorem for groups [144], [406]. Although he did not prove an analog of the Higman embedding theorem, he raised the question of whether such analogs exist (see [84], Problem 1.22). In 1979 Kukin published the paper [216], where he claimed that for any variety $\mathcal{M}$ containing $\mathcal{N}_2\mathcal{A}$, every recursively generated Lie algebra in $\mathcal{M}$ can be embedded into a Lie algebra which is finitely presented in $\mathcal{M}\mathcal{A}^2$. In particular, if $\mathcal{M}$ is the variety of all Lie algebras then we get an analog of the Higman embedding theorem: every recursive Lie algebra is embeddable into a finitely presented algebra. Thus from 1979 till the present time this area was considered well-developed.

The main technique used by Kukin in [216] was an interpretation of partial recursive functions in Lie algebras. He used the same technique in his papers on the word problem [213], [92],[214]. As we show in Section 7.5, this technique contains a serious gap. So now one must consider results from [216], including the Lie algebra analog of the Higman embedding theorem, as only conjectures.

Problem 5.4 *Is the variety of all Lie algebras a Higman variety?*

We are sure that the answer is positive. It can probably be proved either by a modification of Kukin's method or by applying Belyaev's ideas from [33] (see Section 4.8).

Problem 5.5 *Is it true that every recursively presented solvable Lie algebra is embeddable into an algebra that is finitely presented in $\mathcal{A}^n$ for some n.*

This problem is a Lie algebra analog of a Remeslennikov and Roman'kov problem about group varieties [318].

Thus all we know for certain about Higman varieties of Lie algebras is that the locally Noetherian varieties described in Section 5.9 are Higman varieties (see Section 2.6).

We say that a variety is *strongly Higman* if every recursively presented Lie algebra in the variety is embeddable in an absolutely finitely presented Lie algebra that belongs to this variety.

By the result of Baumslag [23] every metabelian variety of Lie algebras over a field of characteristic $\neq 2$ is strongly Higman. This follows from the Baumslag-Remeslennikov Lemma that we will use in Section 7.2.7 (see Lemma BR there). There are, of course, examples of non strongly Higman varieties. For example, the varieties $\mathcal{Y}_c$ constructed in Theorem 6.11 are also not strongly Higman. This is a trivial consequence of Theorem 6.11.

Problem 5.6 *Is there any strongly Higman variety of Lie algebras that is not a subvariety of $\mathcal{A}^2$.*

This problem relates to the following problem, which is an analog of a group theoretic question raised in Noskov, Remeslennikov and Roman'kov [284].

Problem 5.7 *Is it true that any Lie algebra which is finitely presented in the variety $\mathcal{A}^n$ for some n is embeddable into an absolutely finitely presented solvable Lie algebra.*

5.9 Locally Residually Finite Varieties

The class of varieties with residually finite relatively finitely presented algebras is strictly smaller than the class of varieties with hereditary decidable word problem. The authors recently constructed an example of a relatively finitely presented algebra in the variety $\mathcal{N}_2\mathcal{A}$ over a field of char $\neq 2$ which is not residually finite [192]. This variety has hereditary decidable word problem. It is not too hard to show that the 2-generated free $\mathcal{N}_2\mathcal{A}$-algebras are not Noetherian (do not satisfy the maximal condition for ideals).

In the case of absolutely finitely presented algebras the situation differs. Absolutely finitely presented algebras in $\mathcal{N}_2\mathcal{A}$ are residually finite.

We do not know whether the class of varieties where finitely presented algebras are residually finite coincides with the class of locally residually finite varieties. To prove that they are different one has to consider the variety of centre-by-metabelian algebras, which is not locally finite by theorems 5.26 and 5.28.

Problem 5.8 *Describe all varieties of Lie algebras over a field of characteristic 0 in which every relatively finitely presented algebra is residually finite.*

The investigation of locally residually finite varieties of Lie algebras was inspired by the success in the investigation of locally residually finite varieties of associative algebras (see Section 4.7). The first examples of nontrivial locally residually finite varieties of Lie algebras were given by Bakhturin [17]. He proved that over a field of characteristic 0 every metabelian variety satisfies this property. The proof in [17] can be extended to the varieties $\mathcal{N}_c\mathcal{A}$.

Bahturin and Bokut' posed the problem of describing all locally residually finite varieties over a field of characteristic 0 ([84], problem 2.16)

A description of locally residually finite varieties, locally representable varieties, locally Noetherian and locally Hopfian varieties of Lie algebras over infinite fields has been obtained by M.V.Zaitsev [420], [418], [419], [417]. By a Noetherian Lie algebra we mean an algebra that satisfies the ascending chain condition for ideals. A variety is locally Noetherian if and only if it satisfies the familiar property $FP = FG$ (every finitely generated algebra is relatively finitely presented).

The variety of centre-by-metabelian algebras will play a very important role in the description below. One of the essential moments in obtaining this description is an earlier result of Volichenko [412]. He had proved that, over a field K of characteristic zero, a variety $\mathcal{V}$ does not contain all centre-by-metabelian algebras if and only if it satisfies the identity

$$xy^n z = \sum_{i\geq 1} \alpha_i xy^{n-i} zy^i, \tag{22}$$

for some natural number n and some $\alpha_i \in K$.

This identity is the Lie algebra analog of the L'vov identity for associative algebras (16) and the identity (10) for semigroups.

Another essential moment was the result of Zelmanov that every (not only finitely generated) Lie algebra over a field of characteristic zero that satisfies the Engel identity is nilpotent [425].

First we formulate the results of Zaitsev in characteristic 0. For this we need one more definition. Let G be the so called 3-dimensional Heizenberg algebra over a field K, that is the algebra with the basis $\{x, y, z\}$ and the multiplication table $xy = z, xz = yz = 0$. Let Y be the semidirect product of the polynomial algebra $K[t]$, considered as an Abelian Lie algebra, and G where G acts on $K[t]$ as follows: $xf(t) = f'(t),\ yf(t) = tf(t),\ zf(t) = f(t)$. Let $\mathcal{Y}$ be the variety generated by Y.

Theorem 5.26 *(Zaitsev [420], [419], [417]). Let $\mathcal{V}$ be a variety of Lie algebras over a field of characteristic 0. Then the following conditions for $\mathcal{V}$ are equivalent.*

1. *$\mathcal{V}$ is locally residually finite,*
2. *$\mathcal{V}$ is locally representable,*
3. *$\mathcal{V}$ does not contain all centre-by-metabelian algebras and the commutator of every finitely generated algebra in $\mathcal{V}$ is nilpotent.*
4. *$\mathcal{V}$ neither contains all centre-by-metabelian algebras nor the variety $\mathcal{Y}$.*
5. *$\mathcal{V}$ satisfies the identity (22) and does not contain the variety $\mathcal{Y}$.*

In the case of associative algebras and rings the conditions for a variety to be locally residual finite, to be locally representable and to be locally Noetherian are equivalent. In the case of associative algebras and rings without unit they are equivalent to locally Hopfianness (see Section 4.7). In the case of nonperiodic semigroups the conditions for a variety to be locally residually finite, locally Hopfian, Locally representable and locally Noetherian are also equivalent (see Section 3.5).

In the case of Lie algebras over a field of characteristic 0 these conditions are not equivalent.

Theorem 5.27 *(Zaitsev, [418]). Let $\mathcal{V}$ be a variety of Lie algebras over a field of characteristic 0. Then the following conditions for $\mathcal{V}$ are equivalent.*

1. *$\mathcal{V}$ is locally Noetherian $(FG(V) = FP(V))$,*

2. $\mathcal{V}$ is locally Hopfian,

3. $\mathcal{V}$ does not contain all centre-by-metabelian algebras,

4. $\mathcal{V}$ satisfies the identity (22).

The description in Theorems 5.26 and 5.27 is effective. Indeed, in order to check whether the variety of centre-by-metabelian algebras satisfies an identity $f = 0$ it is enough to check if f is equal to 0 in the relatively free algebra of the centre-by-metabelian variety. This can be done since this free algebra is residually finite (see section 5.3). It is also possible to verify if an identity $f = 0$ holds in the algebra Y that generates the variety $\mathcal{Y}$. This problem can be easily reduced to the problem of determining if a finite system of linear differential equations over the algebra $K[t]$ has a solution. This problem is algorithmically decidable, because it can be shown that if the system has a solution then it has a solution of effectively bounded degree (see [181]).

The identity (21) plays the same role in connection with the word problem as Volichenko's identity (22) plays in connection with the residual finiteness of a variety.

It is not difficult to show that if a variety of solvable Lie algebras over a field of characteristic 0 does not contain all centre-by-metabelian algebras then for some c it belongs to $\mathcal{N}_c^2$. In connection with theorem 5.26 the following question arises: *Is any locally residually finite variety of Lie algebras over a field of characteristic 0 contained in $\mathcal{N}_c\mathcal{A}$ for some c?* The analogous result holds for associative algebras over an infinite field. In the case of Lie algebras the answer is negative. The variety given by the identities

$$(x_1x_2x_3)(x_4x_5x_6) = 0, \ (x_1x_2)(x_3x_4)(x_1x_3) = 0$$

is locally residually finite but is not contained in $\mathcal{N}_c\mathcal{A}$ for any c. This variety has been studied in Volichenko [412], [411].

For infinite fields of positive characteristic there is the following result of Zaitsev.

Theorem 5.28 *(Zaitsev, [418]) Let $\mathcal{V}$ be a variety of Lie algebras over an infinite field of positive characteristic. Then the following conditions for $\mathcal{V}$ are equivalent*

1. $\mathcal{V}$ is locally residually finite;

2. $\mathcal{V}$ is locally Hopfian;

3. $\mathcal{V}$ is locally Noetherian;

4. $\mathcal{V}$ is locally representable;

5. The identity (22) holds in $\mathcal{V}$.

If characteristic $\neq 2$ then conditions 1—4 are equivalent to the following condition

7 $\mathcal{V}$ does not contain $Z\mathcal{A}^2$.

If characteristic $= 2$ then conditions 1—4 are equivalent to the following condition:

8 $\mathcal{V}$ does not contain the variety given in $Z\mathcal{A}^2$ by the family of the identities

$$x_1y_1\dots y_k(x_2x_3) + x_2y_1\dots y_k(x_3x_1) + x_3y_1\dots y_k(x_1x_2) = 0, \quad k = 1, 2, \dots,$$

$$x_1x_2y_1\dots y_m(x_1x_2) = 0, \quad m = 1, 2, \dots).$$

In the original formulation of this result, there was an additional restriction that every Engelian algebra in $\mathcal{V}$ be locally nilpotent. Zelmanov's result [428], that every Lie ring with an Engel identity is locally nilpotent, allows one to drop this restriction.

Again, this description is effective: given a finitely based variety one can verify if it satisfies condition 4.

In the case of finite fields the situation is unclear. It was proved by Bahturin in [18] that over any field a finitely generated metabelian Lie algebra is residually finite. The centre-by-metabelian variety contains a non-residually finite finitely generated algebra [18] but we do not even know if this is a minimal non-locally residually finite variety in the case of finite field.

Problem 5.9 *Describe locally residually finite varieties of Lie algebras over finite fields and locally residually finite varieties of Lie rings.*

5.10 Residually finite varieties

Every nilpotent non-Abelian variety of Lie algebras over an arbitrary ring contains a non-residually finite algebra ([18], Section 6.6.6). This simple observation is enough to describe all residually finite varieties of Lie algebras over an infinite field (see [420]). Indeed, it is easy to see that every variety of Lie algebras over an infinite field which does not contain non-Abelian nilpotent algebras is Abelian. This immediately follows from the fact that every variety of Lie algebras over an infinite field may be given by homogeneous identities. Every Abelian algebra over a field is a subdirect product of 1-dimensional algebras. Indeed Abelian Lie algebras are just vector spaces with zero multiplication. Thus there are just two residually finite varieties of Lie algebras over any infinite field: the variety of all Abelian algebras and the 0 variety.

In the case of finite fields the situation is much more difficult but the description of residually finite varieties is known. Some partial results were obtained by Bahturin and Semenov in [21]. Then Premet and Semenov described all residually finite varieties over a finite field.

Theorem 5.29 *(Premet, Semenov, [304]). Let k be a finite field of characteristic $p > 3$. All algebras of the variety $\mathcal{W}$ of Lie algebras over k are residually finite if and only if $\mathcal{W}$ is generated by one finite algebra, such that all of its nilpotent subalgebras are Abelian.*

This theorem is a complete analog of Olshanskii's theorem for groups. In order to make the analogy between these theorems clearer, we note that all Sylow subgroups of a finite group are Abelian if and only if all of its nilpotent subgroups are Abelian. It is interesting that the proof of Theorem 5.29 is not similar to Ol'shanskii's proof. For varieties generated by a finite algebra Theorem 5.29 follows from the result of Freese and McKenzie [102].

5.11 Elementary Theories

Varieties of Lie rings with decidable elementary theory have been described by Zamjatin.

Theorem 5.30 *(Zamjatin, [421]) A variety $\mathcal{V}$ of Lie rings has decidable elementary theory if and only if $\mathcal{V}$ is an Abelian variety.*

Zamjatin told the first author of this survey that the same statement holds for Lie algebras over a field. A similar statement holds in the case of finite traces.

Theorem 5.31 *(Zamjatin) The elementary theory of the finite trace $\mathcal{V}_{fin}$ is solvable if and only if $\mathcal{V}_{fin}$ does not contain nonabelian Lie algebras.*

We do not know of any non-Abelian variety of Lie algebras where every finite algebra is Abelian.

6 Groups

6.1 Basic Definitions

For notation and definitions from the theory of groups we refer the reader to H. Neumann [279], Kargapolov and Merzljakov [163] and Rotman [330]. Let us recall some of them. The analogous notation has been used for varieties of Lie algebras in the previous section.

$\mathcal{V}_1\mathcal{V}_2$ is the Mal'cev product of varieties $\mathcal{V}_1$ and $\mathcal{V}_2$.

$\mathcal{A}^n$ is the variety of all solvable groups of step n.

Groups from $\mathcal{A}^2$ are called *metabelian.*

$\mathcal{N}_c$ denotes the variety of all nilpotent groups of step c.

If $\mathcal{V}$ is a variety of groups then $Z\mathcal{V}$ is the variety of all centre-by-$\mathcal{V}$ groups that is groups G such that $G/Z(G) \in \mathcal{V}$.

$\mathcal{B}_p$ is the variety of all groups of exponent p.

$\mathcal{A}_p$ is the variety of Abelian groups of exponent p.

Recall [279] that a product of two varieties is generated by the (restricted) *wreath product* of free groups of these varieties of countable ranks. The definition of the wreath product is the following. Let G and H be groups and let $G^{(H)}$ be the direct product of $|H|$ copies of G. Then H acts on $G^{(H)}$ in the following natural way. If ϕ is a vector in $G^{(H)}$ and $h \in H$ then for every $t \in H$

$$\phi^h(t) = \phi(th^{-1}).$$

Then the wreath product, $GwrH$, of G and H is the semidirect product of $G^{(H)}$ and H. Thus $GwrH$ is the set of pairs $G^{(H)} \times H$ with the following product:

$$(\phi, a)(\psi, b) = (\phi^b\psi, ab).$$

A group G is called *polycyclic* if it has a series of subgroups

$$1 < G_1 < \ldots < G_n = G \tag{23}$$

where every G_i is normal in G_{i+1} and G_{i+1}/G_i is cyclic. If G is polycyclic then it has a so-called *polycyclic base* $\{g_1, \ldots, g_n\}$ such that $g_i \in G_i \backslash G_{i-1}$ and the coset g_iG_{i-1} generates G_i/G_{i-1}. Polycyclic groups are precisely all the solvable subgroups of $GL(n, \mathbf{Z})$, $n = 1, 2, \ldots$ [366]. Every nilpotent finitely generated group is polycyclic. A group is said to be polycyclic-by-finite, or a PF-group for short, if it has a polycyclic normal subgroup of finite index. By a theorem of P.Hall [137] every PF-group is finitely presented. If one chooses a polycyclic base $\{g_1, \ldots, g_n\}$ of a polycyclic group G as a set of generators, then G can be given by relations which have one of the two forms:

1. $g_i^{n_i} = w(g_1, \ldots, g_{i-1})$, $1 \le i \le n$;
2. $(g_i, g_j) = w(g_1, \ldots, g_{i-1})$, $1 \le j < i \le n$.

Here w is an arbitrary word, (x, y) denotes the commutator $x^{-1}y^{-1}xy$. Such presentations of polycyclic groups will be called *polycyclic presentations*. One can easily find a similarity between polycyclic presentations and presentations of solvable associative algebras which we have discussed in Section 4.10. Polycyclic groups and PF-groups form pseudovarieties (that is each of these two classes is closed under taking subgroups, homomorphic images and finite direct products).

6.2 Overview

Varieties $\mathcal{A}^n$, $\mathcal{N}_c$, $\mathcal{N}_c\mathcal{A}$, $Z\mathcal{N}_c\mathcal{A}$, $\mathcal{B}_p$, $\mathcal{A}_p$ are probably the most important varieties of groups. Several well-known theorems in group theory imply that under some natural conditions a group belongs to one of these varieties. In particular, Kolchin-Mal'cev's theorem says that every solvable matrix group has a normal subgroup of finite index with a nilpotent derived group. Moreover, Platonov [299] proved that every matrix group over a field that satisfies a nontrivial identity is solvable-by-finite, and thus is a finite extension of a group from $\mathcal{N}_c\mathcal{A}$. For fields of characteristic 0 this follows from the well known Tits' alternative.

Varieties $\mathcal{A}^n$ and its subvarieties have nice Burnside properties: every periodic group there is locally finite. Varieties $\mathcal{N}_c\mathcal{A}$ have some nice additional properties. In particular, Krasilnikov [204] proved that these varieties are hereditary finitely based. This makes solvable varieties of groups and in particular nilpotent-by-Abelian varieties a potentially good field to study algorithmic problems.

Nevertheless the class of solvable groups is in many cases more complicated than other classes considered in this survey.

6.3 The Identity Problem

The identity problem for groups was posed by Mal'cev in [200] (Problem 2.40). It has been solved in the negative by Yu. Kleiman.

Theorem 6.1 *(Yu. Kleiman, [197]). There exists a finitely based variety $\mathcal{V}$ of groups such that the identity problem in $\mathcal{V}$ is undecidable.*

The variety $\mathcal{V}$ is contained in $\mathcal{A}^7$. The word problem is undecidable in any free noncyclic group of this variety.

We describe Kleiman's method in detail in Section 7.7.3.

S.V.Aivazyan [8] improved Kleiman's result by providing a similar example within the variety of all solvable groups of derived length at most 5. The proof is based on Kleiman's work.

Recently Storozhev [382] proved that every countable abelian group can be embedded as a verbal subgroup of a center of a relatively free group in some variety. This result provides another way to construct relatively free groups with undecidable word problem. Unfortunately, the varieties constructed by Storozhev are not finitely based. It would be interesting to find similar embeddings into relatively free groups in some finitely based varieties. Storozhev's varieties are far from being solvable.

Another theorem by Yu. Kleiman shows that even "good" groups may have undecidable equational theories.

Theorem 6.2 *(Yu. Kleiman, [197]). There exist a finitely generated solvable of step 4 group with decidable word problem and undecidable equational theory.*

Problem 6.1 *Is there a finitely presented group with an undecidable equational theory?*

Most important varieties have decidable equational theories. If the word problem is solvable in the relatively free groups in the varieties $\mathcal{V}_1$ and $\mathcal{V}_2$ then it is also solvable in the relatively free groups in their product. This follows from Shmel'kin's theorem [374], [373] on the embedding of the free n-generated group in $\mathcal{V}_1\mathcal{V}_2$ into the wreath product of the free n-generated group in $\mathcal{V}_1$ and the free n-generated group in $\mathcal{V}_2$. This embedding is effectively given. This implies the solvability of the equational theories of the varieties $\mathcal{A}^n$, $\mathcal{N}_c\mathcal{A}$ and $\mathcal{A}\mathcal{N}_c$.

Using Krasilnikov's proof of the result that the varieties $\mathcal{N}_c\mathcal{A}$ are hereditary finitely based [205], it seems possible to prove that the equational theory is decidable in any subvariety of the variety $\mathcal{N}_c\mathcal{A}$.

Finitely based varieties with undecidable equational theory have non-residually finite free groups. The property "to be Hopfian" is weaker than the property "to be residually finite" in the case of finitely generated groups. The problem of whether there exists a finitely based variety of groups with non-Hopfian free groups was posed by H. Neumann (see [279], Problem 15). Recently S. Ivanov announced a negative solution of this problem [155]. He constructed a variety given by two relatively simple identities where every non-cyclic free group is not Hopfian. It is still not known whether there exists a solvable variety with this property.

6.4 The Identity Problem For Finite Traces

By Connection 2.9 the identity problem is solvable in the finite trace of any subvariety of the variety $\mathcal{N}_c\mathcal{A}$. It is also solvable in every periodic variety of groups because of the positive solution of the Restricted Burnside problem (see Connection 2.10).

Problem 6.2 *(Rhodes, [9]) Is the identity problem for finite groups solvable?*

This question is very interesting and rather complicated.

6.5 The Word Problem

In this subsection, we first consider varieties of solvable groups and then varieties of periodic groups.

6.5.1 Solvable groups

There are two classes of solvable group varieties where the solvability of the word problem is well known: the varieties of nilpotent groups and the varieties of metabelian groups.

As we mentioned above all nilpotent and metabelian varieties of groups are finitely based. Every finitely generated nilpotent group is finitely presented, representable by matrices over Z and residually finite (see, for example Kargapolov and Merzljakov [163], Hall [137], Mal'cev [236]). This implies the solvability of the word problem in nilpotent varieties. Finitely generated groups in the variety $\mathcal{A}^2$ are finitely presented in this variety and residually finite, hence have solvable word problem (P. Hall [137]). Thus every metabelian variety of groups has solvable word problem.

More recent investigations have been inspired by the following problems.

1. Determine whether or not the word problem is solvable for groups, relatively finitely presented in the variety $\mathcal{A}^n, n \geq 3$ (Mal'cev, [238]).

In our terminology, the question is whether the word problem is solvable in the varieties $\mathcal{A}^n$.

2. Construct a finitely presented group, satisfying a nontrivial identity, with unsolvable word problem (Adian, problem 4.3 in [200]).

   In our terminology, this problem asks to construct a proper subvariety $\mathcal{V}$ of the variety of all groups, in which the word problem is strongly unsolvable. A similar problem posed by Remeslennikov and Romanovskii [318] asked whether there exists a solvable variety with a strongly unsolvable word problem.

3. Determine whether or not every recursively presented group in the variety $\mathcal{A}^n$ is embeddable in a relatively finitely presented group in the variety $\mathcal{A}^m$, for some m (Remeslennikov, problem 5.46 in [200]).

Remeslennikov [315] obtained a solution of Problem 1. He proved that the word problem for the variety $\mathcal{A}^n, n \geq 5$ is unsolvable. This solved Problem 1 in the negative. Using this result, Remeslennikov and Kirkinskii [193] obtained a negative solution of the isomorphism problem for the varieties $\mathcal{A}^n, n \geq 7$.

Kukin and Epanchincev published an article [92] where they claimed the unsolvability of the word problem for any variety $\mathcal{W}$ containing $\mathcal{N}_2\mathcal{A}$. The result is incorrect (see [185], [215]). In [46] it was claimed that the proof can be fixed if one simply replaces $\mathcal{N}_2\mathcal{A}$ by $\mathcal{N}_3\mathcal{A}$ in the formulation of the result and in the proof. But in Section 7.5 we shall show that the proof does not work even after this change.

Later it was proved by the first author [187] that the word problem is undecidable in any variety containing $Z\mathcal{N}_2\mathcal{A}$, in particular in the variety $\mathcal{N}_3\mathcal{A}$.

The solution of Problem 2 was obtained in the article [175] by the first author. It was proved that there exists a finitely presented group with unsolvable word problem that belongs to the variety $\mathcal{A}_2^2\mathcal{A}\cap\mathcal{N}_4\mathcal{A}$. This means that the word problem is strongly unsolvable in any variety containing the variety $\mathcal{A}_2^2\mathcal{A}\cap\mathcal{N}_4\mathcal{A}$. The proof is based on an interpretation of a two-tape Minsky machine with unsolvable halting problem. The method is described in section 7.2.8.

Later G. Baumslag, D. Gildenhuys and R. Strebel [27] found a slightly different approach in constructing finitely presented solvable groups with unsolvable word problem. Namely, they also interpret a two-tape Minsky machine, but the language of matrix groups that they use helps them simplify the proof. Their result is also more general: they constructed finitely presented groups with unsolvable word problem in the varieties $Z\mathcal{N}_3\mathcal{A} \cap \mathcal{A}_p^2\mathcal{A}$ for any $p \geq 2$. They also proved the unsolvability of the isomorphism problem in these varieties. In the case $p = 2$ the group constructed in [27] is isomorphic to a subgroup of the group in [175] generated by all but two of the generators.

These results made it clear that a part of the boundary between varieties with decidable (weakly decidable) and undecidable (strongly undecidable) word problem could be found inside the variety $\mathcal{N}_4\mathcal{A}$.

Then, the variety was made smaller again.

Theorem 6.3 *(Kharlampovich, [176]). The variety $Z\mathcal{N}_2\mathcal{A}$ has a strongly undecidable word problem.*

This variety is given by the following identity:

$$((((x_1, x_2), (x_3, x_4)), (x_5, x_6)), x_7) = 1. \tag{24}$$

We write as usual (a, b, c) instead of $((a, b), c)$.

The proof of this theorem employs an interpretation of differential equations in groups (see Section 7.6).

On the other hand the word problem is decidable in the variety $Z\mathcal{A}^2$. This result was proved by Romanovskii [328].

Romanovskii's result was crucial. It was the first case where the solvability of the word problem was not proved as a consequence of residual finiteness. The variety $Z\mathcal{A}^2$ is not locally residually finite and it is still unknown (and very interesting) if there are groups which are finitely presented in $Z\mathcal{A}^2$ and not residually finite. The result of Romanovskii does not imply the solvability of the word problem in subvarieties of $Z\mathcal{A}^2$.

This result was strongly improved by the following theorem of the first author.

Theorem 6.4 *(Kharlampovich, [185]). If $\mathcal{V} \subseteq \mathcal{N}_2\mathcal{A}$ then the word problem in $\mathcal{V}$ is solvable.*

The weak decidability of the word problem in $\mathcal{N}_2\mathcal{A}$ was proved by Bieri and Strebel in [38]. To be more precise, Bieri and Strebel proved the following important theorem.

Theorem 6.5 *(Bieri and Strebel, [38]). Every absolutely finitely presented group which belongs to the variety $\mathcal{N}_2\mathcal{A}$ is residually finite.*

Earlier Groves [117] proved the same result for the smaller variety of central-by-metabelian groups.

A technique used in the paper [38] is remarkable. The authors use the following concept of a sphere of valuations of a finitely generated Abelian group. A *valuation* of a finitely generated Abelian group H is a homomorphism $v : H \to \mathbf{R}$ into the additive group of $\mathbf{R}$. Two valuations are equivalent if they coincide up to a positive constant scalar multiple. The unit sphere S^{n-1} in $\mathbf{R}^n$ can be identified canonically with the set $S(H)$ of all equivalence classes $[v]$. An explicit description of $S(H)$ is the following. Let T be the torsion subgroup of H. Since H/T is a direct product of finitely many infinite cyclic groups, there exists a homomorphism $\theta : H \to \mathbf{R}^n$, which maps H/T isomorphically onto $\mathbf{Z}^n \subset \mathbf{R}^n$. Fix one of such θ. Every valuation $v : H \to \mathbf{R}$ can be extended to a unique $\mathbf{R}$-linear map $\bar{v} : \mathbf{R}^n \to \mathbf{R}$ such that $v = \bar{v} \circ \theta$. The vector space

$\mathbf{R}^n$ is endowed with the standard inner product. From linear algebra we know that every linear map of $\mathbf{R}^n$ into $\mathbf{R}$ is induced by the product by a vector of $\mathbf{R}^n$. Hence for every valuation v there is a (unique) element $x_v \in \mathbf{R}^n$ with $\bar{v}(y) =< x_v, y >$, for all $y \in \mathbf{R}^n$. Equivalent valuations correspond to parallel vectors. Thus we have a correspondence between the set of equivalence classes of valuations $S(H)$ and the points of the unit sphere S^{n-1}.

For every equivalence class of valuations $[v]$ let us define the monoid $H_v = \{h \in H : v(q) \geq 0\}$ (we identify $[v]$ and its representative). Let us consider finitely generated modules over the commutative ring $\mathbf{Z}H$. A finitely generated $\mathbf{Z}H$-module A may or may not be finitely generated as a module over the semigroup ring $\mathbf{Z}H_v$. If A is any finitely generated $\mathbf{Z}H$-module, the authors study the subset $\Sigma_A = \{[v] : A$ is finitely generated over $H_v\}$ of the topological space $V = S^{n-1}$. This leads to an interesting characterization of finiteness conditions of metabelian groups in terms of the topological behavior of Σ_A. Notice that if G is a metabelian group and A is a normal Abelian subgroup of G such that $H = G/A$ is also Abelian then A may be naturally considered as a $\mathbf{Z}H$-module (H acts on A by conjugation). The main results are the following:

Theorem 6.6 *(Bieri, Strebel [38]). Let $G/A = H$ and let A, H be Abelian. Then G is polycyclic if and only if $\Sigma_A = S^{n-1}$, and G is finitely presented if and only if $\Sigma_A \cup -\Sigma_A = S^{n-1}$.*

This result has many applications. In particular the following results have been obtained.

Theorem 6.7 *(Bieri, Strebel [38]).*

1. *All metabelian homomorphic images of an absolutely finitely presented solvable group are finitely presented,*

2. *If G is a group with the property that G/G'' is absolutely finitely presented then all the homomorphic images of G in $\mathcal{N}_2\mathcal{A}$ satisfy the ascending chain condition for normal subgroups and are residually finite.*

This result implies residual finiteness of finitely presented groups from $\mathcal{N}_2\mathcal{A}$.

Other properties of absolutely finitely presented solvable groups can be found in Bieri and Strebel [36], [37], Strebel [383], Wilson [415], Groves and Wilson [120]. See also [383] for some open problems.

Since the variety $Z\mathcal{N}_2\mathcal{A}$ is hereditary finitely based, the interval of varieties between $\mathcal{N}_2\mathcal{A}$ and $Z\mathcal{N}_2\mathcal{A}$ must contain minimal varieties with unsolvable word problem. It turns out that there are infinitely many such minimal varieties. The following theorem has been proved by the first author of this survey.

Theorem 6.8 *(Kharlampovich). The varieties $Z\mathcal{N}_2\mathcal{A} \cap \mathcal{B}_p\mathcal{A}$, ($p \geq 5$, prime) are minimal varieties with unsolvable word problem.*

The proof of the assertion about unsolvability in this theorem is published in [187], the proof of the minimality has been submitted for publication.

Another series of minimal varieties with undecidable word problem has been found by the second author.

Theorem 6.9 *(Sapir, 1992, unpublished) The varieties $\mathcal{A}_p\mathcal{A}_q\mathcal{A}$ (p, q are distinct. primes) are minimal varieties with unsolvable word problem.*

The ideas of the proof of unsolvability in this theorem are presented in Section 7.2.9. All proper subvarieties of these varieties are locally residually finite, hence have solvable word problem.

In the article [184] by the first author, the word problem for subvarieties of the variety $Z\mathcal{N}_2\mathcal{A}$ was studied. It turned out, as in the case of Lie algebras, that the lower we go in the lattice of subvarieties of $\mathcal{N}_3\mathcal{A}$ the more complicated picture we get. In particular the following surprising result was proved

Theorem 6.10 *(Kharlampovich, [184]). There exists an infinite chain of varieties of groups inside $Z\mathcal{N}_2\mathcal{A}$ in which the varieties with solvable and unsolvable word problem alternate.*

As in the case of Lie algebras this theorem is a corollary of the following two results.

Theorem 6.11 *(Kharlampovich, [184]). The word problem is solvable in the varieties $\mathcal{N}_2\mathcal{N}_c \cap Z\mathcal{N}_2\mathcal{A}$.*

Let $\mathcal{Y}_c$ be a variety defined in $Z\mathcal{N}_2\mathcal{A}$ by the identity

$$((x_1, \ldots, x_{c+2}), (y_1, \ldots, y_{c+2}), (z_1, \ldots, z_c)) = 1.$$

Theorem 6.12 *(Kharlampovich, [184]). The word problem is unsolvable in the varieties $\mathcal{Y}_c$ for any $c \geq 1$.*

Theorem 6.10 follows from these two theorems because $\mathcal{Y}_{c-1} \subset \mathcal{N}_2\mathcal{N}_c \cap Z\mathcal{N}_2\mathcal{A} \subset \mathcal{Y}_{c+1}$.

Before mentioning the subsequent results concerning the variety $Z\mathcal{N}_2\mathcal{A}$, we will formulate the following conjecture that underscores the significance of these results.

Conjecture 6.1 *Any variety of solvable groups with hereditary solvable word problem belongs for some c and k to the variety $\mathcal{N}_c^2\mathcal{B}_k$.*

It is well known that solvable groups with an identity $x^k = 1$ are locally finite. So, if this conjecture is true, then to get a description of all varieties of solvable groups with hereditary decidable word problem we need to consider only groups in $\mathcal{N}_c^2$.

The technique of proving solvability of the word problem developed in Kharlampovich [184], suggest conjecture that for solvable groups the variety $\mathcal{N}_3\mathcal{A}$ is an indicator variety with respect to hereditary solvability of the word problem.

Conjecture 6.2 *A metanilpotent variety of groups $\mathcal{W}$ has hereditary solvable word problem if and only if $\mathcal{W} \cap \mathcal{N}_3\mathcal{A}$ has hereditary solvable word problem.*

So it is very important to consider subvarieties of $\mathcal{N}_3\mathcal{A}$.

Problem 6.3 *Describe all subvarieties of the variety $\mathcal{N}_3\mathcal{A}$ with hereditary solvable word problem.*

We need now some new definitions to describe another series of minimal varieties with unsolvable word problem. These definitions are similar to those for Lie algebras.

Let F be a free group of infinite rank of the variety $Z\mathcal{N}_2\mathcal{A}$. Then the group (F', F', F') is Abelian, and we use the additive notation for this group. If u, v and w are some elements of the group ring $Z[F/F']$, it makes sense to define

$$(\ell_1, \ell_2, z) \circ u\check{v} = (\ell_1^u, \ell_2^v, z),$$

$$(\ell_1, \ell_2, \ell_3) \circ u\check{v}\bar{w} = (\ell_1^u, \ell_2^v, \ell_3^w).$$

We have also a relation $(\ell_1, \ell_2, \ell_3) \circ \bar{x} = (\ell_1, \ell_2, \ell_3) \circ x^{-1}\check{x}^{-1}$.

Let α, β be two coprime integers, $\alpha > 0$, $\beta \neq 0$. Let $< x, \check{x} \mid x\check{x} = \check{x}x >$ be the free Abelian group on the generators $x, \check{x}$. Let $P(\alpha, \beta, x)$ be the polynomial from $Z[< x, \check{x} \mid x\check{x} = \check{x}x >]$ with coefficient 1 at the highest power of x, and containing each factor of the polynomial

$$(x^{\alpha+\beta} - \check{x}^{-\alpha})(x^{\alpha+\beta} - \check{x}^{-\beta})(x^{-\alpha} - \check{x}^{\alpha+\beta})(x^{-\beta} - \check{x}^{\alpha+\beta})$$

exactly once (factors f and $fx^t\check{x}^s$, where $t, s \in \mathbf{Z}$ are considered equivalent). Let $R(\alpha, \beta, x_1, x_2, x_3, x_4)$ be obtained from $P(\alpha, \beta, x)$ by introducing a new variable for each factor. If the number of factors is less then four then x_4 is a fictitious variable. For example

$$R(1, -1, x_1, x_2, x_3, x_4) = (x_1 - 1)(\check{x}_2 - 1).$$

Let $\mathcal{W}_{\alpha,\beta}$ be the subvariety of the variety $Z\mathcal{N}_2\mathcal{A}$ defined by the identity

$$(\ell_1, \ell_2, z) \circ R(\alpha, \beta, x_1, x_2, x_3, x_4)R(\alpha, \beta, x_5, x_6, x_7, x_8) = 0.$$

The following theorem is stronger than Theorem 5.6

Theorem 6.13 *(Kharlampovich, [184]). The word problem is strongly unsolvable in the varieties $\mathcal{W}_{\alpha,\beta}$, $(\alpha, \beta) \neq (1, -1)$, If some subvariety W of the variety $\mathcal{W}_{\alpha,\beta}$ has some additional enveloping identity then the word problem in W is solvable.*

In particular, if $\alpha \neq \alpha_1$ or $\beta \neq \beta_1$, then the word problem is solvable in the varieties $\mathcal{W}_{\alpha,\beta} \cap \mathcal{W}_{\alpha_1,\beta_1}$.

The varieties $\mathcal{Y}_c$ were the first examples of group varieties with unsolvable but weakly solvable word problem. These varieties are not strongly Higman varieties. This means that not every recursively presented group in these varieties is embeddable in an absolutely finitely presented group belonging to the same variety.

Problem 6.4 *Describe as completely as possible solvable group varieties with solvable (weakly solvable) word problem.*

Among other important results about the word problem and varieties of solvable groups, let us mention the following theorem by Yu.Kleiman.

Theorem 6.14 *(Yu. Kleiman, [196]). Let $\mathcal{X}$ be any countable set of groups. Then there exists a solvable group variety of step 4 which is not generated by members of $\mathcal{X}$.*

In particular, let $\mathcal{X}$ be the set of recursively presented groups. This set is obviously countable. Thus there exists a variety of solvable groups which is not generated by its recursively presented groups. In particular, this variety cannot be generated by groups with solvable word problem.

We have mentioned already that the word problem is decidable in relatively free groups in the varieties $\mathcal{A}^n$. We have also mentioned (see Talapov's Theorem 5.3) that in the case of Lie algebras the word problem is solvable in every algebra given by one defining relation in $\mathcal{A}^n$. In the group case the analogous result is not known. Kargapolov posed the corresponding problem as early as 1965.

A partial solution of this problem has been obtained by Romanovskii [324]. Let F be a free group. We say that an element r is primitive modulo the derived series if the inclusion $r \in F^{(n)} \backslash F^{(n+1)}$ implies that r is not a proper power in $F^{(n)} \backslash F^{(n+1)}$. Romanovskii [324] proved that the word problem is solvable in groups given in $\mathcal{A}^n$ by one defining relation, when the relator is a primitive element modulo the derived series. In this case the factors of the derived series of the one-related group are torsion-free, and the situation is close to that for Lie algebras.

To complete the topic of the word problem in solvable varieties, let us remark that it would be very interesting to investigate the complexity of the algorithms solving the word problem in solvable group varieties (see the necessary definitions in section 3.4.5). In particular we have the following conjecture.

Conjecture 6.3 *For every subvariety of the variety $\mathcal{N}_2\mathcal{A}$ and for every subvariety of the variety $\mathcal{A}\mathcal{N}_c$ the word problem is solvable in polynomial time.*

6.5.2 Periodic Groups

The well known Burnside problem can be rewritten in the form: "Is there a variety of groups of finite exponent which is not locally finite?" In 1968 Novikov and Adian solved this problem in the affirmative. They proved that for any odd number $n \geq 4381$ the group variety $\mathcal{B}_n$ is not locally finite. In [4] Adian lowered this bound to 665. In [154] and [153] the analogous result was announced and proved for any $n \geq 2^{48}$; in [230] it was announced for $n \geq 2^{13}$. The property "to have an unsolvable word problem" is stronger than the property "to be non-locally finite". In [5] Adian constructed an example of a group, presented by an infinite recursive set of relations in the variety $\mathcal{B}_n$, which has an unsolvable word problem. In [6] he and Makanin posed a question: *whether there exists such an example with finitely many defining relations?*

The first example of a periodic group variety with unsolvable word problem is given by the following theorem by the second author of the survey.

Theorem 6.15 *(Sapir, [345]). The word problem is unsolvable in the variety $\mathcal{A}_r\mathcal{B}_p$ for every odd $p \geq 665$ and every prime $r \neq p$.*

The ideas of the proof are presented in Section 7.2.9. With the use of methods developed in [345], the following theorem has been recently proved by the first author of this survey.

Theorem 6.16 *(Kharlampovich, [186]). If $n = pm$, where p is prime ≥ 3 and m either has an odd divisor ≥ 665 or $n \geq 2^{48}$, then there exists a group, finitely presented in the variety $\mathcal{B}_n$, with unsolvable word problem.*

The main idea of the proof is the embedding of a group similar to the one constructed in [345] into a finitely presented group inside $\mathcal{B}_n$.

6.6 The Generalized Freiheitssatz

As we already mentioned in Section 5.5, the basic theorems about groups with one defining relation, the Freiheitssatz and the solvability of the word problem, were proved by Magnus in [232], [231] (see also [229]). All the necessary definitions are given in Section 5.5.

The generalized Freiheitssatz for the variety of all groups was conjectured by Lyndon [199]. Romanovskii proved Lyndon's conjecture and the validity of the generalized Freiheitssatz for the varieties $\mathcal{A}^n$ [326] and $\mathcal{N}_c$ [327]. Yabangi proved it for the varieties $\mathcal{A}\mathcal{N}_2$ and $\mathcal{N}_2\mathcal{A}$. C. K. Gupta and N. S. Romanovskii [131] proved it for the variety $Z\mathcal{A}$.

It is interesting that in some sense Romanovskii proved the generalized Freiheitssatz for the variety of all groups and for the variety $\mathcal{A}^n$ simultaneously. He showed that if F is the absolutely free group given by $n+m$ generators $x_1, \dots, x_{n+m}$ and N is a normal subgroup of F generated by m elements, then there exist n generators $x_{i_1}, \dots, x_{i_n}$ with the following property: If H is the subgroup of F generated by $x_{i_1}, \dots, x_{i_n}$ then $H^{(k)} = H \cap NF^{(k)}$ for any k. Here $F^{(k)}$ is the k-th derived subgroup of F.

The generalized Freiheitssatz for the variety of all groups is now just a consequence of the fact that the intersection of all derived subgroups of a free group is trivial.

The generalized Freiheitssatz for the varieties of nilpotent groups follows from the corresponding result of Romanovskii for pro-p-groups from [327]. A pro-p-group is a projective limit of finite discrete p-groups. A free pro-p-group $F(X)$ on a set X is the projective limit of the groups $L(X)/N$, where $L(X)$ is the free (discrete) group on X, and N runs through the normal subgroups of index a power of p, with N containing almost all the elements of X. An identity in the class of pro-p-groups has the form $f = 1$ where f is an element of $F(X)$, where X is a countable set. This identity is said to hold in a pro-p-group G if every continuous homomorphism from $F(X)$ into G sends the element f to 1. By the analogy with discrete groups, one can define the notion of a variety of pro-p-groups, and the notion of a pro-p-group given in a variety by generators and relations. Romanovskii proved that the generalized Freiheitssatz holds in the variety of all pro-p-groups, in the variety of pro-p-groups nilpotent of step c, and in the variety of pro-p-groups solvable of step k.

Finally notice that the Freiheitssatz does not hold in every variety of groups. For example, it does not hold in the variety $\mathbf{B}_{pq}$ where p and q are distinct primes. Indeed, let G be the group given in $\mathbf{B}_{pq}$ be two generators x, y and one defining relation $x^p y^q = 1$. It is easy to see that this relation implies $x^p = 1$ and $y^q = 1$. Thus neither x nor y generate a relatively free group in $\mathbf{B}_{pq}$. A similar argument shows that the Freiheitssatz holds in a periodic variety of groups only if the exponent of this variety is a prime power.

Problem 6.5 *Prove or disprove the (generalized) Freiheitssatz for the variety* $\mathbf{B}_p$ *where* p *is a prime.*

6.7 The Isomorphism Problem And Related Problems

The construction used for the proofs of the results about the unsolvability of the word problem in varieties of solvable groups, as in the cases of Lie and associative algebras, is based on interpretations of either Minsky machines (see Section 7.2) or systems of linear differential equations (see Section 7.6.2). As we have already mentioned in Sections 4.5 and 5.6, a program of the Minsky machine, or a system of linear differential equations can be chosen in such a way that the group with unsolvable word problem will be Hopfian.

Hence by Connection 2.1 the following result is true.

Theorem 6.17 *All varieties of solvable groups with undecidable (resp. strongly undecidable) word problem enumerated in the previous section have undecidable (resp. strongly undecidable) isomorphism problem.*

As in the case of Lie algebras, we do not know very many non-locally finite varieties of groups with decidable isomorphism problem. In fact the largest known class of group varieties with decidable isomorphism problem consists of products of nilpotent and locally finite varieties, and their finitely based subvarieties. Even for such varieties the isomorphism problem is extremely non-trivial.

The isomorphism problem for finitely generated nilpotent groups was solved in the positive by Grunewald and Segal [122], [123]. Later Segal [366], [365] proved the following more general result.

Theorem 6.18 *(Segal, [366], [365]). There exists an algorithm that given two finite presentations of polycyclic-by-finite groups, decides if these groups are isomorphic and if so gives an explicit isomorphism.*

The proof in [365] is based on the same ideas as the proofs in [122] and [123]. These ideas are the following.

To prove the solvability of the isomorphism problem for polycyclic-by-finite groups we first of all reduce this problem to the analogous problem for torsion free polycyclic-by-finite groups. For any two finitely generated torsion-free polycyclic-by-finite groups G_1 and G_2, one can find numbers n_1 and n_2 that satisfy the following two properties:

1. the semidirect product of G_i and $Aut(G_i)$, where $i = 1, 2$ is effectively embeddable into the group of invertible integer matrices of order n_i, $GL_{n_i}(\mathbf{Z})$, (we denote this embedding by β_{G_i}). Here "effectively embeddable" means that we can compute the image of each generator of G_i

2. Groups G_1 and G_2 are isomorphic if and only if $n_1 = n_2 = n$ and there exists an integer matrix $A \in GL_n(\mathbf{Z})$ such that $A^{-1}\beta_G A = \beta_H$.

Now we can consider the set of all polycyclic-by-finite subgroups of the group $GL_n(\mathbf{Z})$ for some n. The group $GL_n(\mathbf{Z})$ acts on this set by conjugations. Our goal is to check if two subgroups belong to the same orbit of this action.

An important feature of the papers of Grunewald and Segal is a (partial) reduction of this problem to the problem of whether two vectors of a vector space with an action of $GL_n(\mathbf{Z})$ are in the same orbit of this action.

This shows that one has to consider actions of the group $GL_n(\mathbf{Z})$ on vector spaces. Groups $GL_n(\mathbf{Z})$ belong to the class of so-called arithmetic groups (see the definition in [55], [366]) which arise in algebraic topology, number theory, etc. There exists a deep theory of actions of arithmetic groups (see, [55]). The first problem was to make this theory "effective", that is to find algorithms where only existence theorems were known. A solution to this problem was an important achievement of Grunewald and Segal.

Using these algorithms they proved that if an action of an arithmetic group on a vector space is in some natural sense explicitly given then it is algorithmically decidable whether two elements are in the same orbit. This result has many applications to different algorithmic problems not only in algebra but also in number theory. We have mentioned some of the applications to the isomorphism problem in Sections 4.5, 5.6. In particular this result allowed Grunewald and Segal to complete the solution of the isomorphism problem for nilpotent and then for polycyclic-by-finite groups.

It is worth mentioning that the solvability of the problem of isomorphism to a fixed finitely presented nilpotent group G had been proved earlier by Pickel [296]. We have mentioned his result in the Introduction. Recall that two universal algebras are called quasi-isomorphic if they have the same finite homomorphic images. Pickel proved that the set of non-isomorphic finitely generated nilpotent groups which are quasi-isomorphic to a fixed group G is finite. Later Grunewald, Pickel and Segal extended this result to arbitrary polycyclic-by-finite groups [124].

By a result of Groves [119], every finitely generated group in a variety is polycyclic-by-finite if and only if this variety does not contain varieties $\mathcal{A}_p\mathcal{A}$ for all prime p.

Thus the varieties $\mathcal{A}_p\mathcal{A}$ are the smallest varieties which are not covered by Theorem 6.18. Unfortunately even for these varieties the solvability of the isomorphism problem is not known.

Problem 6.6 *Is the isomorphism problem decidable in the following varieties:*

a) $\mathcal{A}_p\mathcal{A}$*, where* p *is prime;*

b) $\mathcal{A}\mathcal{A}$ *(the variety of metabelian groups);*

c) $\mathcal{N}_2\mathcal{A}$ *?*

It is quite possible that the answers in cases a), b), c) are different.

The problem of isomorphism to a fixed group G in these varieties is also hard and the answers are not known. Groves and Miller [121] showed that it is decidable whether a finitely presented metabelian group is a free metabelian group. Thus the problem of isomorphism to a free metabelian group is decidable in $\mathcal{A}^2$. Recently Noskov [283] announced that this result can be obtained by a Pickel-type argument: for every finitely generated free metabelian group G there are only finitely many finitely generated metabelian groups which are quasi-isomorphic but not isomorphic to G. Pickel [297] constructed an example of infinitely many non-isomorphic but quasi-isomorphic finitely generated metabelian groups.

Problem 6.7 *Describe varieties of solvable groups in which every set of quasi-isomorphic finitely generated groups is finite. Does the variety* $\mathcal{A}_p\mathcal{A}$ *satisfy this property for some* p*?*

Notice that if the answer to the second half of this problem is negative then the first part has an easy solution: nilpotent-by-locally finite varieties. This follows from the result of Groves [119] mentioned above.

The general isomorphism problem for metabelian groups is actually a commutative algebra problem. Connections between algorithmic problems for metabelian groups and algorithmic problems for modules over commutative rings are discussed, for example, in [26]. These connections are based on the fact that the first derived subgroup G' of any metabelian group G is a module over the commutative ring $\mathbf{Z}(G/G')$. If G is finitely generated then the ring and the module are finitely generated. This follows from the result of P.Hall that every normal subgroup in a finitely generated metabelian group is finitely generated as a normal subgroup [137], [138]. If two metabelian groups are isomorphic then these rings must be isomorphic and the modules must be isomorphic also. The isomorphism problems for finitely generated commutative rings and for finitely generated modules over finitely generated commutative rings are very hard and the answers are unknown.

The preceding results may lead to the impression that varieties of solvable groups are no better algorithmically than arbitrary varieties of groups. Baumslag, Cannonito and Miller showed that this is not so: many algorithmic properties which are undecidable in the variety of all groups turn out to be decidable in varieties of solvable groups. Thus even though we can't say in general whether or not two finitely presented solvable groups are isomorphic, we can decide if these groups have important properties in common. The following fundamental theorem was proved in [25].

Theorem 6.19 *(Baumslag, Cannonito, Miller [25]). There is an algorithm which, given a finite presentation of a group in the variety $\mathcal{A}^n$, decides if the group is polycyclic, and if so, produces the polyciclic presentation of this group.*

This result immediately implies the following striking corollary.

Corollary 6.1 *The following properties of a group finitely presented in the variety $\mathcal{A}^n$ are effectively recognizable: polycyclic; supersolvable; nilpotent; Abelian; finite; cyclic; trivial.*

In the case of metabelian groups one can get even more information.

Theorem 6.20 *(Baumslag, Cannonito, Robinson [26]). Let G be a relatively finitely presented metabelian group. There is an algorithm which finds a finite presentation of the $Z(G/G')$-module G'. Hence there is an algorithm which finds the centre $Z(G)$, and also a finite presentation of $Z(G)$. There is also an algorithm which finds (a finite subset whose normal closure is) the Fitting subgroup $Fit(G)$.*

This theorem has the following corollary.

Corollary 6.2 *Let G be a relatively finitely presented metabelian group. Then there exist algorithms which can:*

decide if G is torsion-free;

decide if a given element of G has finite order;

enumerate all possible orders of elements in G;

find the limit of the lower central series of G;

decide if a finitely presented metabelian group is residually nilpotent;

find the Frattini subgroup of G.

6.8 The Conjugacy Problem

The conjugacy problem is undecidable in varieties with undecidable word problem. From the undecidability of the word problem, it follows that every variety containing, for example, $Z\mathcal{N}_2\mathcal{A}$ has undecidable conjugacy problem. On the other hand, the conjugacy problem for finitely generated metabelian groups was shown to be decidable by Noskov [282]. In [26], Baumslag, Cannonito and Miller presented an algorithm which, given a finitely generated metabelian group G and two sequences $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ of elements of G, decides if there exists an element g of G such that $x_i^g = y_i$ for $i = 1, 2, \ldots, n$.

Varieties which are above the variety of metabelian groups are much less studied. In [130] C. Gupta, Herfort and Levin proved the solvability of the conjugacy problem for some relatively free centre-by-metabelian groups including the free centre-by-metabelian groups of rank 2 and 3. In [132], C. Gupta, N. Gupta and Levin solved the conjugacy problem for all relatively free centre-by-metabelian groups. Earlier Kargapolov and Remeslennikov [164] solved the conjugacy problem for free solvable groups.

Problem 6.8 *Describe solvable group varieties with decidable conjugacy problem. Is it true that the conjugacy problem is decidable in a) every centre-by-metabelian variety b) every subvariety of $\mathcal{N}_2\mathcal{A}$?*

See also the survey articles Gupta [129] and Hurwitz [148] for the results on conjugacy separability. Recall that a group G is conjugacy separable if for every non-conjugate pair of elements x and y in G, there is a homomorphism ϕ onto a finite group H such that $\phi(x)$ and $\phi(y)$ are not conjugate. Finitely presented conjugacy separable groups have solvable conjugacy problem [236]. The survey [148] also contains results on the complexity of solutions to the conjugacy problem and some open problems.

6.9 The Uniform Word Problem For Finite Traces

The uniform word problems for the class of finite groups and for the class of finite nilpotent groups was posed by Mal'cev and written down by Kargapolov in [200] in 1965. In 1978 Slobodskoj proved the unsolvability of this problem for the class of all finite groups [378]. In 1979 the first author of this survey proved the analogous result for finite nilpotent groups [180]. In [181] she investigated some other classes of finite groups. Let $\mathcal{N}$ be the class of all nilpotent groups and $\mathcal{F}$ be the class of all finite groups. Let $\mathcal{X}$ be an arbitrary class of groups such that $Z\mathcal{N}_2\mathcal{A} \subseteq \mathcal{X}$.

Theorem 6.21 *(Kharlampovich, [180], [181]). The uniform word problem is undecidable for the following classes of groups: $\mathcal{F}$; $\mathcal{N}$; $\mathcal{F} \cap \mathcal{N}$; $\mathcal{F} \cap \mathcal{X}$; $\mathcal{N} \cap \mathcal{X}$; $\mathcal{N} \cap \mathcal{X} \cap \mathcal{F}$.*

Notice that a statement similar to Conjecture 5.3 does not hold for groups: there exists a group variety where the word problem is undecidable but the uniform word problem for finite traces is decidable. Indeed, we have mentioned that there exist periodic group varieties with undecidable word problem (Theorems 6.15, 6.16). The solvability of the uniform word problem for finite traces of these varieties follows from the positive solution of the restricted Burnside problem (see Connection 2.10).

Nevertheless it is possible that a statement similar to Conjecture 5.3 may hold for solvable group varieties.

6.10 The Higman Property

In Kukin [214] it was claimed that for any variety $\mathcal{V}$ containing $\mathcal{N}_2\mathcal{A}$ every recursively generated group in $\mathcal{V}$ can be embedded into a group finitely presented in $\mathcal{V}\mathcal{A}^2$. The proof contains a serious gap (see Section 7.5), so this assertion can only be considered a conjecture.

Notice that even if Kukin's theorem were correct, it would not provide an example of a proper Higman variety of groups. In all proper Higman varieties that are currently known, every finitely generated group is finitely presented. On the other hand we do not know of any "sufficiently large" proper variety of groups which has been proved to be non-Higman. All we know is that, as in the case of Lie algebras, not all natural varieties of groups are Higman varieties. For example, the variety $Z\mathcal{A}^2$ is non-Higman. It is a corollary of the fact that the word problem is decidable in this variety (Romanovskii [328]), but there are examples of recursively presented groups in this variety with undecidable word problem.

Thus we formulate the following problems.

Problem 6.9 *Prove or disprove that the varieties $\mathcal{A}^n$ are Higman varieties.*

Problem 6.10 *Prove or disprove that the varieties $\mathcal{B}_n$ are Higman varieties for large n.*

As in the case of Lie algebras we say that a variety is strongly Higman if every recursively presented group in the variety is embeddable in an absolutely finitely presented group belonging to this variety. From Bieri and Strebel, [38] it follows that the variety $\mathcal{N}_2\mathcal{A}$ is not strongly Higman. The varieties $\mathcal{Y}_c$ constructed in Theorem 6.12 are also not strongly Higman.

Problem 6.11 *Prove or disprove that the variety $\mathcal{B}_n$ for large n is strongly Higman.*

Notice that a positive answer to this question would imply the existence of a finitely presented infinite group of finite exponent. This would solve the well known problem of finding such groups. As far as we know, this problem was first posed by P.S.Novikov in the fifties or early sixties.

6.11 Locally Residually Finite Varieties

The following result shows that in the group case (like the cases of associative and Lie algebras and unlike the case of semigroups) there are varieties with solvable word problem which contain relatively finitely presented non-locally finite groups.

Theorem 6.22 *(Kharlampovich, Sapir, [192]). For any prime p, $p \geq 3$ there exists a relatively finitely presented group belonging to the variety $\mathcal{N}_2\mathcal{A} \cap \mathcal{B}_p\mathcal{A}$ which is not residually finite.*

Recall that the varieties $\mathcal{A}_p\mathcal{A}_q\mathcal{A}$ (p, q are distinct primes) are minimal varieties with respect to three properties: to have undecidable word problem, to residually finite relatively finitely presented groups and to be locally residually finite (see Theorem 6.9).

We do not know if there are non-locally residually finite varieties of groups where every relatively finitely presented group is residually finite. It is possible that these two properties coincide in the group case (the situation for other types of algebras is described in the previous sections of the survey).

The study of locally residually finite varieties of groups was initiated by A.Mal'cev [236] and P.Hall [138] in 1959. P.Hall also initiated the study of varieties with the property $FG = FP$ which is, of course, equivalent to the property that every finitely generated group in the variety has the ascending chain condition for normal subgroups. As in all other cases considered in this survey these two properties turned out to be very close. Hall [138] proved that every Abelian-by-nilpotent variety of groups satisfies these properties.

A variety of groups is called *metanilpotent* if it is contained in a product of two nilpotent varieties. A description of metanilpotent varieties of groups which satisfy these properties has been obtained by Groves [118]. The key role in the description is played by the varieties $\mathcal{T}_p$ (p is prime) generated by all 2-generated groups which belong to $\mathcal{B}_p\mathcal{A} \cap Z\mathcal{A}^2$, if p is odd, and which belong to $\mathcal{A}_2^2\mathcal{A} \cap Z\mathcal{A}_2\mathcal{A}$ if $p = 2$.

There are two other descriptions of these important varieties. In [116], Groves found for every p a nice group which generates $\mathcal{T}_p$. He uses the so called crown products of groups introduced by B. H. Neumann in [278]. Let G and H be groups and consider the (restricted) wreath product $GwrH$. Let K be a subgroup of the centre of G. Denote by N the set of all those vectors ϕ of the base group $G^{(H)}$ of $GwrH$ such that

1. all coordinates of ϕ are in K;

2. the product of coordinates of ϕ is 1.

Then N is a normal subgroup of $GwrH$ and we define the crown product, Gcr_KH, of G by H with amalgamated subgroup K to be $(GwrH)/N$. Intuitively the direct product $G^{(H)}$ in $GwrH$ is replaced by the relevant central product in the crown product.

Denote by D_p, for each odd prime p, the non-Abelian group of exponent p and order p^3 and by D_2 the dihedral group of order 8. Let $\mathbf{Z}$ denote the additive group of integers. Let T_p be the crown product $D_p cr \mathbf{Z}$ where the amalgamated subgroup is the centre of D_p. Groves proved [116] that for every prime p T_p generates the variety $\mathcal{T}_p$.

Another description of $\mathcal{T}_p$ has been recently found by Kharlampovich and Gildenhuys in [189]. They proved that for odd primes p the variety $\mathcal{T}_p$ coincides with $Z\mathcal{A}^2 \cap \mathcal{B}_p\mathcal{A}$, and $\mathcal{T}_2$ is the subvariety of $Z\mathcal{A}_2\mathcal{A} \cap \mathcal{B}_4\mathcal{A}$ defined by the identities

$$((x_h, x_1, x_4 \dots x_{h-5}, (x_2, x_3))((x_h, x_2, x_4 \dots x_{h-5}, (x_3, x_1))((x_h, x_3, x_4 \dots x_{h-5}, (x_1, x_2)) = 1$$

and

$$((x_1, x_2, x_4 \dots x_{h-5}, (x_1, x_2)) = 1, h = 4, 5, \dots.$$

This set of identities is infinite. By the result of Krasilnikov [204] this variety is finitely based, but a finite basis of identities of $\mathcal{T}_2$ is not known.

Varieties $\mathcal{T}_p$ were introduced by Ph. Hall in [137]. He proved that $\mathcal{T}_p$ contains non-residually finite finitely generated groups and does not satisfy the condition $FG = FP$. In [65], N. Gupta, C. Gupta and Rhemtulla proved that these are precisely all the minimal non-locally residually finite varieties of centre-by-metabelian groups. Groves [118] extended this result to metanilpotent varieties and obtained the following description of locally residually finite metanilpotent varieties of groups.

Recall that if x, y are elements of a group G then x^y denotes $y^{-1}xy$, and if $t = y^{k_1} + \dots + y^{k_n}$ is an element of the group ring $\mathbf{Z} < y >$ then x^t denotes $x^{y^{k_1}} x^{y^{k_2}} \dots x^{y^{k_2}}$. Recall also that (x, y) denotes the commutator $x^{-1}y^{-1}xy$.

Theorem 6.23 *(Groves, [116], [118]). If $\mathcal{V}$ is a variety of metanilpotent groups then the following conditions are equivalent:*

1. *$\mathcal{V}$ is locally residually finite.*
2. *$\mathcal{V} \cap \mathcal{N}_2\mathcal{A}$ is locally residually finite.*
3. *All finitely generated groups in $\mathcal{V}$ satisfy the ascending chain condition for normal subgroups (that is $\mathcal{V}$ satisfies $FG = FP$).*
4. *The set of finitely generated groups in $\mathcal{V}$ is countable.*
5. *$\mathcal{V}$ does not contain any of the varieties $\mathcal{T}_p$, p is prime.*
6. *$\mathcal{V}$ satisfies an identity of the form*

$$((x, y), (z, t)^{u^k}) = \prod_{i=0}^{k-1} ((x, y), (z, t)^{u^i})^{f_i} w, \qquad (25)$$

where f_i belongs to the group algebra $\mathbf{Z} < u >$ and w is a word from the verbal subgroup of the free group $F = < x, y, z, t, u >$ corresponding to the variety $\mathcal{N}_2\mathcal{A}$.

Notice that conditions 2, 6 are not contained in [118] explicitly, but one can easily extract these conditions from this paper.

Condition 4 is added by the second author. The fact that it is equivalent to the other conditions can be proved as follows. First of all one can show that the centre of the free 2-generated group in $\mathcal{T}_p$ contains an infinitely generated Abelian group of exponent p (this was essentially proved in C. Gupta [128] and F. Cannonito and N. Gupta [67]; using this fact Cannonito and N. Gupta constructed an example of a 2-generated centre-by-free metabelian group with an unsolvable word problem). Thus the free group in $\mathcal{T}_p$ has a continuum of homomorphic images. Since the automorphism group of every finitely generated group is countable, the cardinality of the set of 2-generated groups in $\mathcal{T}_p$ is continuum. Thus any variety with countably many finitely generated groups cannot contain varieties $\mathcal{T}_p$. This gives the implication $4 \to 5$. On the other hand any variety with the property $FG = FP$ contains only countably many finitely generated groups. This gives the implication $3 \to 4$.

The second author of this survey has noticed that there exists an algorithm which checks condition 5 in Theorem 6.23. A sketch of the proof follows. Let Σ be a finite system of identities. We need to check whether there exist p such that the variety $\mathcal{T}_p$ satisfies Σ. Since $\mathcal{T}_p$ is generated by $T_p cr \mathbf{Z}$ it is enough to check whether Σ holds in T_p. As we mentioned above T_p is a semi-direct product of an infinite central product of groups D_p of order p^3 and exponent p if p is odd or 4 if $p = 2$. From this, it is not difficult to deduce that the fact that Σ holds in T_p is equivalent to a formula of the following type.

$$(\forall x_1, \ldots, x_n)(\phi_1(\mathrm{Mod}\ p)) or (\phi_2(\mathrm{Mod}\ p)) or \ldots (\phi_m(\mathrm{Mod}\ p))$$

where ϕ_i is a conjunction of linear equations in $x_1, \ldots, x_n$ with integer coefficients which do not depend on p. This formula is equivalent to a formula of the following form:

$$(\forall x_1, \ldots, x_n)\psi = 0(\mathrm{Mod}\ p)$$

where ψ is a polynomial in $x_1, \ldots, x_n$ with integer coefficients. It is easy to show that this formula holds in the ring $\mathbf{Z}$ if and only if coefficients of ψ are divisible by p. Therefore the algorithm for verifying condition 5 is the following. Given Σ, compute the polynomial ψ. Then compute the greatest common divisor of the coefficients of this polynomial. The variety given by Σ contains $\mathcal{T}_p$ for some p if and only if this greatest common divisor is not equal to 1.

Conjecture 6.4 *The only minimal non-locally residually finite varieties of solvable groups are the varieties $\mathcal{T}_p$ and the varieties A_pA_qA (p, q are distinct primes). A variety of solvable groups is locally residually finite if and only if it does not contain any of these varieties.*

Notice that the identity (25) plays the same role as the identities (10) in the case of semigroups, (16) in the case of associative rings, (22) in the case of Lie algebras. It is easy to see that all these identities have similar forms, although the identity (25) looks more ugly. We don't know whether there exists a nicer identity characterizing locally residually finite metanilpotent varieties of groups.

Other properties from the Club of residually finite varieties have not been thoroughly studied in the group case.

Problem 6.12 *Describe solvable locally Hopfian group varieties.*

By Theorem 6.23 every locally residually finite metanilpotent variety of groups satisfies both these properties (see Section 6.11).

P.M.Neumann noticed [279] that if $\mathcal{V}$ is a variety of solvable groups and the free group $F_k(\mathcal{V})$ is non-Hopfian then there are uncountable many k-generated groups in $\mathcal{V}$. This led him to the following problem.

Problem 6.13 *(P.M.Neumann, [279], Problem 16) Is it true that a locally solvable variety contains finitely generated nonhopfian groups iff it contains uncountably many non-isomorphic finitely generated groups.*

Using a result of Groves [119] it is easy to obtain a description of solvable varieties of groups which are locally representable by matrices over a field of characteristic 0. As far as we know this result has never been published before.

Theorem 6.24 *A solvable variety of groups is locally representable by matrices over a field of characteristic 0 if and only if it is nilpotent-by-locally finite.*

The "if" part follows from the fact that every polycyclic-by-finite group is representable by matrices [366]. The "only if" part follows from the fact that the wreath product $\mathbf{Z}_p wr \mathbf{Z}$ is not representable by matrices over a field of characteristic 0 because it has infinitely many elements of order p (see Mal'cev [235]).

We do not know a description of varieties which are locally representable by matrices over arbitrary fields. But it is easy to prove that this class of varieties is strictly smaller than the class of locally residually finite varieties. Indeed, the variety $\mathcal{A}_{pq}\mathcal{A}$ where p and q are distinct primes is not locally representable by matrices over any field because the group $\mathbf{Z}_{pq} wr \mathbf{Z}$ is not representable (the group of matrices over a field of characteristic p cannot have infinitely many elements of order q).

Problem 6.14 *Describe varieties of solvable groups which are locally representable by matrices over commutative rings. Does this class of varieties coincide with the class of locally residually finite varieties?*

Some algorithmic properties of centre-by-matabelian varieties of groups may be found in N. Gupta and Cannonito [67]).

6.12 Residually Finite Varieties of Groups

Residually finite varieties of groups have been completely described by Ol'shanskii in [289], see 3.43. We recall this theorem here.

Theorem 6.25 *(Ol'shansky, [289]). A variety of groups is residually finite if and only if it is generated by a finite group with Abelian Sylow subgroups.*

Variety $\mathcal{V}$ is called residually small if all subdirectly indecomposable systems in $\mathcal{V}$ form a set, i.e. their cardinalities are bounded by some cardinal. Every residually finite variety is residually small. A group variety is called a *SC-variety* if it is contained in the product of solvable and Cross (generated by a finite group) varieties. Sapir and Shevrin described residually small SC-varieties.

Theorem 6.26 *(Sapir, Shevrin, [354]). An SC-variety of groups is residually small if and only if it either is residually finite or is a join of some residually finite variety and the variety of all Abelian groups.*

Problem 6.15 *(Sapir, Shevrin, [354]). Is it true that a group variety is residually small if and only if it either is residually finite or is a join of some residually finite variety and the variety of all Abelian groups?*

6.13 Elementary Theories

The main result about elementary theories of varieties of groups is the beautiful theorem by Zamjatin.

Theorem 6.27 *(Zamjatin, [422]). The elementary theory of a variety $\mathcal{V}$ is decidable if and only if $\mathcal{V}$ consists of Abelian groups.*

This theorem answered the well-known question of Tarski and Ershov. Earlier Ershov [93] proved that if the variety contains a finite nonabelian group, then the elementary theory of this variety is undecidable. This result is weaker than Zamjatin's result, because there exist nonabelian varieties of groups in which all the finite groups are Abelian [287].

The following theorem is not mentioned in [422], but Zamjatin has observed that it follows from the proof in [422].

Theorem 6.28 *(Zamjatin). The elementary theory of the finite trace $\mathcal{V}_{fin}$ is decidable if and only if all groups in $\mathcal{V}_{fin}$ are Abelian.*

7 Methods

You know my methods. Apply them.
Sir Arthur Conan Doyle "A Study in Scarlet"

Here we would like to present the main methods of proving decidability and undecidability of algorithmic problems. In order not to get lost in insignificant (while technically complicated) details, and to show the genesis of the most general methods and ideas, the results discussed here will not be presented in their strongest form. We will also omit some details, leaving ideas in their purity.

7.1 The Decidability of the Word Problem

The first and the most universal method of proving the decidability of the word problem was discovered by Mal'cev [236]. He noticed a remarkable connection between the residual finiteness and the decidability of the word problem.

We mentioned this connection in the introduction (see Connection 2.4).

In many cases (see, for example, Section 3) a variety has decidable word problem only if all algebras finitely presented in this variety are residually finite. But this is not always so: as we have seen before, the variety $\mathcal{N}_2\mathcal{A}$ of Lie algebras or groups contains non-residually finite relatively finitely presented algebras (groups) and has solvable word problem.

Thus one needs other methods of proving the decidability of the word problem.

There are, of course, methods that apply when we are solving the word problem in an algebra given by a specific system of defining relations. For example the word problem is decidable in the case where one can find a finite terminating Church-Rosser presentation. Then there exist a normal form for every word over the alphabet of generators and an effective procedure which transforms every word to its canonical form (see Section 2.9). We don't know if every variety with solvable word problem is Church-Rosser, thus we need other methods to solve the word problem.

One of these methods is in a sense opposite to the Church-Rosser method. Instead of finding the "canonical" word in the set of words which are equal to a given word u one considers this set as a whole and finds some "hidden" structure on this set. To illustrate this method we will consider two examples: a variety of semigroups and a variety of groups.

7.1.1 Semigroups

Here we would like to present some ideas for proving the solvability of the word problem in varieties of semigroups. This ideas were employed initially in Biryukov [41], and later in Mel'nichuk, Sapir, Kharlampovich [268], Mel'nichuk [266], and Sapir [335]. To illustrate these ideas we will show how they work in the variety of commutative semigroups. Other methods for proving the solvability of the word problem in this variety have been discussed in Section 3.4.1.

First of all recall that the free n-generated commutative semigroup A_n with an identity element is simply the direct product of n semigroups of natural numbers. Therefore every element of this semigroup may be represented by a vector of natural numbers[16]. We can consider A_n as a partially ordered set with the natural coordinate-wise order. It is clear that $u < v$ in A_n if and only if u divides v. It is easy to prove that every subset M of A_n has minimal elements, and every element of M is greater than or equal to a minimal element. The following simple statement is attributed to Dickson.

Lemma 7.1 *(Dickson, [83]). Every infinite set T of elements in A_n contains two comparable elements.*

Let R be a finite set of defining relations, i.e. a finite subset of $A_n \times A_n$. We want to show that the word problem is decidable in the factor semigroup of A_n modulo the congruence generated by R. In other words, we want an algorithm, which, given a pair of elements (u, v) in $A_n \times A_n$, determines if u equals v modulo relations of R.

Take an element u in A_n. We will describe the set $M(u)$ of all elements of A_n which are equal to u modulo R.

For every subset X of A_n let $\min X$ denote the set of minimal elements of X.

Every element v in $M(u)$ is greater than or equal to an element w from $\min M(u)$. Hence $v = we$ for some e. Since both v and w are equal to u modulo R we have that $u = ue$ (Mod R). Such an element e is called a *unit* for u. Let $E(u)$ be the set of all units. Then $E(u)$ is a subsemigroup of A_n and is closed under taking quotients, that is if $e_1 f = e_2$ and e_1, e_2 are units then f is also a unit. This implies easily that the semigroup $E(u)$ is generated by the subset $\min E(u)$of its minimal elements. Notice also that if e is a unit for u and $w \in M(u)$ then, of course, $we \in M(u)$.

By Lemma 7.1 both sets $\min M(u)$and $\min E(u)$are finite. Therefore we have the following description of $M(u)$:

$$M(u) = \{(\prod_{e \in \min E(u)} e^{k_e})w \mid w \in \min M(u), k_e \in \mathbf{N}\}.$$

This description would give us a solution of the word problem, if we had a process of finding the sets $\min M(u)$and $\min E(u)$. This process is almost straightforward. We simply apply the relations from R to u until we find all elements from $\min M(u)$and $\min E(u)$. Of course, the most tricky thing in such processes is stop sign: the sign which shows that we have found all the elements that we need, and we can stop and relax. This is organized as follows.

Denote the maximal length of words from R by $\ell(R)$. If M is a subset of A_n then let $T(M)$ be the set of all elements of A_n which can be obtained from elements of M by applying relations of R (at most one application for each relation and for each element of M).

[16]Zero is also natural.

Let us construct a sequence of sets $M_k \subseteq M(u)$. Let $M_0 = \{u\}$. Suppose we have constructed the set M_k. Let E_k be the set of all quotients of elements from M_k. Notice that $E_k \subseteq E(u)$.

Let us apply T to M_k many times until we obtain all elements of the form

$$(\prod_{e \in \min E_k} e^{k_e})w$$

where $w \in \min M_k$ and the sum of k_e does not exceed $\ell(R)$. This set is finite and each element in it is equal to u modulo R, so we will find all of these elements sooner or later. Then let M_{k+1} denote the union of these sets.

By Lemma 7.1 there is a number k such that

$$\min M_k = \min M_{k+1}, \min E_k = \min E_{k+1}.$$

We claim that then $\min M(u)$is equal to $\min M_k$, $\min E(u)$=$\min E_k$. Indeed, it is enough to show that if we apply a relation from R to a word

$$v = (\prod_{e \in \min E_k} e^{k_e})w \tag{26}$$

where $w \in \min M_k$ then we obtain a word w of the same form. If the sum of k_e does not exceed $\ell(R)$ then this follows from the definition of M_k and from the equalities $\min M_k$=$\min M_{k+1}$, $\min E_k$=$\min E_{k+1}$.

Let this sum be greater than $\ell(R)$. Any application of a relation from R touches at most $\ell(R)$ units e from the right hand part of (26). Therefore $v = (\prod_{e \in \min E_k} e^{m_e})v_1$, and $w = (\prod_{e \in \min E_k} e^{m_e})w_1$ where v_1 belongs to M_n and w_1 is obtained from v_1 by applying a relation from R. But then w_1 is of the form (26), and so is w. This completes the proof.

7.1.2 Nilpotent-by-Abelian Groups

Let us consider the group variety $\mathcal{N}_2\mathcal{A}$ and prove that the word problem is decidable there (Theorem 6.4).

Let us take any group $G =< X >$, finitely presented in $\mathcal{N}_2\mathcal{A}$. The group G may be represented as a factor of the relatively free group $\bar{F} =< X >$ of $\mathcal{N}_2\mathcal{A}$.

Therefore $G = \bar{F}/R$ for some normal subgroup R which is finitely generated as a normal subgroup. Given a word w over X, we want to find out if $w = 1$ in G. We can consider w as an element of $\bar{F}$. Then $w = 1$ in G if and only if w belongs to R as an element of $\bar{F}$. Thus we have the first (simple) reduction:

The word problem in G is decidable if and only if the membership problem for R in $\bar{F}$ is decidable

This reduction allows us to consider $\bar{F}$ instead of G. Usually free groups in varieties have much better structure than other members of these varieties. For example $\bar{F}$ certainly has a decidable word problem. So it is easier to work with $\bar{F}$ than with G.

Still, it is not clear how to solve the membership problem for R. But we can find a somewhat bigger normal subgroup for which the membership problem *is* decidable. For example consider the subgroup $\bar{F}''R$ (here $\bar{F}''$ is the second derived subgroup of $\bar{F}$). The factor group $\bar{F}/\bar{F}''R$ is finitely presented in the variety of metabelian groups. It is known that the word problem in the variety of metabelian groups is decidable (P. Hall [138]). Therefore the membership problem for the subgroup $\bar{F}''R$ is decidable.

So we can check if our word w belongs to $\bar{F}''R$. If we are lucky then $w \notin \bar{F}''R$ and so $w \notin R$. In the worst case $w \in \bar{F}''R$ and our question of whether w belongs to R remains unanswered. But, nevertheless, we can gain some profit from these considerations. Indeed, since $w \in \bar{F}''R$, we can represent w as a product pr with $p \in \bar{F}''$, $r \in R$. This may be done effectively in the following way. We can lift $\bar{F}''$, R, and w up to the absolutely free group F over X. Then $\bar{F}''$ will lift to F'', R will lift to some normal subgroup T of F generated as a normal subgroup by a finite set (for each generator of R we take one pre-image of it in F), and w will lift to some reduced word w'. We can enumerate all elements of F'' and T and their products. One of these products $p'r'$ is equal to w' (in the absolutely free group). When we return down to $\bar{F}$, p' and r' will turn into the desired p and r. Since $r \in R$, it is enough to decide if $p \in R$. The word p is better than an arbitrary w since $p \in \bar{F}''$.

Thus we may suppose that $w \in \bar{F}''$

This is the second reduction. Now we have to consider the membership problem for the subgroup $R \cap \bar{F}''$.

Consider $\bar{F}''$ first. This is an Abelian subgroup of F. If $\bar{F}''$ were finitely generated then $R \cap \bar{F}''$ would be a finitely generated Abelian subgroup and the membership problem for $R \cap \bar{F}''$ would be trivially decidable. Unfortunately $\bar{F}''$ is not finitely generated as an Abelian subgroup (if $\bar{F}$ is not a cyclic group).

But we should recall that not only finitely generated Abelian groups have nice algorithmic properties. Finitely generated modules[17] over "good" (commutative, Noetherian, etc.) rings are almost as nice. In particular the membership problem is decidable for every submodule of a finitely generated module over such a ring. So why not define a module structure on $\bar{F}''$? There is a standard way of doing this.

Any Abelian normal subgroup A in a group H may be viewed as a right module over the group algebra $\mathbf{Z}H$ where the action of $\mathbf{Z}H$ on A is defined by the following formula:

$$a \circ (\sum \pm g_i) = \Pi a^{g_i^{\pm 1}}.$$

[17]Here and after we consider only right modules.

It is clear that this module has a big annihilator. For example $\mathbf{Z}A$ and even $\mathbf{Z}C_H(A)$ (here $C_H(A)$ is the centralizer of A in H) is in this annihilator. Therefore A may be considered as a module over $\mathbf{Z}H/N$ for every normal subgroup $N \leq C_H(A)$.

In our case $\bar{F} \in \mathcal{N}_2\mathcal{A}$. Therefore $\bar{F}''$ is Abelian and $C_{\bar{F}}(\bar{F}'') \geq \bar{F}'$. Hence $\bar{F}''$ may be considered as a right module over the ring $K = \mathbf{Z}\bar{F}/\bar{F}'$. This ring is just the ring of polynomials over Z with $X \cup X^{-1}$ as the set of unknowns, factorized by the ideal generated by elements $xx^{-1} - 1$ for all $x \in X$. Thus this is a commutative, finitely generated domain which is certainly good.

Now how about $\bar{F}''$? Is it finitely generated as a module over K? Unfortunately, it is not. But - an important thing - we can *make* it into a finitely generated module by increasing K! Indeed, by definition, the second derived subgroup $\bar{F}''$ consists of all elements which are products of double commutators $[[a, b], [c, d]]$ where $a, b, c, d \in \bar{F}$. Let us define a new action of generators from X on $\bar{F}''$:

$$[[a, b], [c, d]] \circ x = [[a, b]^x, [c, d]][[a, b], [c, d]^x].$$

It is amazing that this wild (but beautiful) action is well defined and can be extended to another action of $K = Z\bar{F}/\bar{F}'$ on F''. Moreover this action commutes with the first action of K. Thus we can consider $\bar{F}''$ as a module over the tensor product $K \otimes K$. The last ring is, of course, a finitely generated domain, and so it is as good as K. It can be verified that $\bar{F}''$ is a finitely generated module over $K \otimes K$. You see: *we have found a hidden module structure on* $\bar{F}''$ and now $\bar{F}''$ looks very nice.

But $\bar{F}''$ is not the main character of our play. We are mainly interested in R, or more precisely, in $R \cap \bar{F}''$. This is a normal subgroup and thus a K-submodule under the first action of K (normal subgroups are closed under conjugations!). But we are not lucky enough: this is not necessarily a $K \otimes K$-submodule. This seems to be a big difficulty. And indeed this is the most involved and technical point of our scheme.

The idea is *to split one "beast" thing ($R \cap \bar{F}''$) into a number of "beauty" things.* In our case $R \cap \bar{F}''$ turns out to be a sum of three Abelian groups $A_1 + A_2 + A_3$ where A_1 is finitely generated as a subgroup ($\mathbf{Z}$-module), A_2 is finitely generated as a normal subgroup (K-module) and A_3 is finitely generated as a $K \otimes K$-module. Moreover generators of A_1, A_2, A_3 may be found effectively (see Kharlampovich [184] for details). Therefore *the membership problem for R has been reduced to the membership problem for the sum of finitely generated modules A_1, A_2, A_3 over different rings.*

Now we can apply the following powerful result.

Lemma 7.2 *Let M be a $D-$module where D is a finitely generated commutative domain. Let $D_1 < D_2 < \ldots < D_m < D$ be a sequence of finitely generated subrings of D. Let N_i $(i = 1, \ldots, m)$ be a finitely generated D_i-submodule of M. Then the membership problem for $N = N_1 + N_2 + \ldots + N_m$ is decidable.*

This Lemma gives us the decidability of the membership problem for $R \cap \bar{F}''$, which, in turn, implies the decidability of the word problem in $\mathcal{N}_2\mathcal{A}$.

Lemma 7.2, the key Lemma in the above proof, has its origin in an article by Romanovskii (see [325]). It was subsequently modified by other authors who adapted it to their needs. For example, it turned out (see Romanovskii [325], Sapir [343]) that the restrictions on D may be weakened. Without big troubles the commutative domain D may be replaced by a group algebra of a nilpotent finitely generated group, or by an arbitrary finitely generated ring which satisfies an identity of the form $[x_1, x_2, \ldots, x_n] = 0$.

Sometimes one has to drop the restriction that the subrings D_i form a chain. In such cases the situation needs a much more careful investigation.

Even if the subrings do not form a chain, there are usually some polynomial connections between them (connections induced by the identities of the variety under consideration). Let us clarify this a little bit.

The standard situation is the following. The ring D is generated by two sets of the same cardinality $\{x_i \mid i \leq m\}$ and $\{\bar{x}_i \mid i \leq m\}$, D_1 is generated by the first set, D_2 is generated by the second set, and $p(x_{i_1}, \ldots, x_{i_k}, \bar{x}_{i_1}, \ldots, \bar{x}_{i_k}) = 0$, $i_j \leq m$, for a polylinear polynomial $p(t_1, \ldots, t_k, \bar{t}_1, \ldots, \bar{t}_k) = \prod(t_i - \alpha_i \bar{t}_i)$ and any rearrangement $\{i_1, \ldots, i_k\}$ of $\{1, \ldots, k\}$. Here the elements α_i belong to an algebraic extension of the ground field. The word problem is reduced to the membership problem in a free finitely generated D-module for the sum of three finitely generated modules A_1, A_2, A_3 where A_1 is a $\mathbf{Z}$-module (a finitely generated Abelian subgroup), A_2 is a module over D_1, A_3 is a module over D_2.

Then the decidability/undecidability of this membership problem depends on properties of the α_i. If $\alpha_i = \alpha_j$ for some $i \neq j$ then one can find A_1, A_2, A_3 such that the membership problem is undecidable (see an example in Section 7.6.3 below). If all α_i are different then the situation very much depends on deep number theoretic properties of these roots (see Kharlampovich [184]).

One of the number theoretic problems which one has to deal with is the Problem 5.3 mentioned above.

7.1.3 Other Hidden Structures

It is interesting that a hidden module structure appears in the case of semigroups also. We will call a semigroup S a *semi-module* over a semigroup A if there exists an action $\circ$ of A on S (i.e. a function $\circ : A \times S \to S$ with $(a_1 a_2) \circ s = a_1 \circ (a_2 \circ s)$) which "agrees" with the operation in S: $a \circ (s_1 s_2) = (a \circ s_1) s_2 = s_1 (a \circ s_2)$ for every $a \in A, s_1, s_2 \in S$. The following result was proved by the second author.

Theorem 7.1 *(Sapir, [348]) Let T be a semigroup finitely presented in a variety $\mathcal{V}$ satisfying one of the following identities*[18]

$$x^{nk} y (z^k t^k)^p z^{(n-p)k} = x^{(n-p)k} (t^k x^k)^p y z^{nk};$$

[18]As it was mentioned in Section 3, this condition is necessary for the decidability of the word problem.

$$xy^n z = y^k xy^m zy^p, n > m.$$

and let every periodic group in $\mathcal{V}$ be locally finite. Then there exists a semi-module S over a commutative semigroup A such that T is "almost" a subsemigroup of S. More precisely there is a semigroup S_1 such that S is an ideal of S_1 with S_1/S finite and nilpotent and T is a factor-semigroup of S_1 over a congruence with all congruence cosets finite.

This Theorem is one of the main tools in describing semigroup varieties with decidable word problem.

The module structure is not the only possible hidden structure on the set of words equal to a given word. For example if we are dealing with semigroups, groups, or inverse semigroups, it is useful to draw these words on the Cayley graph of a relatively free semigroup or group. Then the subgraph formed by these words may have nice geometric properties, as in [263], which helps to solve the word problem (or to prove its undecidability). Another possibility was discovered by Margolis and Meakin in [247]. They considered inverse semigroups[19] given by finitely many defining relations of the form $e = f$, where e and f are words which are idempotents in the free inverse semigroup (equivalently which are equal to 1 in the free group). It turned out that if we take any such semigroup S, and a word u, and draw all words which are equal to u in S on the Cayley graph of the free group, which is, of course, a labeled tree, then we will get a set of vertices which may be given by a formula from a decidable fragment of the second order theory of this tree. This implies the decidability of the word problem in S. A similar method works if we consider certain subvarieties of the variety of inverse semigroups. Only instead of the free group one has to consider relatively free groups in the corresponding varieties of groups. Also, as we have mentioned in Section 3.3.4, the set of words which are equal to a given word in the free Burnside semigroup of index ≥ 3 is a language recognizable by a finite non-deterministic automaton.

7.2 Minsky Machines and the Undecidability of the Word Problem in Varieties

7.2.1 Minsky Machines

One of the most powerful tools in proving the undecidability of the word problem is the so-called Minsky machine. It was invented by Marvin Minsky in 1961[20] (see Minsky [272], [271], Mal'cev [233]). In the Computer Science literature Minsky machines are

[19]See Section 3.3.5 for the definitions of inverse semigroups and for a discussion of their role in the theory of semigroups and groups.

[20]It is written in [272] that the concept of a two-tape non-writing machine was inspired by some ideas of Rabin and Scott [309], and that it was suggested to M.Minsky by J.McCarthy that such a machine might be equivalent to the ordinary Turing machine.

usually called *counter machines*. Yu. Gurevich was the first to use Minsky machines to prove the undecidability of an algorithmic problem in algebra [133] (see Sections 2.5, 3.4.2).

Let us give the "canonical" definition of a Minsky machine.

The hardware of a (two-tape) Minsky machine consists of two tapes and a head. The tapes are infinite to the right and are divided into infinitely many cells numbered from the left to the right, starting with 0. The first cells on both tapes always contain 1, all other cells have 0. The head may acquire one of several internal states: $q_0, \ldots, q_N$; q_0 is called *the terminal state*. At every moment the head looks at one cell of the first tape and at one cell of the second tape. So the *configuration* of the Minsky machine may be described by the triple (m, q_k, n) where m (resp. n) is the number of the cell observed by the head on the first (resp. second) tape, q_i is the state of the head.

Every command has the following form:

$$q_i, \epsilon, \delta \longrightarrow q_j, T^\alpha, T^\beta.$$

where $\epsilon, \delta \in \{0,1\}, \alpha, \beta \in \{-1,0,1\}$. This means that if the head is in the state q_i and it observes a cell containing ϵ on the first tape and a cell containing δ on the second tape then it acquires the state q_j and the first (the second) tape is shifted α (resp. β) cells to the left relative to the head. If, say, $\alpha = -1$ then the first tape is shifted one cell to the right.

The machine always starts working at state q_1 and ends at the *terminal state* q_0.

The program (software) for a Minsky machine is a set of commands of the above form.

One says that a Minsky machine *calculates* a function $f(m)$ if for every m starting at the configuration $(m, q_1, 0)$ it ends at the configuration $(f(m), q_0, 0)$. If m does not belong to the domain of f then the machine works forever and never gets to the terminal state.

The main property of Minsky machines is contained in the following theorem.

Theorem 7.2 *(Minsky, [272]). For every partial recursive function $f(m)$ there exists a Minsky machine which calculates the partial function $g_f : 2^m \to 2^{f(m)}$.*

Remark. It is interesting that, in this theorem, the function g_f cannot be replaced by the function f. In particular, there is no Minsky machine which calculates the function m^2 (see [233]).

The "canonical" definition of a Minsky machine seems to be very long and complex while in fact it is very simple and can be understood by a high school or even elementary school student. Let us give a "high school" definition of a Minsky machine.

Consider two glasses. We assume that these glasses are of infinite height. Another (more restrictive!) assumption is that we have infinitely many coins. There are four *operations*: "Put a coin in a glass", "Take a coin from a glass if it is not empty". We are able to check if a glass is/isn't empty. A *program* is a numbered sequence of instructions.

An *instruction* has one of the following forms:

- Put a coin in the glass # n and go to instruction # j;
- If the glass # n is not empty then take a coin from this glass and go to instruction # j otherwise go to instruction # k.
- Stop.

A program starts working with the command number 1 and ends when it comes to the Stop instruction which will always have number 0.

We say that a program calculates a function $f(m)$ if, starting with m coins in the first glass and empty second glass, we end up with $f(m)$ coins in the first glass and empty second glass[21].

This "high school" version of a Minsky machine is also known as a Minsky algorithm [272].

One can prove that Minsky machines are equivalent to Minsky algorithms, that is, given a program for Minsky machine (resp. given a Minsky algorithm), it is easy to construct a Minsky algorithm (resp. a program for a Minsky machine) which calculates the same function.

A configuration of a Minsky algorithm is a triple (m, k, n), where m is the number of coins in the first glass, n is the number of coins in the second glass, and k is the number of the instruction we are executing. So the number of an instruction in the algorithm plays the role of an inner state!

Another, elementary school, version of Minsky algorithms was invented by the second author of this paper. It was implemented by M.Sapir, V.Klyachin, and E.Linetsky in LOGO-type software and has been used in thousands of schools in the former U.S.S.R., as well as in some other schools throughout the world, to teach kids programming and problem-solving skills.

Imagine a part of the plane bounded from the left and from the bottom by two orthogonal lines. Imagine further a small kangaroo, named Roo, living on this part of the plane. It can only "Hop" (hop 1 cm forward), "Step" (draw a straight line 1 cm long[22]), and "Turn" (turn 90 degrees to the left). Roo can also check if the wall (one of the two boundary lines) is 1 cm ahead of him. The program for Roo is a non-numbered sequence of commands. This sequence may contain branchings (If-instructions) like

---

[21]If you want to teach Minsky machines for business students you should change glasses to bank accounts and coins to dollars.

[22]This command is unnecessary, it is included just for fun.

If the wall is (isn't) ahead then

...

otherwise

...

The end of branching

and loops like

While the wall is (isn't) ahead repeat

...

The end of cycling

Roo stops when it gets to the last command of the program.

It is easy to understand what it means for Roo to calculate a function, and it is easy to prove that Roo is equivalent to Minsky algorithms and machines.

Notice that although Minsky machines and Minsky algorithms are equivalent, Minsky algorithms are sometimes better tools in proving the undecidability of algorithmic problems. The main reason: there are four possible commands of a Minsky machine which correspond to the same inner state q_k, while there is only one instruction of a Minsky algorithm with a given number k (see also Section 7.2.9 below).

M.Minsky himself gave the following definition of his algorithms (see [272]):

We work with a number denoted by t. We can suppose that $t = 2^m 3^n$ for some nonnegative integers m and n. Every instruction has one of the following forms

- $[\times 2|j]$ - multiply t by 2 and pass to the j-th command;
- $[\times 3|j]$ - multiply t by 3 and pass to the j-th command;
- $[: 3|j|k]$ - check if t is divisible by 3 and if "yes" then divide and pass to the $j-$th command, otherwise pass to the k-th command;
- $[: 2|j|k]$ - check if t is divisible by 2 and if "yes" then divide and pass to the $j-$th command, otherwise pass to the k-th command;
- Stop.

It is easy to establish a correspondence between this definition and the glass-coin definition. The number $t = 2^m 3^n$ corresponds to the situation when the first glass contains m coins and the second glass contains n coins.

It is interesting that a variant of the "glass-coin" machine was invented independently by J. Lambek [219], though his machine used an infinite number of glasses and so it was technologically more difficult to build.

Now we will show how to apply Minsky machines to prove the undecidability of word problems in different types of algebras.

7.2.2 Minsky Machines and the Word Problem for General Algebras

Here we will show how to apply Minsky machines (algorithms) to prove the undecidability of the word problem. All applications of Minsky machines (algorithms) are based on the following idea. We will show this idea for Minsky machines; exactly the same idea works for Minsky algorithms. Take a Minsky machine M calculating a partial function g_f with a non-recursive domain X and a constant value 1.

One can define an "equality" on the set of all possible configurations of the Minsky machine M: two configurations (m, q_k, n) and $(m', q_{k'}, n')$ are "equal" or "equivalent" (we will write $(m, q_k, n) \equiv (m', q_{k'}, n')$) if there exists another configuration $(m'', q_{k''}, n'')$ such that M transforms both configurations (m, q_k, n) and $(m', q_{k'}, n')$ into $(m'', q_{k''}, n'')$ in zero or more steps. It is easy to see that this "equality" is symmetric, transitive and reflexive. By the choice of M, $(m, q_1, 0) \equiv (1, q_0, 0)$ if and only if $m \in X$ (recall that we always assume that q_0 is the stop state. Therefore we have a sequence of configurations γ_m and a special configuration γ_0 such that $\gamma_m \equiv \gamma_0$ iff $m \in X$.

Suppose now that we want to construct a finitely presented universal algebra $A(M)$ with an undecidable word problem.

The idea of doing this has its origin in the works of Markov [249], [250] and Post [303]. First, with every configuration ψ one associates a word $w(\psi)$. This word is usually called a *canonical word.*

Then with every command κ of the Minsky machine M one associates a finite set of defining relations R_κ. The algebra $A(M)$ will be defined by the relations from the union R of all R_κ (which is finite since we have only a finite number of commands) and usually some other relations Q which are in a sense "independent" of R. We need Q, for example, to make $A(M)$ satisfy a particular identity.

The algebra $A(M)$ will have an undecidable word problem if the following property holds:

$$\psi_1 \equiv \psi_2 \text{ if and only if } w(\psi_1) = w(\psi_2) \text{ in } A. \tag{27}$$

Indeed, in this case one cannot algorithmically decide if $w(m, q_1, 0) = w(1, q_0, 0)$ for the given number m.

We will say that we have an *interpretation* of the Minsky machine M in the algebra $A(M)$ if we have an assignment $\psi \to w(\psi)$ with the property (27).

Usually, in order to prove the property (27) one has to prove two Lemmas.

Lemma 7.3 *If we can proceed from configuration ψ_1 to configuration ψ_2 using command κ, then we can proceed from the word $w(\psi_1)$ to the word $w(\psi_2)$ using relations from R_κ.*

Lemma 7.4 *If we can proceed from the word $w(\psi_1)$ to the word $w(\psi_2)$ by using relations from the union of all R_κ, then $\psi_1 \equiv \psi_2$.*

It is easy to see that Lemmas 7.3 and 7.4 imply property (27). Lemma 7.4 in most cases is more difficult to prove than Lemma 7.3.

It is worth mentioning also that in order to prove Lemmas 7.3 and 7.4 we usually do not need any information about the function that is calculated by M.

Let us also make a remark about the case where we are constructing an algebra with undecidable word problem which is finitely presented in a variety $\mathcal{V}$. In this case we are allowed to use identities of $\mathcal{V}$ when we deduce relations of $A(M)$. Notice that unlike the relations of R_κ corresponding to commands of M, relations obtained from identities of $\mathcal{V}$ have no connection with the Minsky machine, and can spoil the canonical words. So we have to make the canonical words resistant to applications of identities of $\mathcal{V}$. In the best case, the identities of $\mathcal{V}$ are not applicable to canonical words.

The procedure for constructing an algebra, finitely presented in a variety $\mathcal{V}$, with undecidable word problem is roughly the following. First we temporarily forget about $\mathcal{V}$ and construct an interpretation of a Minsky machine M in an "absolutely" finitely presented algebra $A(M)$. We prove Lemma 7.3 for $A(M)$. Then we consider the factor algebra $\hat{A}(M)$ of $A(M)$ by the verbal congruence corresponding to $\mathcal{V}$, that is we identify all pairs of terms in $A(M)$ which are identically equal in $\mathcal{V}$. The algebra $\hat{A}(M)$ is finitely presented in $\mathcal{V}$. Then we have to prove Lemmas 7.3 and 7.4 for $\hat{A}(M)$. Fortunately we have Lemma 7.3 for free because the statement of this Lemma is stable under homomorphic images. To prove Lemma 7.4 we usually need the above mentioned independence of canonical words from the identities of $\mathcal{V}$.

7.2.3 Why Minsky Machines?

The concept of an interpretation of a machine in an algebra can be applied to any kind of Turing machine. Why did we choose Minsky machines? The first answer is: because Minsky machines are in some important sense the simplest universal Turing machines possible.

Indeed, the first and the main step in any interpretation of a machine in an algebra is the choice of canonical words. The canonical words must encode the configurations of the machine. Therefore the smaller the number of parameters which determine the configurations, the more freedom we have in simulating the parameters. A configuration of a Minsky machine is determined by just three numbers: m, i, n. Here i runs over a finite set. Therefore one can encode each i by a separate letter q_i. It is also important that the commands of Minsky machines change those three numbers in a natural way. They add 0, 1 or -1 to m and n, and change i according to some simple rule. Therefore we can simulate m and n by, say, powers of different letters, say, a and b, and the relations corresponding to the relations of the Minsky machine will increase (decrease) the powers of these letters. Therefore we can encode the configuration (m, i, n) by a canonical word $a^m q_i b^n$ (we suppose for simplicity that we have a binary associative operation).

True, after a little pondering one can conclude that something is missing in this encoding. Indeed, recall that the relations will simulate the commands. The action of a command depends on whether m or n is equal to 0 or not, so there are four different situations ($m \neq 0, n \neq 0$; $m = 0, n \neq 0$; $m \neq 0, n \neq 0$, and $m = 0, n = 0$). Therefore for each one of these 4 situations, the corresponding canonical word must have a special small subword which tells us that this situation occurs (then the corresponding relation will replace this subword by some other word).

Notice that our canonical words corresponding to these 4 situations have the following form: $a^n q_i b^m$, $q_i b^m$, $a^n q_i$, q_i where $m, n > 0$. All these words are subwords of the first one. So every subword of the second (the third or the fourth) word is a subword of the first word. Thus we cannot distinguish between these situations.

The solution to this problem is simple: we have to add two more letters A and B, which we will call "locks", and encode the configuration (m, i, n) by the word $Aa^m q_i b^n B$. Then each of the 4 situations is characterized by a small subword of the canonical word: The canonical word has a subword:

aq_ib iff $m \neq 0, n \neq 0$;

Aq_ib iff $m = 0, n \neq 0$;

aq_iB iff $m \neq 0, n = 0$;

Aq_iB iff $m = 0, n = 0$.

Thus, as one can see, it is very easy to find an interpretation of the Minsky machine. But this is not the only reason why one has to use it.

Recall that we want to simulate a machine in an algebra which satisfies as many identities as possible. And as everybody knows, those identities tend to change words. For example suppose that we have an identity $xy = yx$, and we simulate the configuration (m, i, n) by the word $Aa^m q_i b^n B$. Then the words $Aa^m q_i b^n B$, $a^m A q_i b^n B$, $Aa^m q_i B b^n$, and $a^m A q_i B b^n$ are equal and, again, we cannot find small subwords which distinguish the four situations from each another.

The important feature of Minsky machines and their interpretations is that the canonical words, which we obtain, are very stable with respect to identities.

For example, in the case of semigroups, Theorem 3.28 means the following. There exist a few basic encodings of the configurations of Minsky machines, and if a non-periodic semigroup variety with locally finite nil-semigroups satisfies identities which can change all these encodings, then the word problem is solvable in this variety, so the interpretation of a Minsky machine (and any other universal Turing machine) in this variety is impossible.

It is much more difficult to find stable interpretations of other kinds of machines. For example, the configuration of the general (one-tape) Turing machine is a triple u, q_i, v where q_i is an inner state of the head, u is the word written on the tape to the left of the head, and v is the word written to the right of the head. So if we want to interpret the general Turing machine, we need to somehow encode arbitrary words u and v. Since any identity can change some words, we cannot encode the word by itself, thus the encoding must be unnatural. This leads to difficulties in simulating commands of the Turing machine, and so on.

The so called *modular machine* (see Aanderaa and Cohen [2]) are also more difficult to simulate than Minsky machines. While the configurations of a modular machine is described by 3 natural numbers like in the case of Minsky machines, commands of a modular machine induce more complicated operations on these numbers.

It is interesting that even the number of tapes of the Minsky machines (two) is important. For example, if we try to simulate a Minsky machine with many tapes (one can readily define such machines) in a variety $\mathcal{N}_k\mathcal{A}$ of Lie algebras or groups by means of the technique in Sections 7.2.6, 7.2.8, then it is possible to show that the minimal k for which such a simulation is possible, is greater than the number of tapes. So if the number of tapes of the Minsky machine were 3 we could not simulate it in the variety $\mathcal{N}_3\mathcal{A}$. We will return to this discussion at the end of Section 7.5.

A curious reader may ask here: "What if we take a Minsky machine with only one tape?". We will discuss this question also at the end of Section 7.5.

7.2.4 The Word Problem for Semigroup Varieties. The Nonperiodic Case

There are two important semigroup interpretations of Minsky machines: the semigroups S_1 and S_2 below. Let M be a Minsky machine with internal states $q_0, \ldots, q_N$. Then both S_1 and S_2 are generated by the elements $q_0, \ldots, q_N$ and a, b, A, B. The correspondences between commands of M and relations of S_1 and S_2 are given by the following tables. Every command corresponds to one relation in S_1 and one relation in S_2.

Command	S_1
$q_i, 0, 0 \to q_j, T^\alpha, T^\beta$	$aq_ib = a^{1+\alpha}q_jb^{1+\beta}$
$q_i, 1, 0 \to q_j, T^\alpha, T^\beta$	$Aq_ib = Aa^\alpha q_jb^{1+\beta}$
$q_i, 0, 1 \to q_j, T^\alpha, T^\beta$	$aq_iB = a^{1+\alpha}q_jb^\beta B$
$q_i, 1, 1 \to q_j, T^\alpha, T^\beta$	$Aq_iB = Aa^\alpha q_jb^\beta B$

(28)

Command	S_2
$q_i, 0, 0 \to q_j, T^\alpha, T^\beta$	$q_i ab = q_j a^{1+\alpha} b^{1+\beta}$
$q_i, 1, 0 \to q_j, T^\alpha, T^\beta$	$q_i Ab = q_j a^\alpha A b^{1+\beta}$
$q_i, 0, 1 \to q_j, T^\alpha, T^\beta$	$q_i aB = q_j a^{1+\alpha} b^\beta B$
$q_i, 1, 1 \to q_j, T^\alpha, T^\beta$	$q_i AB = q_j a^\alpha A b^\beta B$

(29)

The canonical words in S_i are the following:

Configuration	S_1	S_2
(m, q_k, n)	$Aa^m q_k b^n B$	$q_k a^m A b^n B$

(30)

To make these interpretations work and to make these semigroups satisfy as many identities as possible we need also some additional relations independent of the commands of M.

In the semigroup S_2 we need the following commutativity relations:

$$ab = ba,\ aB = Ba,\ bA = Ab,\ AB = BA. \tag{31}$$

Also we need all relations of the type [23]

$$xy = 0$$

where xy is a two letter word which is not a subword of $w(m, q_k, n)$ for some m, n or of any word obtained from $w(m, q_k, n)$ by the commutativity relations above.

These relations "kill" all "wrong" words. Basically only canonical words and their subwords are distinct from zero in S_1 and S_2. Even a semigroup-illiterate reader would agree that "Wanna more laws — kill more words". A more literate reader would say that this is not quite true, and (s)he would be absolutely right: sometimes it is better not to kill words but to equalize them[24]. But for the semigroups S_1 and S_2, killing words is the right thing to do.

Thus we have the following additional relations in S_1: all two letter words are equal to 0 except Aa, Aq_i, a^2, aq_i, q_ib, q_iB, b^2, bB. And we have the following additional relations in S_2: all two letter words are equal to 0 except q_ia, q_ib, q_iA, q_iB, a^2, aA, ab, aB, ba, b^2, bB, bA, Ab, AB, Ba, BA.

[23]The equality $u = 0$ is an abbreviation for the following system of equalities: $uz = u, zu = u$ where z runs over all the generators of the semigroup.

[24]Doesn't it remind us of a difference between dictatorship and democracy?

These semigroups are very convenient for demonstrating the standard proofs of Lemmas 7.3 and 7.4.

To prove Lemma 7.3 one needs to show that if we pass from the configuration (m, q_k, n) to another configuration $(m', q_{k'}, n')$ by a command κ then we pass from the word $w(m, q_k, n)$ to the word $w(m', q_{k'}, n')$ by the relation corresponding to κ.

Let us prove this only for the case of the semigroup S_2 and the command κ : $q_k, 1, 0 \to q_{k'}, T^\alpha, T^\beta$. All other cases are similar.

Since the command κ is applicable to the configuration (m, q_k, n), in this configuration the head observes the first cell of the first tape and some cell other than the first on the second tape. Thus $m = 0$, $n \neq 0$. Then $m' = \alpha$, $n' = n + \beta$ (in this case α cannot be negative). Now $w(m, q_k, n) = q_k b^n AB$ and the relation corresponding to κ is $q_k bA = q_{k'} a^\alpha b^{1+\beta} A$. Since $AB = BA$ and $aB = Ba$ we have $w(m, q_k, n) = q_k b^n AB = q_k bAb^{n-1}B$. Thus we can apply our relation and replace $q_i bA$ by $q_{k'} a^\alpha b^{1+\beta} A$. As a result we obtain the word $q_{k'} a^\alpha b^{1+\beta} Bb^{n-1} A$ which is equal to $q_{k'} a^\alpha Ab^{n+\beta} B$ since $bA = Ab$. The last word is equal to $w(m', q_{k'}, n')$ as desired.

The proof of Lemma 7.4 is based on the following two standard observations. First, for every canonical word $w(m, q_k, n)$ there exists at most one relation corresponding to a command of M which is applicable to this word from the left to the right (this means that one replaces the left hand side of this relation by the right hand side of it).

Second, any application of a relation from tables (28) or (29) to any canonical word — from the left to the right or from the right to the left — gives us another canonical word (we do not distinguish words in S_2 which are obtained from each other by the commutativity relations[25]).

Now consider two words $w(m, q_k, n)$ and $w(m', q_{k'}, n')$ in S_1 or S_2. Suppose these words are equal in this semigroup. Therefore there exists a sequence of words

$$w(m, q_k, n), w_1, \ldots, w_n, w(m', q_{k'}, n')$$

where each word is obtained from the previous one by applying a defining relation corresponding to a command of the machine M. By the second observation each w_i corresponds to a configuration of M.

These relations may be applied from the left to the right and from the right to the left. Now suppose that in the passage $w_{r-1} \to w_r$, a relation was applied from the right to the left and in the passage $w_r \to w_{r+1}$, a relation was applied from the left to the right. Then w_{r-1} and w_{r+1} are obtained from w_r by applying relations from the left to the right. By the first observation these two relations must coincide and the words w_{r-1} and w_{r+1} must coincide also. Therefore in this case we can shorten

[25]Actually — important! — we assign to each configuration not a single word $w(m, q_k, n)$ but a set of words which may be obtained from each other by the commutativity relations. It is worth mentioning that it is almost always better to consider $w(m, q_k, n)$ as a set of words; relations corresponding to commands of M connect these sets of words; auxiliary relations connect words in the same set.

our sequence of words w_i. Thus we can suppose that in our sequence, there is a word $w_r = w(m'', q_{k''}, n'')$ such that all relations before w_r are applied from the left to the right and all relations after w_r are applied from the right to the left. But this means that the machine M passes from both configurations (m, q_k, n) and $(m', q_{k'}, n')$ to the configuration $(m'', q_{k''}, n'')$. Therefore $(m, q_k, n) \equiv (m', q_{k'}, n')$, as desired.

Thus we have proved that S_1 and S_2 have undecidable word problems provided the Minsky machine that we started with computes a non-recursive function. These semigroups are important because every (finitely based) semigroup variety with undecidable word problem, whose periodic semigroups are locally finite, contains either S_1 or S_2 or the semigroup anti-isomorphic to S_2 (see Sapir [348]). Identities of S_1 and S_2 do not depend on what Minsky machine is simulated. Therefore we do not need any other semigroup with undecidable word problem to treat non-periodic varieties with good periodic semigroups. But if the periodic semigroups are not locally finite we need something else, and we will discuss it in the next subsection.

7.2.5 The Word Problem for Semigroup Varieties. The Periodic Case

Theorem 3.8 of the second author of this survey implies that a finitely based variety of semigroups in which not every periodic semigroup is locally finite contains either a periodic group or a nil-semigroup which is not locally finite. The periodic group case will be considered later (see Section 7.2.9). Now let us consider the nil-case. So suppose that we have a finitely based variety $\mathcal{V}$ containing a non-locally finite nil-semigroup. Then by virtue of Theorem 3.29 $\mathcal{V}$ has an undecidable word problem, and now we are going to explain how to prove this using Minsky machines.

In order to show only the principal details of the proof, let us take the semigroup variety given by the identity $x^3 = 0$ and prove that the word problem is undecidable there. This variety consists of nil-semigroups and it was proved by Morse and Hedlund [274] that it contains an infinite finitely generated semigroup.

We will need a slight modification of the Morse and Hedlund construction [26].

Let us start with the following Thue endomorphism ϕ [398] of the free semigroup with generators a, b:

$$\phi(a) = ab, \ \phi(b) = ba.$$

Now let us iterate ϕ and consider the words $a, \phi(a), \phi^2(a), \phi^3(a), \ldots$. For every n $\phi^n(a)$ is an initial segment of the word $\phi^{n+1}(a)$, so all these words are initial segments of an infinite sequence. Let us denote this sequence by $T(\phi)$. Thue [398] proved that these words do not have subwords of the form uuu where u is any non-empty word. Given $T(\phi)$, Morse and Hedlund construct a semigroup $S(\phi)$ as follows. Let $S(\phi)$ be the set consisting of all subwords of the words from this sequence, and a special

[26] Actually the author of this construction was Dilworth, as was mentioned in [274].

symbol 0. Define an operation on this set by the following rule:

$$u \cdot v = \begin{cases} uv & \text{if } uv \text{ is a subword of } \phi^n(a) \text{ for some } n; \\ 0 & \text{if } uv \text{ is not a subword of any } \phi^n(a). \end{cases}$$

Here uv is the result of concatenation of u and v. [27]

It is easy to prove that $S(\phi)$ is an infinite semigroup generated by a and b. This semigroup satisfies the identity $x^3 = 0$. Indeed, as we have mentioned above, for every word u the word u^3 is not a subword of any $\phi^n(a)$ and so it is equal to 0 in $S(\phi)$.

One can easily see that it is not possible to use the interpretations from the previous subsection to simulate a Minsky machine in a semigroup satisfying $x^3 = 0$. Indeed, we can no longer encode the number of the cell observed by the head of the machine by a power of a letter: there is a shortage of powers (only 3 is available). The idea is to use powers of the Thue endomorphism ϕ instead of powers of letters.

Thus we want to encode the configuration (m, q_k, n) by a word like $\phi^m(a)q_k\phi^n(a)$. Notice that such a word will be cube-free (will not contain subwords of the form uuu) for any m and n, so we won't be able to apply our identity $x^3 = 0$ to this word. This is good because the more "independent" the canonical words are from the identities of the variety, the better (see the remark at the end of Section 7.2.2).

Now we have to assign a relation to every command of M. This relation must increase (decrease) m and n in $w(m, q_k, n)$ if the command shifts the tapes to the left (to the right). Unfortunately it is impossible to pass from $\phi^m(a)q_k\phi^n(a)$ to $\phi^{m+1}(a)q_{k'}\phi^n(a)$ by using one relation independent of m and n. Indeed, for "large" m it is impossible to proceed from $\phi^m(a)$ to $\phi^{m+1}(a)$ by replacing a "small" subword by another "small" subword.

But we notice that $\phi^{m+1}(a) = \phi(\phi^m(a))$, so we need to find some auxiliary relations which will simulate the iteration of ϕ. This can be done by adding one letter, say, c_1 and relations $ac_1 = c_1\phi(a), bc_1 = c_1\phi(b)$. Indeed, for every m we will then have $\phi^m(a)c_1 = c_1\phi^{m+1}(a)$.

A practical realization of this idea is the following (see Section 7.2.9 for another realization).

Our semigroup, — let us denote it by $S(M, \phi)$,— will be generated by the set $\{q_0, \ldots, q_N,\ a, b,\ c_1, c_0, c_{-1},\ d_1, d_0, d_{-1},\ A, B\}$ where A, B are, of course, locks.

For every configuration (m, q_k, n) let

$$w(m, q_k, n) = A\overline{\phi^m(a)}c_0q_kd_0\phi^n(a)B$$

where $\bar{u}$ is the word u written from the right to the left.

The correspondence between commands of the Minsky machine M and relations in $S(M, \phi)$ is given by the following table.

[27] As we mentioned in Section 3.3.1 there is a general construction which associates a semigroup $S(D)$ with every symbolic dynamical system D. The semigroup $S(\phi)$ is equal to $S(D)$ where D is the symbolic dynamical system generated by the limit of words $\phi^n(a)$.

Command	$S(M,\phi)$
$q_i, 0, 0 \to q_j, T^\alpha, T^\beta$	$bac_0q_id_0ab = bac_\alpha q_jd_\beta ab$
$q_i, 1, 0 \to q_j, T^\alpha, T^\beta$	$Aac_0q_id_0ab = Aac_\alpha q_jd_\beta ab$
$q_i, 0, 1 \to q_j, T^\alpha, T^\beta$	$bac_0q_id_0aB = bac_\alpha q_jc_\beta aB$
$q_i, 1, 1 \to q_j, T^\alpha, T^\beta$	$Aac_0q_kd_0aB = Aac_\alpha q_jd_\beta aB$

(32)

The auxiliary relations are the following:

(i) $ac_0 = c_0a$, $bc_0 = c_0b$;

(ii) $ac_1 = c_1\phi(a)$, $bc_1 = c_1\phi(b)$, $Ac_1 = Ac_0$;

(iii) $\phi(a)c_{-1} = c_{-1}a$, $\phi(b)c_{-1} = c_{-1}b$, $Ac_{-1} = Ac_0$;

(iv) $d_0a = ad_0$, $d_0b = bd_0$;

(v) $d_1a = \phi(a)d_1$, $d_1b = \phi(b)d_1$, $d_1B = d_0B$;

(vi) $d_{-1}\phi(a) = ad_{-1}$, $d_{-1}\phi(b) = bd_{-1}$, $d_{-1}B = d_0B$.

The role of the new generators c_i and d_i is clear from these relations: d_1 and c_1 increase the power of ϕ, c_{-1} and d_{-1} decrease the power of ϕ.

It is not very difficult to prove Lemmas 7.3 and 7.4. Therefore we have obtained an interpretation of the machine M. Now, since all words $w(m, q_k, n)$ and all words which can be obtained from these words by the defining relations of $S(M,\phi)$ are cube-free, the identities of our variety won't change these words. Therefore the statements of Lemmas 7.3 and 7.4 hold for the factor-semigroup $\hat{S}(M,\phi)$ of the semigroup $S(M,\phi)$ over the verbal congruence corresponding to the identity $x^3 = 0$. Indeed, the statement of Lemma 7.3 is stable under homomorphic images. The statement of Lemma 7.4 holds because canonical words which are distinct in $S(M,\phi)$ are also distinct in $\hat{S}(M,\phi)$. It remains to notice that the semigroup $\hat{S}(M,\phi)$ is finitely presented in the variety given by the identity $x^3 = 0$. Therefore this variety has an undecidable word problem.

In the general case, when the identities of the variety $\mathcal{V}$ are more (sometimes much more) complicated, one has to use the endomorphisms constructed in Sapir [346] instead of ϕ, and the interpretation is slightly different also (see Sapir [335]).

7.2.6 The Word Problem for Varieties of Lie Algebras

For every variety of Lie algebras containing $\mathcal{A}^3$ (the variety of all 3-solvable Lie algebras), it is not very difficult to construct a Lie algebra which is finitely presented in this variety and has an undecidable word problem.

All that we basically need is to take the semigroup S_2 from Section 7.2.4 and replace the semigroup product by the Lie algebra product.

In more detail, let us consider the following Lie algebra L. It has generators $q_0, \dots, q_N$, a, b, A, B. The relations assigned to the commands of the machine M are the same as in the semigroup S_2. We also need the following commutativity relations similar to (31):

$$ab = aB = bA = BA = 0.$$

The word $w(m, q_k, n)$ is defined as in Section 7.2.4:

$$w(m, q_k, n) = q_k a^m A b^n B.$$

Then the proof of Lemma 7.3 is similar to the proof of this Lemma in the semigroup case.

Again, we have to prove that if we pass from the configuration (m, q_k, n) to another configuration $(m', q_{k'}, n')$ by a command κ then we pass from the word $w(m, q_k, n)$ to the word $w(m', q_{k'}, n')$ by the relation corresponding to κ.

As in Section 7.2.4, let us prove it only in the case when the command κ is of the form $q_k, 1, 0 \to q_{k'}, T^\alpha, T^\beta$ (it is instructive to compare the Lie algebra proof with the semigroup proof in Section 7.2.4):

Since the command κ is applicable to the configuration (m, q_k, n), in this configuration the head observes the first cell of the first tape and some cell other than the first on the second tape. Thus $m = 0, n \neq 0$. Then $m' = \alpha, n' = n + \beta$ (in this case α cannot be negative). Now $w(m, q_k, n) = q_k b^n AB$ and the relation corresponding to κ is $q_k bA = q_{k'} a^\alpha b^{1+\beta} A$. Since $AB = 0$ and $aB = 0$, by virtue of the Jacoby identity and the anti-commutativity identity[28] we have $w(m, q_k, n) = q_k b^n AB = q_k bAb^{n-1}B$. Thus we can apply our relation and replace $q_k bA$ by $q_{k'} a^\alpha b^{1+\beta} A$. As a result we obtain the word $q_{k'} a^\alpha b^{1+\beta} A b^{n-1} A$, which is equal to $q_{k'} a^\alpha A b^{n+\beta} B$ since $aB = 0$. The last word is equal to $w(m', q_{k'}, n')$ as desired.

Notice the role of commutativity in this proof. In order to deduce the relation $w(n, q_k, m) = w(n', q_{k'}, m')$ we need to permute neighboring letters, say, A and B, regardless of the place in the word where these letters occur. Since we do not have associativity as in the case of semigroups, this is not a completely trivial thing. In order to achieve these permutations we need the Jacoby identity. We will meet this problem once again in the case of groups, where we will replace the Lie algebra product by the group commutator. Unfortunately the group commutator does not satisfy the

[28] By these identities $xyz + yzx = xzy$ for every x, y, z, so if $yz = 0$ then $xyz = xzy$.

Jacoby identity, and this is precisely the reason why the group case is more difficult than the Lie algebra case.

To prove Lemma 7.4 we will construct a homomorphic image $\hat{L}$ of L with an explicit structure such that words $w(m, q_k, n)$ which are distinct in S_2 (as semigroup words) are distinct in $\hat{L}$ (as Lie algebra words). The algebra $\hat{L}$ is constructed in a canonical way from the semigroup S_2.

Let us take the Abelian Lie algebra T, freely generated by the elements x_u, where u runs over all non-zero elements of S_2. By definition let $x_0 = 0$. Define linear transformations of this algebra corresponding to the elements a, b, A, B:

$$x_u t = x_{ut}$$

where $t = a, b, A, B$ and ut is the product in S_2. Since T is Abelian these linear functions are derivations. Now take the semi-direct product $\hat{L}$ of T and the algebra generated by the transformations a, b, A, B. It is almost obvious that $\hat{L}$ is generated by $x_{q_0}, \dots, x_{q_N}, a, b, A, B$ and that the mapping $q_i \to x_{q_i}$, $a \to a$, $b \to b, A \to A$, $B \to B$ is extendable to a homomorphism. It is clear that in $\hat{L}$ two words $w(m, q_k, n)$ and $w(m', q_{k'}, n')$ are equal if and only if the corresponding words in S_2 are equal. Therefore if the words $w(m, q_k, n)$ and $w(m', q_{k'}, n')$ are equal in L then these words are equal in S_2. Thus Lemma 7.4 for the semigroup S_2 implies a similar statement for the Lie algebra L.

Now let us prove that the algebra $\hat{L}$ belongs to the variety $\mathcal{A}^3$, i.e. it is a solvable algebra of degree 3. Indeed, since T is an Abelian ideal of $\hat{L}$, it is enough to show that the algebra generated by the derivations a, A, b, B is metabelian.

By the commutativity relations this algebra is a direct product of the algebra E_a generated by a, A and the algebra E_b generated by b, B, so it is enough to prove that, say, the algebra E_a is metabelian. To this end notice that every Lie monomial containing more than one A is equal to zero in E_a. This immediately follows from the fact that in the semigroup S_2, every word containing more than 2 A-s is equal to 0. Therefore the monomials containing A form an Abelian ideal in E_a. The factor-algebra is generated by a, so it is also Abelian. Thus E_a is a metabelian algebra. Therefore, indeed, $\hat{L}$ is a 3-solvable algebra.

Now let us take any variety $\mathcal{V}$ of Lie algebras containing all 3-solvable algebras. The algebra $\hat{L}$ is a homomorphic image of the factor-algebra $L_{\mathcal{V}}$ of L over the verbal ideal corresponding to $\mathcal{V}$. Therefore the statement of Lemma 7.4 holds for $L_{\mathcal{V}}$. The statement of Lemma 7.3 holds too since this statement is stable under homomorphic images. Therefore the word problem in $L_{\mathcal{V}}$ is undecidable provided the Minsky machine simulates a non-recursive function.

This proves that the word problem is undecidable in any variety containing all 3-solvable algebras[29].

[29]More careful considerations would show that the algebra $\hat{L}$ belongs to the variety $\mathcal{Z}\mathcal{N}_2\mathcal{A}$. This implies that the word problem is undecidable in any variety containing $\mathcal{Z}\mathcal{N}_2\mathcal{A}$.

7.2.7 The Word Problem for Varieties of Lie Algebras. Strong Undecidability.

In the previous subsection we constructed a Lie algebra with undecidable word problem, finitely presented in a variety $\mathcal{V}$ containing all 3-solvable algebras.

Now we are going to show how to construct a finitely presented Lie algebra with undecidable word problem such that the identities of $\mathcal{V}$ follow from the defining relations of this algebra.

Notice that if the algebra $\hat{L}$, constructed in the previous subsection, was finitely presented, we would win immediately, since this algebra belongs to $\mathcal{V}$. Unfortunately (the reader surely expected this word here) it is not finitely presented. The idea roughly is the following: we want to embed $\hat{L}$ into a finitely presented algebra which belongs to $\mathcal{A}^3$.

To this end we have to add new elements to the set of generators of L (which is the set $\{q_i, a, A, b, B\}$) and add new relations to the set of defining relations of algebra L in such a way that

1. the elements q_i generate an Abelian ideal, and

2. the elements a, A, b, B generate a metabelian subalgebra.

Notice that we can add any relation of $\hat{L}$ because it will not prevent $\hat{L}$ from being embeddable in the resulting algebra.

Let us start with the second task. From the relations of L, it follows that the subalgebras $E_a =< a, A >$ and $E_b =< b, B >$ form a direct product. Thus we have to make each of them belong to $\mathcal{A}^2$.

To make E_a metabelian we have to make all words like $(Aa^n)(Aa^m)$ equal to 0 (the variety of metabelian algebras satisfies the identity $(xy)(zt) = 0$) by adding finitely many relations to L. It would be enough, of course, if we make $Aa^nA = 0$ for every n. Fortunately, we can use the embedding of a finitely generated metabelian Lie algebra into a finitely presented metabelian Lie algebra first found by G. Baumslag in 1977 [23]. Earlier, the group theoretic analogue of this embedding was found independently by Baumslag [24] and Remeslennikov [316]. Since the formulation and the proof of this statement is almost the same in the group case and in the Lie algebra case, we will call it the Baumslag-Remeslennikov Lemma or simply the BR Lemma.

Lemma BR. *Suppose that a Lie algebra over a field of characteristic $\neq 2$ is generated by three sets $X, K = \{a_i \mid i = 1, \ldots, m\}, K' = \{a'_i \mid i = 1, \ldots, m\}$ such that*

(1) The subalgebra generated by $K \cup K'$ is Abelian;

(2) For every $a \in K$ and every $x \in X$ we have $xa^2 = \gamma x a'$ (for some constant γ);

(3) $x_1 a_1^{\alpha_1} \ldots a_m^{\alpha_m} x_2 = 0$, for every $x_1, x_2 \in X$, and every $\alpha_1, \ldots, \alpha_m \in \{0, 1\}$.

Then the ideal generated by X in the subalgebra $< X \cup K \cup K' >$ is Abelian and this subalgebra itself is metabelian.

If the statement of this lemma holds for the sets $K = \{a_i \mid i = 1, \ldots, m\}, K' = \{a_i' \mid i = 1, \ldots, m\}$ and X, then we will call the elements a_i' *BR-conjoint* to elements a_i with respect to X, and we will call pairs (a_i, a_i') *BR-pairs*.

In our case $X = \{A\}, K = \{a\}$. Using BR Lemma, let us add a BR-conjoint a' to a. This means that we add a new generator a' and new relations:

$$Aa^2 = \gamma Aa', aa' = AA = AaA = 0. \tag{33}$$

for some γ which we do not want to specify now. Similarly, let us add a BR-conjoint b' to b with respect to $\{B\}$:

$$Bb^2 = \gamma Bb_1, bb_1 = BB = BbB = 0. \tag{34}$$

Now let us turn to the first problem. We have to add new generators and finitely many relations to make the ideal generated by q_i Abelian. This means that $(q_iW)(q_jU)$ must be equal to 0 for all possible words W and U in the alphabet $\{A,\ a,\ a',\ B,\ b,\ b'\}$.

By the Jacoby identity, this will follow if we prove that for any word W_1 in the alphabet $\{A,\ a,\ a'\}$ and any word W_2 in the alphabet $\{B,\ b,\ b'\}$ one has $q_iW_1W_2q_j = 0$. This seems to be similar to the problem which we have just solved (instead of A we now have q_i). But the analogy is not complete because now the letters outside the set $X = \{q_i \mid i = 0, \ldots, N\}$ do not commute. In particular, $aA \neq 0$. So we have to use some new ideas besides the Baumslag-Remeslennikov Lemma.

For the sake of simplicity let us forget about letters b, b', B, and about the word W_2.

We can represent the word W_1 in the form $W_1 = U_1AU_2A...U_nA$, where $U_1, \ldots, U_n$ are words in the alphabet a, a'. We have to make the word $R = q_iU_1AU_2A...U_nAq_j$ equal 0.

Notice that if A does not occur in R then this word is equal to $q_iU_1q_j$, which may be made 0 by adding relations which make a' a BR-conjoint to a with respect to the set of q's (so far it was a BR-conjoint to a with respect to A).

If A occurs in R, U_1 is empty, and U_2 is not empty, then R begins with the subword q_iAa, which may be made 0 by adding the relation

$$q_iAa = 0. \tag{35}$$

This relation holds in $\hat{L}$, so we can safely add it (see the remark above).

If U_1 is empty and U_2 is empty then R has a subword q_iAA or q_iAq_j. Both are equal to 0 in $\hat{L}$. So to take care of this case we can add the relations

$$q_iAA \;=\; q_iAq_j \;=\; 0 \tag{36}$$

Therefore the case when U_1 is empty and A occurs in R may be handled easily by adding finitely many relations of $\hat{L}$.

But what if U_1 is not empty? The solution is straightforward: make it empty! Let us add a new letter $\tilde{a}$, such that this letter commutes with all letters except q_i, and $q_i a = q_i \tilde{a}$.

Now in the case when U_1 is not empty we can make it empty by replacing all a' by a (using the BR relations) and a by $\tilde{a}$, and by squeezing $\tilde{a}$ through A. For example, if we had the word qaa_1Aq, then we can transform it into γqa^2aAq by using a BR relation, then we can transform it into $\gamma q\tilde{a}^3Aq$, and then we can move $\tilde{a}$ through A, and get $\gamma qA\tilde{a}^3q$. Now we have nothing between q and A.

To make such words zero, we can use the BR Lemma again. It is enough to add a BR-conjoint to $\tilde{a}$ with respect to $\{q_iA, q_i \mid i = 0, \ldots, N\}$. We will denote this pair by $\tilde{a}'$. In order not to spoil the picture that we have already drawn, let us also add relations which say that this new letter commutes with all letters except q_i.

Therefore, by adding finitely many new letters and defining relations we can make all products $q_iW_1q_j$ equal to 0. If we also add the twins of the above relations, replacing a by b, and A by B, then the ideal generated by all q_i will be Abelian.

Now let's see how these new elements and relations co-exist with the relations corresponding to the Minsky machine.

For example, consider the word $w(1, q_k, 1) = q_kabAB$ corresponding to a configuration of the Minsky machine. We can represent it in the form $q_kabAB = q_k\tilde{a}bAB = q_kbAb^{m-1}\tilde{a}B$.

Therefore, for $w(1, q_k, 1)$ there are two possible ways to apply relations corresponding to commands of the Minsky machine: one with the left part q_kab and another one with the left part q_kbA. So by adding new letters $\tilde{a}$ and $\tilde{b}$ we destroyed the main property of the algebra L, that this algebra simulates the Minsky machine. This is a disaster!

Fortunately this is the last disaster in this section, and its consequences are easy to overcome. We will show how to save everything by adding just one new generator d.

This element plays the role of a lock for the relations corresponding to the Minsky machine. In all these relations, let us replace q_i by q_id and make d commute with everything except q_i's and $\tilde{a}$, $\tilde{b}$, $\tilde{a}'$, $\tilde{b}'$. So $\tilde{a}$, for example, cannot squeeze through d. One can see that this new letter d, indeed, stops the illegal activities of letters with tildes.

Of course we now again have to take care of the ideal generated by q_i. It should still be Abelian. This is easy to do. One can repeat the analysis that we did before and conclude that it is enough to add relations $q_idd = 0$ for all i, and relations which are obtained from relations (35) and (36) by replacing q_i by q_id, and to make $\tilde{a}'$ a BR-conjoint to $\tilde{a}$ with respect to d. It is also clear that we need similar relations for the b's.

Let us take generators and defining relations of the algebra L and add the new generators and relations that we discussed above. The resulting algebra will be denoted by $\bar{L}$. This algebra belongs to $\mathcal{A}^3$.

To prove the undecidability of the word problem in $\bar{L}$ we have to prove Lemmas 7.3 and 7.4. Lemma 7.3 is a consequence of the fact that the subalgebra of the algebra $\bar{L}$ generated by the elements $q_i d, a, b, A, B$, is a homomorphic image of the algebra L from the previous Section (all the relations of L are satisfied).

To prove Lemma 7.4 we construct a homomorphic image of algebra $\bar{L}$ as we did for L.

Let S_3 be the semigroup obtained from S_2 by adding an element d which satisfies $d^2 = 0$ and commutes with a, b, A, B, and by replacing all q_i in the relations (29) by $q_i d$.

Let us take the Abelian Lie algebra T_1, freely generated by the elements q_u, where u runs through all non-zero elements of S_3. By definition let $q_0 = 0$. Using the same idea as in the previous section, we want to build a homomorphic image $\tilde{L}$ of $\bar{L}$ which is a semidirect product of T_1 and an algebra generated by $\{a,\ b,\ A,\ B,\ d,\ a',\ b',\ \tilde{a},\ \tilde{b},\ \tilde{a}',\ \tilde{b}'\}$.

In order to do this we have to define linear transformations of T' corresponding to all these generators. These linear transformations must satisfy the relations discussed above (otherwise $\tilde{L}$ won't be a homomorphic image of $\bar{L}$).

The actions of $a,\ b,\ A,\ B,\ d$ are easy to predict:

$$x_u t = x_{ut}$$

where $t \in \{a, b, A, B, d\}$, and ut is the product in S_3.

The actions of a' and b' are determined by the BR relations:

$$x_u a' = x_{ua^2}, x_u b' = x_{ub^2}.$$

The actions of $\tilde{a},\ \tilde{a}',\ \tilde{b},\ \tilde{b}'$ are also determined by the relations described above, so we will leave the definitions of these actions to the reader.

Notice that since T_1 is Abelian every linear operator of T_1 is a derivation, so we do not have to check that all our actions are in fact derivations of T_1.

Now it is almost obvious that $\hat{L}$ is isomorphic to the subalgebra of $\tilde{L}$ generated by $q_{0d}, \ldots, q_{Nd},\ a,\ b,\ A,\ B$.

This proves the statement of Lemma 7.4, since we know that in the algebra $\hat{L}$ two canonical words are equal if and only if the corresponding configurations of the Minsky machine are equivalent.

7.2.8 The Word Problem for Varieties of Groups. The Nonperiodic Case

In order to compare the applications of Minsky machines in group varieties with their applications in Lie algebra varieties, let us take the variety $\mathcal{A}^3$ of all solvable groups of degree 3. This is, of course, the group analogue of the Lie algebra variety which we considered in the previous subsections.

Let us start with an example of a group with undecidable word problem which is finitely presented in this variety.

The standard analogy between groups and Lie algebras dictates the use of the same constructions as in Section 7.2.6, replacing the Lie product by the group commutator.

So let us take the Lie algebra L constructed in Section 7.2.6 and replace the Lie algebra product by the group commutator. Then we obtain the group G generated by the elements $q_0, \ldots, q_N, a, b, A, B$, with defining relations

$$(a, b) = (a, B) = (A, b) = (A, B) = 1,$$

where (x, y) denotes the commutator $x^{-1}y^{-1}xy$, and relations given by table (29) where the product is replaced by the commutator. The canonical words are also similar [30]:

$$w(m, q_k, n) = (q_k, a^{(m)}, b^{(n)}, A, B).$$

Now, to prove Lemma 7.3 we need to be able to permute adjacent letters, say, A and B in any canonical word, regardless of the place where these letters occur.

In the case of Lie algebras it was enough to have the relation $AB = 0$, because if $uv = 0$, then for every x we had $xuv = xvu$ by the Jacoby identity.

In the group case we do not have this. It is easy to give an example of a group and three elements x, u, v in it such that $(u, v) = 1$, but $(x, u, v) \neq (x, v, u)$[31].

So we have to construct a finite family of relations which would imply all the desired permutations. We would succeed if we could find finitely many relations which make the normal subgroup generated by all the q's Abelian. Indeed, then we could consider this normal subgroup as a module over the group ring of the group generated by $\{a, b, A, B\}$. Now if x belongs to this normal Abelian subgroup then the commutator (x, u, v) will correspond to the element $x^{(u-1)(v-1)}$ of this module. And, the equality $(u, v) = 1$ implies $(u - 1)(v - 1) = (v - 1)(u - 1)$ and so $x^{(u-1)(v-1)} = x^{(v-1)(u-1)}$.

One of the ways of making this subgroup Abelian is almost obvious. Since we want to construct a group which is finitely presented in the variety $\mathcal{A}^3$, we can do, for example, the following. To each q_i we assign four new generators $p_{i,1}, \ldots, p_{i,4}$ and a relation $q_i = ((p_{i,1}, p_{i,2}), (p_{i,3}, p_{i,4}))$. These relations will automatically push q_i down to the second derived subgroup of the group that we are constructing. Since this second derived subgroup is Abelian, we guarantee that the normal subgroup generated by all q's is also Abelian.

But this method is not good enough. First of all it works only for the variety $\mathcal{A}^3$. Secondly, this trick makes Lemma 7.3 easy to prove, but it makes the proof of Lemma 7.4 very messy. Indeed, in order to build the group analogue of the Lie algebra $\hat{L}$, we will have to define linear transformations corresponding to those new generators $p_{i,j}$, and this is not trivial at all.

[30] We have agreed to read the commutator (x, y, z) as $((x, y), z)$ and $(x, y^{(n)})$ as $(x, y, \ldots, y)$.

[31] Take the free group with two generators a, b and elements $x = a, u = b, v = b^2$.

So we choose another way. Recall that when we were proving the strong undecidability of the word problem in $\mathcal{A}^3$ in the Lie case we made an ideal generated by q's Abelian. It is interesting that almost the same method allows one to solve the group analogue of this problem.

Following the Lie algebra method, we add new generators $\{d,\ a',\ b',\ \tilde{a},\ \tilde{b},\ \tilde{a}',\ \tilde{b}'\}$ and the relations saying that a' is a BR-conjoint to a with respect to the set of q's and with respect to A; b' is a BR-conjoint to b with respect to the set of q's and with respect to B; $\tilde{a}'$ is a BR-conjoint to $\tilde{a}$ with respect to the set $\{q,(q,A)\}$ and with respect to d; $\tilde{b}'$ is a BR-conjoint to $\tilde{b}$ with respect to the set $\{q,(q,B)\}$ and with respect to d;

In order for the previous paragraph to make sense we have to define what exactly a BR-conjoint in the group case is. Fortunately there exists a group analogue of the Baumslag-Remeslennikov Lemma [24], [316].

Lemma BR$_G$. *Suppose that a group G is generated by three sets $X, K=\{a_i \mid i=1,\dots,m\}, K'=\{a'_i \mid i=1,\dots,m\}$ such that*

(1) The subgroup generated by $K\cup K'$ is Abelian;

(2) For every $a\in K$ and every $x\in X$ we have $x^{f(a)}=x^{a'}$ (for some monic polynomial f of a which has at least two terms;);

(3) $(x_1^{a_1^{\beta_1}\dots a_m^{\beta_m}},x_2)=1$, for every $x_1,x_2\in X$, and every $\beta_1,\dots,\beta_m\in\{0,1,-1\}$.

Then the normal subgroup generated by X in the subgroup $< X\cup K\cup K' >$ is Abelian and G is metabelian.

If the elements a_i and a'_i and the set X satisfy this Lemma we will call a'_i a BR-conjoint to a_i with respect to X.

In addition to these relations we have to add analogues of other relations of the algebra $\bar{L}$:

$(q_i,a)=(q_i,\tilde{a})$, $(q_i,a')=(q_i,\tilde{a}')$, $(q_i,b)=(q_i,\tilde{b})$ and $(q_i,b')=(q_i,\tilde{b}')$. Also we add the analogues of relations (35) and (36): $(q_i,A,a)=1$, $(q_i,A,a')=1$, $(q_i,B,b)=1$, $(q_i,B,b')=1$ and $(q_i,A,A)=(q_i,A,q_j)=1$, $(q_i,B,B)=(q_i,B,q_j)=1,(q_i,q_j)=1$. We add the corresponding relations for $d,\tilde{a},\tilde{a}',\tilde{b},\tilde{b}'$ too. Now we are able to prove that the normal subgroup generated by $q_0,\dots,q_n$ is Abelian and hence to prove Lemma 7.3.

To prove Lemma 7.4 we construct a homomorphic image of the group G as we did for the algebras L and $\bar{L}$. We will use a semidirect product again. Let S_3 be the semigroup from the previous subsection.

Let us continue to use the Lie algebra analogy and take the direct product T_2 of cyclic groups generated by the elements x_u where u runs over all non-zero elements of S_3. By definition let $x_0=1$. We want to define automorphisms corresponding to letters $A,B,d,a,b,a',b',\tilde{a},\tilde{b},\tilde{a}',\tilde{b}'$ of the group T_2 in such a way that the subgroup, generated by the set $\{x_u,u\in S_3,A,B,d,a,b,a',b',\tilde{a},\tilde{b},\tilde{a}',\tilde{b}'\}$ in the semidirect product of T_2 and this group of automorphisms, becomes a homomorphic image of G.

There is not much freedom in defining these automorphisms. Like in the Lie algebra case for every $v\in\{a,\ b,\ A,\ B,\ d\}$ we should have $(x_u,v)=x_{uv}$, i.e.

$x_u^{-1}v^{-1}x_u v = x_{uv}$. From this we immediately deduce that x_u^v should be equal to $x_u x_{uv}$. So if we denote by ϕ_v the automorphism corresponding to v, we should have $\phi_v(x_u) = x_u x_{uv}$. We can similarly deduce the definitions of automorphisms corresponding to other letters.

But how can we prove that these are actually automorphisms?

Recall that in the case of Lie algebras we did not have to prove that our mappings were derivations, since any linear operator on a vector space is a derivation of the corresponding Abelian Lie algebra. Here we also do not worry about these mappings being endomorphisms. The main difficulty is to prove that they have inverses. And in fact it can be proved that these endomorphisms *do not* have inverses.

In particular, $\phi_a^{-1}(x_u)$ must be a vector with infinitely many non-zero coordinates, which does not belong to the direct product T_2. This is similar to the well-known fact that the function $\frac{1}{1-t}$ cannot be represented by a finite sum of powers of t (while it is representable by an infinite series).

One solution of this problem (see Kharlampovich [175]) is to replace T_2 by the Cartesian (full) product of cyclic groups. Then it becomes possible to prove that each of our endomorphisms has an inverse, but this is very non-trivial. This is like working with analytic functions instead of polynomials.

Another, easier, solution is to change G itself. To do this, let us note that the commutator is not the only operation which can be defined on a group[32]. We chose the commutator because it is the traditional analogue of the Lie algebra operation. But there are lots of other nice operations. So the idea is to choose one of these other operations.

To choose a suitable operation we start the construction of G from the end. We first define $\hat{G}$, and then define G itself. Namely, we define automorphisms of a suitable direct product of cyclic groups, then find out which operation $*$ satisfies the property $q_u * v = q_{uv}$, and replace the commutator by this operation $*$.

Of course, we have to choose the automorphisms corresponding to letters in such a way that automorphisms corresponding to members of a BR-pair form a BR-pair themselves. Practically this means that if (u, v) is a BR-pair, and $x^{f(u)} = x^v$ (see condition 2 of Lemma BR_G), then we have to define an automorphism u and then only check that $f(v)$ is also an automorphism for some monic polynomial f of degree > 1 which has at least 2 terms.

The solution is the following (see Kharlampovich [181]).

Instead of T_2 let us take T_3, a free Abelian group generated by the elements $x_{i,j,u}$ where u runs over all non-zero elements of S_3 and $i, j \in 1, 2, 3$. By definition let $x_{i,j,0} = 1$. We see that instead of one element x_u (for every u), as we had in the Lie algebra case, we have now 9 "brothers" $x_{i,j,u}$.

Let us define the automorphisms. For simplicity we will denote automorphisms corresponding to letters $a,\ a',\ b,\ b',\ A,\ B$ by the same letters.

[32] Specialists in universal algebra would definitely agree with us here.

Let us start with automorphisms $a,\ a'$. We have to define $x^a_{i,j,u}$ and $x^{a'}_{i,j,u}$ for every $i,\ j,\ u$. First suppose that u does not contain A. Then let

$$x^a_{i,j,u} = \begin{cases} x_{i,j,u}x_{i,j+1,u}x_{ij+2,u}x_{i,j,ua}, & \text{if } j=1; \\ x_{i,j,u}x^{-1}_{i,j-1,u}, & \text{if } j=2; \\ x_{i,j-2,u}, & \text{if } j=3. \end{cases}$$

$$x^{a'}_{i,j,u} = x^{-1}_{i,j,u}x^a_{i,j,u}.$$

If u contains letter A, then let $x^a_{i,j,u} = x^{a'}_{i,j,u} = x_{i,j,u}$.

It is not difficult to find automorphisms a^{-1} and $(a')^{-1}$. For example, let us find $x^{a^{-1}}_{1,j,u}$ where u does not contain letter A. By the definition of a we have

$$x^a_{1,1,u} = x_{1,1,u}x_{1,2,u}x_{1,3,u}x_{1,1,ua},$$

$$x^a_{1,2,u} = x_{1,2,u}x^{-1}_{1,1,u},$$

$$x^a_{1,3,u} = x_{1,1,a}.$$

If we apply a^{-1} to all these equalities, we get

$$x_{1,1,u} = x^{a^{-1}}_{1,1,u}x^{a^{-1}}_{1,2,u}x^{a^{-1}}_{1,3,u}x^{a^{-1}}_{1,1,ua},$$

$$x_{1,2,u} = x^{a^{-1}}_{1,2,u}(x^{a^{-1}}_{1,1,u})^{-1},$$

$$x_{1,3,u} = x^{a^{-1}}_{1,1,a}.$$

We get something like a triangular system of equations. The third equality gives us $x^{a^{-1}}_{1,1,u}$. We can use the third equality to transform the second and the first ones. Then we can use the second equality to transform the first one. Finally we get:

$$x^{a^{-1}}_{1,3,u} = x^{-1}_{1,2,u}x^{-1}_{1,3,u}x^{-1}_{1,3,ua}$$

$$x^{a^{-1}}_{1,2,u} = x_{1,2,u}x_{1,3,u},$$

$$x^{a^{-1}}_{1,1,a} = x_{1,3,u}.$$

Automorphisms $b,\ b'$ are defined similarly. Let u not contain B. Then

$$x^b_{i,j,u} = \begin{cases} x_{i,j,u}x_{i+1,j,u}x_{i+2,j,u}x_{i,j,ub}, & \text{if } i=1; \\ x_{i,j,u}x^{-1}_{i-1,j,u}, & \text{if } i=2; \\ xx_{i-2,j,u}, & \text{if } i=3. \end{cases}$$

$$x^{b'}_{i,j,u} = x^{-1}_{i,j,u}x^b_{i,j,u}.$$

If u contains B, then $x^b_{i,j,u} = x^{b'}_{i,j,u} = x_{i,j,u}$.

If $w \in \{A, B, d\}$ then let $x^w_{i,j,u} = x_{i,j,u}x_{i,j,uw}$.

It is easy to see that a works with the second additional indexes (j), and b works with the first additional index (i). This, by the way, automatically makes the mappings a and b commute. The idea of several additional indexes solving independent parts of a problem will appear again in the next subsection (though in a completely different situation).

Now we can define a partial operation $*$. We know that for every $w \in \{a, b, A, B\}$ we should have

$$x_{1,1,u} * w = x_{1,1,uw}.$$

From this relation we can deduce the form of the operation $*$:

For every $f \in G$ let $f * a = f^{-1}f^a f^{-a^{-1}} f^{(a')^{-1}}, f * b = f^{-1}f^b f^{-b^{-1}} f^{(b')^{-1}}$, for $z \in A, B$ let $f * z = (f, z)$.

It is easy to check that we have all the BR-pairs that we wanted to have. Now if we take the relations which define each of these BR-pairs, and the additional relations listed above (which also hold in $\hat{G}$), we will get a finitely presented group G which belongs to $\mathcal{A}^3$ and has an undecidable word problem.

7.2.9 The Word Problem for Varieties of Groups. The Periodic Case

Here we present the method from Sapir [345] of constructing a group with undecidable word problem which is finitely presented in the variety $\mathcal{A}_r\mathcal{B}_p$ for every odd $p \geq 665$ and every prime $r \neq p$.

First let us try to naively use the method from Section 7.2.6, replacing the Lie product by the group commutator, plus the identities of the variety $\mathcal{A}_r\mathcal{B}_p$. So let us take a Minsky machine M and define a group G by generators $q_0, \ldots, q_N, a, A, b, B$ and defining relations $(a, B) = (a, b) = (B, A) = (A, b) = 1$ and the relations of the table (29), where the product is replaced by the group commutator. A canonical word will be of the form $w(m, q_i, n) = (q_i, a^{(m)}, b^{(n)}, A, B)$. As we have shown already, in order to prove the statement of Lemma 7.3, we need certain commutativities. We would get these commutativities if the q'-s generated an Abelian normal subgroup. This is not a big problem. Indeed, since our variety satisfies the identity $x^py^p = y^px^p$, every element of exponent r generates an Abelian normal subgroup. Therefore it would be enough to push our relations and canonical words down to the normal $r-$subgroup, as we did in Section 7.2.8. For this we can add one letter Q with a relation $Q^r = 1$, and replace every q_i in the relations and canonical words by the commutator (Q, q_i). This will give us Lemma 7.3.

Let us turn to Lemma 7.4. The group G is a semidirect product of an Abelian $r-$group R and a group L of exponent p (because $r \neq p$). It is easy to see that L is a $\mathcal{B}_p$-free product of the free $\mathcal{B}_p$-group L_1, generated by the q's, and the direct product of two free $\mathcal{B}_p$-groups $L_1 =< a, A >$ and $L_2 =< B, b >$. Every homomorphic image of G is a semidirect product of a homomorphic image of R and a homomorphic image of L. Unfortunately we do not understand homomorphic images of the free Burnside groups well enough. Indeed, it takes a lot of work to prove that the free Burnside group does not collapse even if we do not add any non-trivial relations. So it would be better not to add any more relations to L. Also the group L is so complicated that it would be extremely difficult to find a handy representation of L by automorphisms of an Abelian $r-$group like T_1 or T_2 in the previous subsection. Therefore we have to reject on the spot the idea of constructing a homomorphic image of group G which is easy to deal with.

The only other idea which comes to mind is to prove syntactically the absence of "wrong" equalities of canonical words in G. For this purpose we can use the fact that R is a right $\mathbf{Z}_r L-$module. Then for every $x \in L$, $y \in R$ we have $(y,x) = y^{-1}y^x = y \circ (x-1)$ where $\circ$ is the module action. Now let us rewrite the canonical words:

$$w(m, q_k, n) = (Q, q_i, a^{(m)}, b^{(n)}, A, B) = Q \circ (q_i - 1)(a-1)^m (b-1)^n (A-1)(B-1).$$

Relations of the form $(Q, u_i) = (Q, v_i)$ may be rewritten in the form:

$$Q \circ (\hat{u}_i - \hat{v}_i) = 0$$

where $\hat{w}$ is the result of substitution $\hat{}: x \to x-1$ in w.

Every element in the normal subgroup generated by the left parts of these relations will have the form

$$Q \circ \sum_i (\hat{u}_i - \hat{v}_i) \cdot f_i$$

where the f_i are elements from L.

Now we see that two canonical words $w(m, q_k, n)$ and $w(m', q_{k'}, n')$ are equal in G if and only if we have the following equality in the group algebra $\mathbf{Z}_r B$

$$\hat{w}(m, q_k, n) - \hat{w}(m', q_{k'}, n') = \sum_i (\hat{u}_i - \hat{v}_i) \cdot f_i \tag{37}$$

for some $f_i \in L$, where $u_i = v_i$ are relations from the table (29).

Therefore we can state that *two canonical words are equal in G if and only if the difference*

$$\hat{w}(m, q_k, n) - \hat{w}(m', q_{k'}, n') =$$
$$(q_i - 1)(a-1)^m (b-1)^n (A-1)(B-1) - (q_j - 1)(a-1)^{m'} (b-1)^{n'} (A-1)(B-1)$$

belongs to the right ideal of the group algebra $\mathbf{Z}_p L$ generated by the elements $\hat{u}_i - \hat{v}_i$.

Now, we can easily see that there are many "wrong" equalities of canonical words. Indeed, since L is a periodic group, there are only finitely many different powers of a and b. This and the fact that $\mathbf{Z}_r$ is finite, gives us that there are only finitely many different powers of $a-1$ and $b-1$ in $\mathbf{Z}_r L$. Therefore there exist only finitely many different canonical words, and the problem of whether two canonical words are equal can not be undecidable.

This shows that our "naive" approach was indeed too naive. Nevertheless, we do get something positive from the above considerations. First of all we now understand that we win if we find a finitely generated right ideal in $\mathbf{Z}_p L$ with an undecidable membership problem. Secondly, we understand that we have to fight "big" powers as in the case of periodic semigroup varieties.

In order to fight the "big" powers we can employ methods from Sections 7.2.5 and 7.2.8. The first method consists in taking powers of an endomorphism instead of powers of letters. The second method consists in using artificial operations.

Let us start with the second method. If we replace the commutator by another operation then we must replace $x-1$ in the substitution $x \to x-1$ by another polynomial. It is easy to see that this cannot be a polynomial of one variable x for the same reason: we will get only finitely many powers.

So this polynomial must contain other variables as well. The simplest such polynomial is $x+x'$ where x' is a "brother" of x. Of course, we would have to add x' to the set of generators of the corresponding group L_i $(i=1,2,3)$. If we replace $x-1$ by $x+x'$, we will solve the "finite number of powers" problem. Indeed, it is possible to show that there are infinitely may powers of $x+x'$ in $\mathbf{Z}_r L$.

We can suppose that the newly born letters behave with respect to other letters just as their older brothers, i.e. if x commutes with y then x' commutes with y and y'. The semigroup given by these commutativity relations will be denoted by S'.

Now let us again try to prove that there are no "wrong" equalities of canonical words. As above, we have to consider the equality (37).

We need to prove that if this equality holds then the configurations (m, q_k, n) and $(m', q_{k'}, n')$ of the machine M are equivalent. The idea for proving this is the following. Let us represent both sides of this equality as sums of monomials. Take a monomial U in the sum $(q_k' + q_k)(a'+a)^m(b'+b)^n(A'+A)(B'+B)$. This monomial must coincide with a monomial from the right side (we are in a group ring!).

Hence we have $U = u_i' f_i$ or $U = v_i' f_i$. Here w' (for w equal to u_i or v_i) is obtained from w by replacing some letters by their "brothers". Suppose that the first equality holds. Imagine that u_i' is an initial part of the word U, i.e. the equality $U = u_i' f_i$ is an equality in the semigroup S'. Then we can take a monomial v_i' in the sum $\hat{v}_i$ and replace U by $U_1 = v_i' f_i$. The monomial U_1 must cancel with some other monomial in a sum $(\hat{u}_s - \hat{v}_s) f_s$. Again imagine that $U_1 = u_s' f_s$ in the semigroup S', and replace U_1 by $U_2 = v_j' f_s$. Finally we will hit all sums $(\hat{u}_s - \hat{v}_s) f_s$ in the right side of the equality (37) and so one of U_g will coincide with a monomial from the sum $\hat{w}(m', q_{k'}, n')$. Now look at the sequence of monomials $U,\ U_1,\ U_2, \ldots, U_g$. If we identify letters and their doubles then U will coincide with $w(m, q_k, n)$, U_k coincides with $w(m', q_{k'}, n')$, and every step in this sequence of words is obtained by replacing one side of relations from the table (29) by another side of this relation. Therefore we have a deduction of the relation $w(m, q_k, n) = w(m', q_{k'}, n')$ in the semigroup S_2. But we already know that if such relation holds in S_2 then the corresponding configurations of the Minsky machine are equivalent.

This is only an idea for a proof. Everybody can see that it has very much of a utopia. Just look at all the "let us imagine"s. But at least we can see what to do to make this proof work:

1. If w_1, w_2 are different canonical words then monomials in $\hat{w}_1$ must be different and none of these monomials can occur among monomials of $\hat{w}_2$ (and vise versa)[33]. This will force the monomials from $\hat{w}(m, q_k, n)$ to cancel with monomials from the right side of the equality (37) and not with other monomials from the left side of this equality.

2. We have to understand what to do if U, the monomial we are dealing with, is equal to $u_i' f_i$ but u_i' is not an initial part of U. Of course, we still can take a monomial v_i' and replace U by $U_1 = v_i' f_i$ but first of all it may eventually lead to big powers of letters, which is bad since we are working with periodic groups. Secondly, the transformations $U \to U_1 \to U_2 \dots$ will not simulate semigroup transformations, which is also not good.

The first problem is not very difficult. We know from Adian [3] that if p is odd and bigger than 665 then any cube-free word in the free $\mathcal{B}_p$-group is not equal to any other word of smaller length, and no two cube-free words are equal. We also notice that if w is a cube-free word then every monomial in $\hat{w}$ is also cube-free, since if we identify x and x' in this monomial, we get w. Therefore, in order to solve the first problem, it is enough to make all canonical words cube-free. We have done it in Section 7.2.5.

The second problem is harder. To solve it, we define a set of words E over the alphabet of generators of L with the following properties.

E1. Every word in E is cube-free.

E2. If w is a canonical word then one and only one monomial in $\hat{w}$ belongs to E.

E3. If the word f_i is contained in E and the word $u_i' f_i$ is also contained in E, then there exists a unique v_i' such that $v_i' f_i$ belongs to E. [34]

E4. If the word f_i does not belong to E but $u_i' f_i$ belongs to E then there exists a unique monomial u_i'' in $\hat{u}_i$ which is distinct from u_i' and is such that $u_i'' f_i$ belongs to E.

E5. If w is a positive word from E, $f \in E$, and $w = u_i' f$, then u_i' is an initial part of w.

[33] Here we consider monomials as elements of the group L.

[34] Here $u_i = v_i$ is a relation from (29).

We will continue the discussion of the second problem a little bit later. Now we would like to note that, given this set E, we can describe our transformations $U \to U_1 \to U_2 \to \dots$ precisely. Indeed, suppose again that we have the equality (37). We know that there exists a (unique) monomial U in $\hat{w}(m, q_k, n)$ which belongs to E. Since all monomials in the left part of (37) are cube-free, U is not equal to any monomial in the left side, and so it must be equal to some $u_i' f_i$ from the right side.

Now if f_i belongs to E, we choose a (unique by E3) word v_i' in $\hat{v}$, and replace U by $U_1 = v_i' f_i$. Since U is a positive word, u_i' is an initial part of U, and so U_1 is also a positive word. In this case we use all monomials from the sum $(\hat{u}_i - \hat{v}_i) f_i$ which belong to E. Let us call this transformation $U \to U_1$ an $R1$-transformation.

If f_i does not belong to E we choose a (unique by E4) monomial u_i'' in $\hat{u}_i$ which is distinct from u_i' and such that $U_1 = u_i'' f_i$ belongs to E. In this case we will call our transformation $U \to U_1$ an $R2$-transformation.

Now if U_1 belongs to the sum $\hat{w}(m', q_{k'}, n')$, our process ends. If not, we can find another sum on the right side of (37) where it belongs. Then we can proceed from U_1 to U_2 by an $R1$- or $R2$-transformation.

Let us return to our second problem: why does this process simulate the process of semigroup deductions? Notice that if all transformations in the sequence $U \to U_1 \to U_2 \to \dots$ are $R1-$transformations then all these words are positive and the sequence indeed simulates a deduction of the relation $w(m, q_k, n) = w(m', q_{k'}, n')$.

But what should we do if some of these transformations are $R2$-transformations? The idea is almost standard. *We prove that if there exists a sequence of $R1$- and $R2$-transformations which connects a monomial U from $\hat{w}(m, q_k, n)$ with a monomial U' from $\hat{w}(m', q_{k'}, n')$ then there exists another (perhaps even shorter) sequence of $R1$-transformations only, which connects these monomials.*

In truth, this last statement does not hold for any cube-free interpretation of a Minsky machine. Moreover, it turns out that we can not use the general Minsky machine at all: it is too rough an instrument for our goals. Minsky algorithms are much better. The main difference: a Minsky machine has 4 different commands with the same q_k, while a Minsky algorithm has only one. Another difference: Minsky algorithms allow shorter interpretations, i.e. the left and right sides of the corresponding relations are shorter.

The following semigroup interpretation of Minsky algorithms is used in Sapir [345]. The alphabet consists of letters $\{ q_i, q_{i1}, a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, A, B, C, D \mid i = 0, 1, \dots, N\}$. We need commutativity relations:

$$x_i y_j = y_i x_j, \quad x_i Y = Y x_i \tag{38}$$

where $x, y \in \{a, b, c, d\}, x \neq y, Y$ is the capital y for $y \in \{a, b, c, d\}$.

Recall that a Minsky algorithm deals with a number of the type $2^m 3^n$ and can multiply and divide by 2 and by 3 (see Section 7.2.1 for the definition of Minsky algorithms). A configuration of the algorithm is a triple (m, i, n) where i is the number of the command which is being executed, 2^m3^n is the current value of the number, with which the algorithm works.

Let ϕ be the Thue-like substitution (see Section 7.2.5):

$$\phi(x_1) = x_1x_2,\ \phi(x_2) = x_2x_1,\ x \in \{a, b, c, d\}.$$

Then the canonical word $w(m, i, n)$ is equal to $q_i\phi^m(a_1)\phi^n(b_1)ABCD$.

The relations corresponding to the commands of the Minsky algorithm are the following.

Command	Semigroup relations
$[\times 2\|j]$	$q_ia_t = q_i\phi(c_t), t = 1, 2$ $q_iA = q_{i1}A$ $q_{i1}c_t = q_{i1}a_t, t = 1, 2$ $q_{i1}C = q_jC$
$[\times 3\|j]$	$q_ib_t = q_i\phi(d_t), t = 1, 2$ $q_iB = q_{i1}B$ $q_{i1}d_t = q_{i1}b_t, t = 1, 2$ $q_{i1}D = q_jD$
$[: 2\|j\|k]$	$q_ia_1A = q_ka_1A$ $q_i\phi(a_t) = q_ic_t, i = 1, 2$ $q_iA = q_{i1}A$ $q_{i1}c_t = q_{i1}a_t$ $q_{i1}C = q_jC$
$[: 3\|j\|k]$	$q_ib_1B = q_kb_1B$ $q_i\phi(b_t) = q_id_t, i = 1, 2$ $q_iB = q_{i1}B$ $q_{i1}d_t = q_{i1}b_t$ $q_{i1}D = q_jD$

(39)

It is not difficult to check that these relations indeed simulate the Minsky algorithm. Suppose, for example, that the i-th command of the Minsky algorithm is of the type $[\times 2|j]$, which means that we have to multiply our number 2^m3^n by 2 (i.e. replace m by $m + 1$) and proceed to command # j. The canonical word corresponding to the configuration (m, i, n) is $q_i\phi^m(a_1)\phi^n(b_1)ABCD$. Using the relation $q_ia_t = q_i\phi(c_t)$ and the commutativity relations (38) we transform all a's into c's, and move c's through A, until there is no a between q_i and A. As a result we transform $\phi^m(a_1)$ into the word $\phi^{m+1}(c_1)$ written from the right to the left. Then we apply the relation $q_iA = q_{i1}A$, changing the index of q. The role of the letter q_{i1} is to restore the a's. We do this using the relation $q_{i1}c_t = q_{i1}a_t$ until there is no c between q_{i1} and C. As a result we transform $\phi^{m+1}(c_1)$ into $\phi^{m+1}(a_1)$, so we get the word $q_{i1}\phi^{m+1}(a_1)\phi^n(b_1)ABCD$, which is equal to the word $q_{i1}C\phi^m(a_1)\phi^n(b_1)ABD$ by the relations (38). Now we can replace $q_{i1}C$ by q_jC, and we are done: we have got the canonical word corresponding to the configuration $(m+1, j, n)$.

As one can see, there are some significant differences between the cube-free interpretation (32) and the interpretation (39). Here we do not have special letters which scan the word $\phi^m(a_1)$ and increase (decrease) the power of ϕ. All relations involve q, so they apply to a prefix of the word. We need this because we need to have that if two canonical words are equal modulo relations (38) and (39) then the difference between the "hats" of these words belongs to the *right* ideal generated by the "hats" of relations from (39).

Another feature of these relations is that we cannot apply them for a very long time if the word is not canonical: the relations check the "canonicalness" of the word all the time. For example, if there is no A in the word or there is a q between the a's and A, then q_i will never transform into q_{i1}, and furthermore it will never transform into q_j. This is also important because we want to get rid of "bad" $R2$-transformations. These $R2$-transformations "spoil" the word and we have to "undo" them after a short time.

It turned out also that we need more than one "brother" of every letter in order to solve the second problem mentioned above. In fact we need three "brothers". So instead of the polynomial $\hat{x} = x + x'$ we have to use the polynomial $\hat{x} = x_{11} + x_{12} + x_{21} + x_{22}$. Each of the new additional indexes plays its own role when we define $R1$- and $R2$-transformations. The first index keeps a word cube-free (the first indexes of letters always form a part of the Thue sequence). The second index is needed to preserve the "bad" things which happened to the word after an $R2$-transformation, and not to allow this word to become "good" before this $R2$-transformation is undone.

We must stop here. We have explained all the main ideas of the proof; for technical details we refer the reader to Sapir [345].

7.3 Minsky Machines and The Complexity of The Uniform Word Problem

As we mentioned in Section 3.4.5, the uniform word problem for commutative semigroups is exponential space complete. Here we would like to present a sketch of the Mayr-Meyer [253] proof of this result. Let us denote the uniform word problem for commutative semigroups by D.

The fact that there exists a space-exponential algorithm solving the problem D follows from Hermann's Theorem 3.24.

Indeed, if $(u, v, R) \in S_D$ then there exists a deduction

$$u = w_0 \to w_1 \to \ldots \to w_n = v \tag{40}$$

where each transition is made by applying one of the relations from R. By Hermann's Theorem the lengths of the w_i may be bounded by

$$(s(u,v,R)^2)^{2^{s(u,v,R)}} < 2^{2^{cs(u,v,R)}}$$

where $s(u,v,R)$ is the size of the triple (u,v,R), c is a constant. Since we are working in a commutative semigroup, we need no more than $s(u,v,R)2^{cs(u,v,R)}$ cells of the tape to write down each of w_i. Indeed, we only need to write the numbers of occurrences of each generator in w_i, and we can write these numbers in binary form; therefore we will use no more than $2^{cs(u,v,R)}$ cells for each exponent. The number of these exponents does not exceed the number of generators, which is less than $s(u,v,R)$.

The deduction (40) may be considered as a non-deterministic algorithm. At every step we guess a relation from R which is to be applied to w_i to get w_{i+1}. We do not need to remember all previous w_j, $j < i$, in order to guess this relation. Thus we only need enough memory to store one w_i at a time. This means that the space complexity function for this non-deterministic algorithm does not exceed $s(u,v,R)2^{s(u,v,R)}$. Therefore the problem D may be solved in exponential space by a non-deterministic machine. Now we can refer to the powerful result of Savitch [362], which states that every problem which can be solved in exponential space by a non-deterministic machine can also be solved in exponential space by a deterministic machine. This completes the proof.

The proof that every exponential space problem can be reduced to D is harder. Similarly to the proofs of the unsolvability of algorithmic problems, one has to take a problem, say, Q which is known to be exponentially space complete and reduce it to D.

The problem Q chosen by Mayr and Meyer is the following.

- B_Q is the set of all programs for a Minsky glass-coin machine with three glasses.
- The size $s(p)$ of the program $p \in B_Q$ is the length of this program.
- S_Q is the set of programs from B_Q such that starting at the configuration $(1;0,0,0)$ (the first command of the algorithm, all three glasses are empty) they end at the configuration $(0;0,0,0)$ (the Stop-command, all three glasses are empty again) and at any time the number of coins in any glass does not exceed $2^{2^{s(p)}}$.

The fact that Q is exponentially space complete was proved in Fisher, Meyer and Rosenberg [99].

The reduction of Q to D is constructed in two steps.

Take any program p from B_Q. Let n be the size of p. Let us denote $2^{2^{s(p)}}$ by e_n. The first semigroup which we will construct, will, in fact be similar to the semigroup S_1 from Section 7.2.4. Its alphabet consists of the following letters: $q_0, q_1, \ldots, q_N, a, b, c, A, B, C$ where N is the number of commands in the program p.

Since we do not care about the configurations where some glasses contain more than e_n coins, we can define canonical word as follows:

$$w(i;\ell,k,m) = q_i a^\ell b^k c^m A^{e_n-\ell} B^{e_n-k} C^{e_n-m}.$$

Now it is easy to understand how to define the system of relations $R(p)$ which correspond to commands of the program p. For example, if a command i is "Put a coin in the first glass and go to command number j", then the relation is:

$$q_i A = q_j a.$$

If a command i is "If the first glass is not empty then take a coin from this glass and go to the command number j, otherwise go to the command number k", then the relations are:

$$q_i A^{e_n} = q_j A^{e_n},\ q_i a = q_k a.$$

It can be proved, just like Lemmas 7.3, 7.4, that $w(1;0,0,0)$ is equal to $w(0;0,0,0)$ modulo $R(p)$ if and only if p belongs to S_Q. Therefore the function ϕ which is required by the definition of a reduction of one problem to another problem (see Section 2.8) is the following:

$$\phi(p) = (w(1;0,0,0), w(0;0,0,0), R(p)).$$

This function satisfies the first condition from the definition of a interpretation (in the sense of Karp):

$$p \in S_Q \text{ if and only if } \phi(p) \in S_D. \tag{41}$$

But unfortunately it does not satisfy the second condition: the size of $R(p)$ is too big. Indeed, each relation $q_i A^{e_n} = q_j A^{e_n}$ adds $e_n = 2^{2^n}$ to the size of $\phi(p)$.

In order to overcome this difficulty Mayr and Meyer have shown that each relation $q_i A^{e_n} = q_j A^{e_n}$ can be replaced by a "small" number of small relations which do the same thing.

This is not at all trivial. The idea is the following (see details in Mayr and Meyer [253]).

For each of these big words, Mayr and Meyer construct a small Minsky machine which "calculates" this word. It is easy to see that in order to calculate e_n for every n on a glass-coin machine with 4 glasses, one needs at most cn commands where c is a constant. Then they define a system of relations $R'(p)$ as the union of all the relations corresponding to these small machines. This set of relations will have size at most $c_1 n^2$ where c_1 is a constant. The hard thing is to prove that (41) holds for $R'(p)$. This is done by methods similar to those used in proving correctness of computer programs. A simpler and more efficient, but conceptually similar, way of computing e_n in commutative semigroups has been found by Jap in [416].

It is interesting that the semigroup given by relations $R(p)$ turns out to be a subsemigroup of the semigroup given by relations $R'(p)$. Thus a commutative semigroup of "large size" can be embedded into a commutative semigroup of "small size". As is mentioned in Mayr and Meyer [253], this idea is a computational complexity analogue of the Higman embedding idea. Mayr and Meyer prove this only for semigroups given by relations $R(p)$. We think that it would be interesting to solve the following problem.

Problem 7.1 *Let $A =< X \mid R >$ be a commutative semigroup of with presentation of size $s(R)$. Let $t(R)$ be the minimal size of a presentation of a semigroup containing A. Is there an algorithm to compute $t(R)$? Describe the sets of relations R such that $t(R) < log\ s(R)$.*

7.4 Minsky Machines and The Uniform Word Problem for Finite Algebras

As we wrote in the Introduction, the uniform word problem for finite algebras in a class K is the following:

> *Find an algorithm which, given a set of equalities $\{u_i = v_i \mid i \in I\}$, and an equality $u = v$, where all the words u_i, v_i, u, v are written in an alphabet X, determines if the last equality holds in every finite algebra $A =< X >$ in K that satisfies all the equalities $u_i = v_i$.*

If K is closed under homomorphic images, finite direct product, and subalgebras, this problem is equivalent to the problem of the decidability of the Q-theory (universal theory) of the the class K_{fin}.

In this subsection we describe how to prove undecidability of the uniform word problem for finite semigroups (Gurevich [133]) and finite groups (Slobodskoii [378]).

7.4.1 General Scheme

Let K be a class of finite algebras closed under finite direct products, homomorphic images, and subalgebras (i.e. a *pseudovariety*). Let f be a partially recursive function with two values, say, 1 and 2, such that the sets $f^{-1}(1)$ and $f^{-1}(2)$ are *recursively inseparable*, i.e there is no recursive set X with $f^{-1}(1) \subseteq X$ and $X \cap f^{-1}(2) = \emptyset$. Such functions exist (see Mal'cev [233] or Rogers [322]).

Suppose that we have constructed a finitely presented algebra $A =< X \mid R >$, where X is the set of generators, R is the set of defining relations, and we have a computable set of terms u_n, $n = 0, 1, \ldots$ over X such that the following two statements hold (these statements play the role of Lemmas 7.3 and 7.4).

Lemma 7.5 *If $f(n) = 1$ then $u_n = u_0$ in A $(n \geq 1)$;*

Lemma 7.6 *If $f(n) = 2$ $(n \geq 1)$ then there exists a homomorphism $\phi : A \to B_n$ from A into an algebra from K such that $\phi(u_n) \neq \phi(u_0)$.*

Then the uniform word problem in K is undecidable. Indeed, consider the following universal formula:

$$(\forall X)\&R \to u_n = u_0. \tag{42}$$

If $f(n) = 1$ then formula (42) holds in any algebra, because if all equalities from R hold in some algebra B then there exists a natural homomorphism from A to B. Since $u_n = u_0$ in A (by Lemma 7.5), the equality $u_n = u_0$ holds in B too.

If $f(n) = 2$ then by Lemma 7.6 there exists a homomorphism $\phi : A \to B$, $B \in K$, where $\phi(u_n) \neq \phi(u_0)$. Since B is a homomorphic image of A, all equalities of R hold in B, but the conclusion of the implication (42) does not hold. So B does not satisfy our implication.

The final possibility is that $f(n)$ is not defined, i.e. n is outside the domain of f. In this case (42) may or may not hold.

Therefore the implication (42) holds in K if $f(n) = 1$, and it does not hold in K if $f(n) = 2$. Hence the set of n for which (42) holds in K contains $f^{-1}(1)$ and is disjoint from $f^{-1}(2)$. Since the sets $f^{-1}(1)$ and $f^{-1}(2)$ are recursively inseparable, there is no algorithm which, given the implication (42), tells us if this implication holds in K. Therefore the uniform word problem in K is undecidable.

Remark. Moreover, the uniform word problem is undecidable in the class of finite algebras from the variety generated by A. And we can even restrict ourselves to the class of $|X|$-generated finite algebras of this variety.

Thus, in the case of finite algebras, as before, our task is to construct finitely presented (infinite) algebras with certain properties. To construct such algebras, we will use Minsky machines again.

7.4.2 Semigroups

Here we will show how to prove that the uniform word problem in the class of all finite semigroups is undecidable.

Let us take the two-valued function f from the previous subsection and construct a Minsky machine calculating the function $g_f(n) = 2^{f(n)}$. The function g_f also has two values: 2 and 4, and the preimages of 2 and 4 are recursively inseparable.

Now let us take one of the semigroups constructed in Section 7.2.4, say S_2. Notice that this semigroup solves half of our task. Indeed, let u_n be the canonical word $w(n, q_1, 0)$ for $n = 1, 2, \ldots$, and $u_0 = w(2, q_0, 0)$. Then if $f(n) = 1$ then $u_n = u_0$ in S_2. So the statement of Lemma 7.5 holds.

Unfortunately, the statement of Lemma 7.6 does not hold. Indeed, if $f(n) = 2$ then $u_n = w(4, q_0, 0)$ in S_2. If there were a homomorphism from S_2 into a finite semigroup B which separates u_0 and $w(4, q_0, 0)$, then the uniform word problem in B would be undecidable, because we cannot algorithmically decide if u_i is equal to u_0 or $w(4, q_0, 0)$. The word problem in every finite semigroup is, of course, decidable, so such B does not exist.

Therefore we have to modify the semigroup S_2. To make S_2 have many homomorphisms into finite semigroups we define a "graded" analogue of S_2. The grading parameter will be the number of steps of the Minsky machine.

In order to do this let us make an important observation: *For every number m there is only a finite number of configurations (n, q_i, n') from which the Minsky machine gets to the stop state q_0 in m steps.*

Now let us introduce a new generator, c, which will play the role of a step counter. We will suppose that c commutes with all other generators except q_i. In order to make this generator a step counter, we include it to all the relations of the table (29):

Command	S_2'
$q_i, 0, 0 \to q_j, T^\alpha, T^\beta$	$q_i ab = q_j a^{1+\alpha} b^{1+\beta} c$
$q_i, 1, 0 \to q_j, T^\alpha, T^\beta$	$q_i Ab = q_j a^\alpha A b^{1+\beta} c$
$q_i, 0, 1 \to q_j, T^\alpha, T^\beta$	$q_i aB = q_j a^{1+\alpha} b^\beta B c$
$q_i, 1, 1 \to q_j, T^\alpha, T^\beta$	$q_i AB = q_j a^\alpha A b^\beta B c$

(43)

Of course, we have to keep all the commutativity relations of S_2.

It is easy to check that for every natural number n, if $f(n)$ exists, then u_n is equal to $w(2^{f(n)}, q_0, 0)c^k$ where k is the number of steps needed for the Minsky machine to get to the stop state from $(2^n, q_1, 0)$.

Now to make all words u_n with $f(n) = 1$ again equal to u_0 (and to make the statement of Lemma 7.5 hold), it is enough to add one more relation:

$$u_0 = w(2, q_0, 0) = 0.$$

To prove the second Lemma suppose that $f(n) = 2$, and our Minsky machine transforms the configuration $(2^n, q_1, 0)$ into the configuration $(4, q_0, 0)$.

All elements of the semigroup S_2' which are equal to the word $w(4, q_0, 0)c^m$ have the form $w(n, q_i, n')c^r$, where $0 \le r \le m$ and $n, n' \le m + 4$. Indeed, we can only apply relations from the table (43) to the word $w(4, q_0, 0)c^m$ from the right to the left. Every such application reduces the exponent of c, and changes the exponents of a and b by at most 1.

Now to find a finite homomorphic image of S_2' where u_n ($f(n) = 2$) and u_0 are different, we can do the following. Since $f(n) = 2$ we have $u_n = w(4, q_0, 0)c^m$ for some m. Now let us consider the set W of words equal to u_n in S_2' and all subwords of these words. This set is finite, and u_n, of course, belongs to W. The complement $I = S_2' \backslash W$ is an ideal, and $u_0 = 0$ belongs to I. Now the natural homomorphism $S_2' \to S_2'/I$ separates u_n and u_0. This proves Lemma 7.6.

7.4.3 Groups

The idea is almost straightforward. It is clear that one has to take the group from Section 7.2.8, and modify it by introducing a step-counting generator c.

We can even avoid the pain of long calculations, because everything we need has been done in Section 7.2.8.

Let us take the semigroup S_3 from Section 7.2.7 (this is the semigroup S_2 with an additional generator d), add a generator c, as we did in the previous subsection (do not forget the relation $u_0 = 0$): we will get a semigroup S_3'. Then we can construct a group $G(S_3')$ as we did it in Section 7.2.8 for the semigroup S_3. Let u_n be the canonical words $w(n, q_1, 0)$ in $G(S_3')$. We can also assume that $u_0 = w(2, q_0, 0) = 1$ in $G(S_3')$.

The proof of Lemma 7.5 is similar to the proof of Lemma 7.3, outlined in Section 7.2.8.

To find the finite homomorphic images of our group which are required by Lemma 7.6, let us consider the homomorphic image $\hat{G}(S_3')$ of the group $G(S_3')$, similar to $\hat{G}$ in Section 7.2.8. This group is a semi-direct product of a free Abelian group T_3' constructed using the semigroup S_3', and a finitely generated group generated by automorphisms corresponding to all generators of $G(S_3')$ except the q's.

Now let us take a natural number n such that $f(n) = 2$ and construct the ideal I of the semigroup S_3' as we did in the previous subsection. From the definition of $\hat{G}(S_3')$, it is easy to deduce that for every ideal I of S_3' the subgroup $H(I)$ of T_4, generated by the elements $x_{i,j,u}$ with $u \in I$, is normal in $\hat{G}(S_3')$. In our case the ideal I has a finite index (i.e. S_3'/I is finite). Therefore the factor-group $T_4/H(I)$ is a finitely generated Abelian group. Hence $\hat{G}(S_3')/H(I)$ is a semi-direct product of a finitely generated Abelian group and a finitely generated group from $\mathcal{A}^2$. Such semi-direct products are known to be residually finite (see Mal'cev [236], and Hall [138]). The element u_n does not belong to $H(I)$, so it is not equal to 1 in $\hat{G}(S_3')/H(I)$. Then there exists a homomorphism from $\hat{G}(S_3')/H(I)$ into a finite group B which separates u_n from $1 = u_0$. Now to finish the proof of Lemma 7.6 it is enough to consider the composition of homomorphisms from $G(S_3')$ onto $\hat{G}(S_3')$, from $\hat{G}(S_3')$ onto $\hat{G}(S_3')/H(I)$, and from $\hat{G}(S_3')/H(I)$ onto B.

7.5 An Analysis of Kukin's Method of Simulating Recursive Functions

Minsky machines and algorithms are not the only tools for proving the undecidability of the word problem. In this Section we shall discuss a method of simulating recursive functions proposed and extensively used by G.P.Kukin [213], [92],[216],[214]. Kukin used this method to prove the undecidability of word and isomorphism problems, and to obtain Higman-type results for varieties of groups and Lie algebras. Some of the proofs contained gaps and some results were later proved to be false (see Kukin

[215]). The results about the word problem which turned out to be correct have been later rigorously proved and improved by using Minsky machines and interpretations of systems of differential equations (see Kharlampovich [187], Sapir and Kharlampovich [353], Melnichuk, Sapir and Kharlampovich [268], Kharlampovich [185]). However the Higman-type results have yet to be proved by methods different from Kukin's. Therefore there is a widespread opinion that Kukin's method is an alternative to Minsky machines.

So we obviously could not ignore this method in our survey. We wanted to present the main ideas of the method, and then, as an example, simulate a particular recursive function, say, $f(n) = 2n$. We must say that papers by G.P.Kukin make difficult reading. There are many misprints and statements without proofs. The notation is sometimes strange and difficult to decipher. But finally we extracted the main ideas of the method and constructed, following Kukin's scheme, relations intended to simulate the function $f(n) = 2n$ in a Lie algebra.

But then we noticed that these relations do not work properly. Instead of simulating our nice function $f(n) = 2n$, they just glue together all canonical words. After a careful analysis of Kukin's method we came to the conclusion that this method *does not work at all.* It is impossible to simulate any more or less complicated function using this method.

So we changed the plan of this section. Now we are going to show why Kukin's method does not work. We will also show how to fix it. And we will show that after we fix the method, it turns into a method of simulating Minsky machines with many (more than 2) tapes.

We recall the definition of a primitive recursive function (see Rogers [322]).

The class of primitive recursive functions is the smallest class $\mathcal{C}$ of functions from N^s $(s = 1, 2, \ldots)$ to N such that

- All constant functions $c(x_1, \ldots, x_k) = m$ are in $\mathcal{C}$;

- The successor function $s(x) = x + 1$ is in $\mathcal{C}$;

- All projection functions $I_m^k(x_1, \ldots x_m) = x_k$ are in $\mathcal{C}$;

- If ϕ is a function of k variables in $\mathcal{C}$ and $\phi_1, \ldots, \phi_k$ are (each) functions of m variables in $\mathcal{C}$, then the function

$$\psi(x_1, \ldots x_m) = \phi(\phi_1(x_1, \ldots, x_m), \ldots, \phi_k(x_1, \ldots, x_m))$$

  is in $\mathcal{C}$. This function f is called a composition of ϕ_i and ϕ;

- If h is a function of $k + 1$ variables in $\mathcal{C}$, and g is a function of $k - 1$ variables in $\mathcal{C}$, then the unique function f of k variables satisfying

$$f(x_1, \ldots, x_{k-1}, 0) = g(x_1, \ldots, x_{k-1}),$$

$$f(x_1,\ldots,x_{k-1},y+1) = h(f(x_1,\ldots,x_{k-1},y),x_1,\ldots,x_{k-1},y)$$

is in $\mathcal{C}$. And we say that f is obtained by primitive recursion from g and h.

For example, let $f(x,y) = x+y$, then this function is obtained by primitive recursion from the functions $g(x) = I_1^1(x) = x$ and $h(x,y,z) = s(x) = x+1$. Indeed, $f(x,0) = I_1^1(x) = x$ and $f(x,y+1) = h(f(x,y),x,y) = f(x,y)+1 = (x+y)+1 = x+(y+1)$.

The function $\bar{\psi}(x) = 2x$ is obtained by composition of $\phi_1(x) = \phi_2(x) = I_1^1(x) = x$ and $\phi(x,y) = x+y$.

Let $\phi(x)$ be a recursive function. Following Kukin [213] we shall say that a Lie algebra $L =< b_{init},\ldots,b_{fin},x_{init},\ldots,x_{fin},y,y_*,z_1,\ldots >$ simulates the function ϕ if it satisfies the commutativity relations (44) below, and for every natural number m the equality $b_{init}x_{init}^{1+n}y_*yR(z) = b_{fin}x_{fin}^{1+m}y_*y$, where $R(z)$ is a word in the alphabet of $z_1,\ldots$, holds if and only if $m = \phi(n)$ for some n. We also say that the relations of this algebra simulate the function.

We shall say that an algebra

$$L =< b_{init},\ \ldots,\ b_{fin},\ x_{init},\ \ldots,\ x_{fin'},\ x_{fin},\ y,\ y_*,\ z_1,\ldots >$$

simulates the graph of the function ϕ if it satisfies relations

$$z_iz_j = yz_i = y_*z_i = x^*z_i = x_kz_i = x_kx_{k'} = x^*x_k = y_*x_k = 0. \tag{44}$$

and the equality

$$b_{init}x_{init}^{1+n}y_*yR(z) \ = \ b_{fin}x_{fin'}^{1+n}x_{fin}^{1+m}y_*y$$

holds if and only if $m = \phi(n)$. In this notation we will call x_{init}, b_{init} initial letters, and $b_{fin}, x_{fin}, x_{fin'}$ terminal letters.

The idea of constructing such an algebra is the following. List all steps in the construction of ϕ as a primitive recursive function. Each step is either a concrete function (a constant, a projection, or the successor), a composition, or a primitive recursion. Then for each step write down the list of relations which simulates this step (that is the list which simulates the function which we obtain at this step). Then the algebra L defined by the last of these lists of relations will simulate the function ϕ.

For example, if we have a list of relations π_α which simulates the function $\alpha(x)$ and a list of relations π_β which simulates the function $\beta(x)$, and we want to make relations which simulate the composition $\beta(\alpha(x))$ then we can do the following. Write down relations of π_α and π_β using disjoint alphabets. Let, say, b_{init}, x_{init} be the initial letters, and b_{fin}, x_{fin} be the final letters of ϕ_α, and let b'_{init}, x'_{init} be the initial letters, and b'_{fin}, x'_{fin} be the final letters of ϕ_β Then the list corresponding to the composition is the union of π_α and π_β plus relations which identify the terminal letters of π_α with the initial letters of π_β: $x_{fin} = x'_{init}, b_{fin} = b'_{init}$. It is easy to see that if $b_{init}x_{init}^{1+n}y_*yR_1(z) = b_{fin}x_{fin}^{1+\alpha(n)}y_*y$ modulo relations in ϕ_α and

$b'_{init}(x'_{init})^{1+\alpha(n)}y_*yR_2(z) = b'_{fin}(x'_{fin})^{1+\beta(\alpha(n))}y_*y$ then $b_{init}x_{init}^{1+n}y_*yR_1(z)R_2(z) = b'_{fin}(x'_{fin})^{1+\beta(\alpha(n))}y_*y$ modulo the joint set of relations constructed above. Notice that the converse implication is not so obvious, but probably can be proved.

One can show that each of the concrete functions (constants, the successor, and the projections) can be simulated this way. The simulation of primitive recursion (Lemma 3.6 in Kukin [213]) is where Kukin's method fails. Note that, to the best of our knowledge, concrete relations simulating recursive functions may be found only in [213]. Kukin never repeated these relations in his other papers. Even in Epanchintsev and Kukin [92] devoted to interpretations of recursive functions in groups, the authors say (p.172 in the translation and p.282 in the original article) that the relations and proofs carry over verbatim [from the Lie algebra situation] to the situation for groups. Thus we will use the relations and the scheme from [213].

We will construct a counter-example to Theorem 4.1 in [213], the key statement in this paper. Similar statements may be found in other papers [92], [216], [214]. This Theorem says that if we have a Lie algebra which simulates a primitive recursive function $f(n)$ with initial letters b_{init}, x_{init}, and final letters b_{fin}, x_{fin}, and add the relation $b_{init}x_{init}^2z = 0$ and the other two relations from the next to the last line on page 276 of the translation of [213] (the 7th and 8th lines of page 412 in the original article), then in the resulting algebra $b_{fin}x_{fin}^{1+m}yy_*$ is equal to 0 if and only if $m = f(n)$ for some n.

Let us take the function $f(n) = 2n$. Following Section 3 of [213], the algebra L_f (in [213] it is denoted by $\tilde{A}$), which simulates $f(n)$, is generated by the set

$$C = \{b_i, z_j, x_t, x^*, y, y_*;\ i = 0, \dots 9,\ j = 0, \dots 16, t = 1, \dots 8, \},$$

$x_{init} = x_1, x_{fin} = x_5, b_{init} = b_0, b_{fin} = b_8$.

The defining relations of L_f are listed below.

First we include the relations (44) (see relations (*) from Section 2 of [213].

Then, following the recipes of Lemmas 3.3 and 3.4 of [213] we add a set of relations π (Lemmas 3.3, 3.4) which simulates the function $I_1^1(x)$:

$$b_2x_2z_{11} = b_2x_5x_7,\ b_2x_7z_{12} = b_9x_7,\ b_9x_7z_{13} = b_9x_2,\ b_9x_2z_{14} = b_3x_2;$$

We also need the successor, so add some relations ρ from Lemma 3.2 of [213] and from the scheme of Lemma 3.6 (page 411 of the original article):

$$b_4z_{15}x_5 = b_4x_5x_6, b_4z_{16}x_6 = b_5x_6^2.$$

The next step is the primitive recursion. So we need the relations from Lemma 3.6 of [213]:

$$b_1z_1x_8 = b_1x_3x_4,\ b_1z_2x_3 = b_2x_3,\ b_3z_3x_4 = b_3x^*,\ b_3z_4x^{*^2} = b_4x^*,$$

$$b_4z_5x^* = b_4x_4,\ b_5z_6x_5x_3 = b_5x_3,\ b_5z_7x_3 = b_6x_3,$$

$$b_6 z_8 x_6 = b_6 x_5,\ \ b_6 z_9 x_3 = b_3 x_3,$$

$$b_3 x^* y_* = b_7 y_*, b_7 = b_8$$

To finish the construction we have to add the relations defining the composition of the functions $\phi_1(n) = \phi_2(n) = n$ and the function $\phi_3(x, y) = x + y$:

$$b_0 z_{10} x_1 = b_0 x_8 x_2, b_0 z_{17} x_8 x_2 = b_1 x_8 x_2.$$

OK, the meal is almost ready. Now if the proof of the main result of [213], Theorem 4.1, were correct, we would add the relations

$$b_8 x_2 x_3 x_5 z_{14} = b_8 x_5,\ \ b_0 x_1^2 z_0 = b_0 x_1^3, b_0 x_1^2 y_* y = 0$$

(see the relations from the next to the last line on page 276 of the translation of [213] or the 7th and 8th lines of page 412 in the original article) and get an algebra where

$$b_8 x_5^{1+m} y_* y = 0$$

if and only if m is a value of the function $f(n) = 2n$, that is if and only if m is even.

The next sequence of transformations shows that $b_8 x_5^{1+3} y_* y$ is also equal to zero in this Lie algebra. At each step in this sequence we use relations of the algebra.

$$0 = b_0 x_1^2 y_* z_{10}^2 z_{17} z_1^2 z_2^2 z_{11}^2 z_{12} z_{13}^2 z_{14} z_3^2 z_4 z_5 z_{15}^2 z_{16} z_6 z_7 z_8^3 z_9 z_3 z_{14}^2 =$$
$$b_0 x_8^2 x_2^2 y_* z_{17} z_1^2 z_2^2 z_{11}^2 z_{12} z_{13}^2 z_{14} z_3^2 z_4 z_5 z_{15}^2 z_{16} z_6 z_7 z_8^3 z_9 z_3 z_{14}^2 =$$
$$b_1 x_8^2 x_2^2 y_* z_1^2 z_2^2 z_{11}^2 z_{12} z_{13}^2 z_{14} z_3^2 z_4 z_5 z_{15}^2 z_{16} z_6 z_7 z_8^3 z_9 z_3 z_{14}^2 =$$
$$b_1 x_2^2 x_3^2 x_4^2 y_* z_2^2 z_{11}^2 z_{12} z_{13}^2 z_{14} z_3^2 z_4 z_5 z_{15}^2 z_{16} z_6 z_7 z_8^3 z_9 z_3 z_{14}^2 =$$
$$b_2 x_2^2 x_3^2 x_4^2 y_* z_{11}^2 z_{12} z_{13}^2 z_{14} z_3^2 z_4 z_5 z_{15}^2 z_{16} z_6 z_7 z_8^3 z_9 z_3 z_{14}^2 =$$
$$b_2 x_3^2 x_4^2 x_5^2 x_7^2 y_* z_{12} z_{13}^2 z_{14} z_3^2 z_4 z_5 z_{15}^2 z_{16} z_6 z_7 z_8^3 z_9 z_3 z_{14}^2 =$$
$$b_9 x_3^2 x_4^2 x_5^2 x_7^2 y_* z_{13}^2 z_{14} z_3^2 z_4 z_5 z_{15}^2 z_{16} z_6 z_7 z_8^3 z_9 z_3 z_{14}^2 =$$
$$b_9 x_3^2 x_4^2 x_5^2 x_2^2 y_* z_{14} z_3^2 z_4 z_5 z_{15}^2 z_{16} z_6 z_7 z_8^3 z_9 z_3 z_{14}^2 =$$
$$b_3 x_2^2 x_3^2 x_4^2 x_5^2 y_* z_3^2 z_4 z_5 z_{15}^2 z_{16} z_6 z_7 z_8^3 z_9 z_3 z_{14}^2 =$$
$$b_3 x_2^2 x_3^2 x^{*^2} x_5^2 y_* z_4 z_5 z_{15}^2 z_{16} z_6 z_7 z_8^3 z_9 z_3 z_{14}^2 =$$
$$b_4 x_2^2 x_3^2 x^* x_5^2 y_* z_5 z_{15}^2 z_{16} z_6 z_7 z_8^3 z_9 z_3 z_{14}^2 = b_4 x_2^2 x_3^2 x_4 x_5^2 y_* z_{15}^2 z_{16} z_6 z_7 z_8^3 z_9 z_3 z_{14}^2 =$$
$$b_5 x_2^2 x_3^2 x_4 x_5^2 x_6^3 y_* z_6 z_7 z_8^3 z_9 z_3 z_{14}^2 = b_5 x_2^2 x_3^2 x_4 x_5 x_6^3 y_* z_7 z_8^3 z_9 z_3 z_{14}^2 =$$
$$b_6 x_2^2 x_3^2 x_4 x_5 x_6^3 y_* z_8^3 z_9 z_3 z_{14}^2 = b_6 x_2^2 x_3^2 x_4 x_5^{1+3} y_* z_9 z_3 z_{14}^2 =$$
$$b_3 x_2^2 x_3^2 x_4 x_5^{1+3} y_* z_3 z_{14}^2 = b_3 x_2^2 x_3^2 x^* x_5^{1+3} y_* z_{14}^2 =$$
$$b_7 x_2^2 x_3^2 x_5^{1+3} y_* z_{14}^2 = b_8 x_2^2 x_3^2 x_5^{1+3} y_* z_{14}^2 = b_8 x_5^{1+3} y_*$$

We can also show that for any natural m the element $b_8x_5^{1+m}y_*y$ is equal to 0 in this algebra.

Thus the construction is incorrect. Now let us explain what's wrong with it. The main mistake in the construction is the following.

Consider, for example, the relations $b_5z_6x_5x_3 = b_5x_3$ and $b_5z_7x_3 = b_6x_3$ from Lemma 3.6 in [213]. We can apply both of these relations to the canonical word $b_5z_6^kz_7^tx_5^sx_3^py_*y$. Kukin considers only one alternative. He supposes that only the first relation is applied until the letter x_5 disappears. However, one can apply the second relation first, and get an unexpected result.

We have met the same difficulty in Section 7.2.7. We have seen that it is always bad when different relations can be applied to the same canonical word. And we know the cure: locks. Suppose we have a canonical word $b_ix_j^k\dots$ and suppose we want to apply to this word a relation until x_j disappears, and then replace b_i by b_ℓ; we can do the following. Add a new letter y_j (a lock for x_j) to the set of generators and multiply the canonical word by this letter. This y_j must commute with all other y's, and with all x_s for $s \neq j$. Then we need relations like $b_ix_5^2 = b_ix_5$, $b_iy_5 = b_\ell y_5$. These relations will do what we need.

But if we add all these locks y_j, the construction will become equivalent to an interpretation of a Minsky machine with many tapes, similar to what was used in Section 7.2.6.

This would be easier to observe if we replace b_i, x_i, y_i by the letters which play similar roles in the Minsky machine interpretations:

$$\begin{array}{ccc} b_i & \to & q_i \\ x_i & \to & a_i \\ x^* & \to & b \\ y_i & \to & A_i \\ y_* & \to & B \end{array}$$

As the reader may recall, A and B were letters which marked the ends of the tapes of the Minsky machine. Using this similarity, we can conclude that what the author in [213] did was an interpretation of a Minsky machine with a huge number of tapes where only one tape had a left end, and the other tapes were infinite in both directions. We can prove that no recursive function with a non-recursive range can be calculated using such a machine. Actually, we will provide sketches of two proofs. The first proof is based on the fact that such a machine can be simulated in a Lie algebra which is finitely presented in the variety $\mathcal{N}_2\mathcal{A}$. The construction is the same as that used in Section 7.2.6. But this variety has a decidable word problem by Theorem 5.7. Therefore the machine cannot compute a recursive function with a non-recursive range.

The following direct proof was sent to us by M.Minsky [270]. With his permission, we present this proof here.

If one of the tapes of the machine is infinite, with no end marker, then it might as well not exist, because it will always appear the same to the reading head.

Therefore any such machine is equivalent to a 1-tape non-writing machine. Any 1-tape non-writing machine is equivalent to a finite-state machine (hence it cannot be universal). To prove this, suppose the machine has N states. Then it must always repeat a state within N steps. Therefore, if you start the machine at the end of the tape, then either it will never get further than N boxes away from the end — and therefore will be a finite state machine — or else it will at some time get further than N boxes away. In the latter case, though, it will also be finite state, because it must forever keep moving further away, by repeating whatever state-sequence got it that far. This state sequence is a finite loop, hence the reading head will never return to the end of the tape.

Finally notice that even if we add all these locks y_i it would be impossible to construct either a group or a Lie algebra in the variety $\mathcal{N}_3\mathcal{A}$ with unsolvable word problem. Indeed, the class of nilpotency of the derived subgroup (subalgebra) must be greater than the number of tapes of the machine that we simulate (see also Section 7.2.3). So even if we use the modified version of Kukin's method we will only obtain algebras and groups from $\mathcal{N}_k\mathcal{A}$ for very big k, and not from $\mathcal{N}_3\mathcal{A}$ as was claimed in [213], [92],[216],[214].

We do not know if the Higman-type results obtained by using Kukin's method can also be obtained by using the modified method, or, for that matter, any other method. Thus all these results must be considered as only conjectures (see Section 6).

7.6 Differential Equations

We are going to present a method of simulating differential equations which has proved to be more powerful than the Minsky machines method in the cases of groups, associative and Lie algebras. This method allowed us to go deep down in the lattice of varieties and find minimal varieties with undecidable word problem — a task inaccessible to Minsky machines.

7.6.1 Where Differential Equations Come From

As we have mentioned in Section 7.1, in order to prove the decidability or undecidability of the word problem in a variety of groups, Lie or associative algebras, we often need to consider the membership problem for a sum of two (right) modules over different subrings of a "big" ring.

Let us look at an example which shows how these considerations lead to systems of differential equations.

Let K be a commutative algebra over a field $\mathcal{F}$ of characteristic 0, generated by elements $\{a_i, b_i \mid i = 1, \dots, e\}$ subject to the following polynomial relations:

$$(a_i - b_i)(a_j - b_j) = 0, i, j = 1, \dots, e.$$

Let K_1 be the subring generated by the a's, K_2 be the subring generated by the b's. Let M be a free module over K, freely generated by $\{q_1, \dots, q_s\}$. Finally let $M_1 =< t_j \mid j = 1, \dots >$ be a finitely generated K_1-submodule of M, and $M_2 =< s_j \mid j = 1, \dots >$ be a finitely generated K_2-submodule of M. The membership problem is the following:

Given an element m in M, determine whether or not m belongs to $M_1 + M_2$.

We can rewrite the relation $m \in M_1 + M_2$ in the "coordinate" form. We have that m, each of the t_j, and each of the s_j are sums of q_i with coefficients from K. The inclusion $m \in M_1 + M_2$ means that m is equal to $\sum s_j f_j(a_1, \dots, a_e) + \sum t_j g_j(b_1, \dots, b_e)$ where f_j and g_j are polynomials. This is equivalent to a system of equations like

$$\begin{cases} c_{11}f_1 + c_{12}f_2 + \ldots + d_{11}g_1 + d_{12}g_2 + \ldots &= r_1 \\ c_{21}f_1 + c_{22}f_2 + \ldots + d_{21}g_1 + d_{22}g_2 + \ldots &= r_2 \\ \ldots\ldots\ldots \end{cases} \tag{45}$$

Where c_{kj}, d_{kj}, r_i are fixed elements from K. So the problem is the following:

Is there an algorithm for solving systems like (45)?

We can further simplify this problem by using the following observation. Recall that a_i and b_i are connected by the relation $(a_i - b_i)^2 = 0$. This means $a_i^2 = 2(a_i - b_i)b_i + b_i^2$. Multiplying this equality by a_i and transforming it a little bit, we get $a_i^3 = 3(a_i - b_i)b_i^2 + b_i^3$, and so on; for every n we have $a_i^n = n(a_i - b_i)b_i^{n-1} + b_i^n$. For those who have forgoten Calculus, recall that nx^{n-1} is the derivative of x^n. Since the derivative is a linear operator, we have

$$f(a_i) = (a_i - b_i)f'(b_i) + f(b_i),$$

where f is a polynomial in one variable.

If we turn to polynomials of several variables and use the relation $(a_i - b_i)(a_j - b_j) = 0$, then similar calculations give us the following formula:

$$f(a_1, \dots, a_e) = f(b_1, \dots, b_e) + \sum (a_i - b_i) \frac{\partial f}{\partial x_i}(b_1, \dots, b_e) \tag{46}$$

for every polynomial $f(x_1, \dots, x_e)$.[35]

[35] This formula looks like (and exactly is!) the Taylor expansion of the polynomial f around the point $(b_1, \dots, b_e)$.

This implies, by the way, that the $\mathcal{F}$-algebra K is spanned by monomials in the b's and monomials in the b's multiplied by $(a_i - b_i)$, $i = 1, 2, \ldots, N$. Actually these elements form a basis of the vector space K.

If we now rewrite the system (45) using representations of all coefficients and unknowns in this basis, we will get a system of linear differential equations

$$\begin{cases} D_{11}f_1 + D_{12}f_2 + \ldots d'_{11}g_1 + d'_{12}g_2 + \ldots & = r'_1 \\ D_{21}f_1 + D_{22}f_2 + \ldots d'_{21}g_1 + d'_{22}g_2 + \ldots & = r'_2 \\ \ldots\ldots\ldots \end{cases} \tag{47}$$

where the D_{ij} are linear differential operators, $f_i(b_1, \ldots, b_e), g_i(b_1, \ldots, b_e)$ are unknown functions, d'_{ij} and r'_j are polynomials.

It is important to mention that the left-hand side of this system depends on the modules M_1 and M_2, but not on m, and the right side depends on the element m.

Of course, we cannot say that we can get all such systems, but this shows that it is worth considering the following *Differential Equations Problem*:

> *Given the left-hand side of a linear system of differential equations (47) with polynomial coefficients, is there an algorithm which for every polynomial right-hand side determines whether or not the system (47) has a solution in* $K[x_1, \ldots, x_n]$?

7.6.2 Undecidability of the Differential Equations Problem

One of the simple but crucial ideas in proving the undecidability of the Differential Equation Problem was its reformulation in terms of the Weyl algebra.

Let n be a natural number and K a domain of characteristic 0. The Weyl algebra $W_n = W_n(x_1, \ldots x_n)$ over K is an associative algebra with the presentation

$$\begin{aligned} W_n \quad &=< x_1, \ldots, x_n, d_1, \ldots, d_n \mid \\ &d_i x_i - x_i d_i = 1, x_i x_j = x_j x_i, d_i d_j = d_j d_i, x_i d_j = d_j x_i (i \neq j) > . \end{aligned}$$

The algebra W_n acts on the ring of polynomials $K[x_0, \ldots, x_n]$ by the operation $*$ defined by

$$x_i * f = x_i f, d_i * f = \frac{\partial f}{\partial x_i}.$$ [36]

Therefore every linear system of differential equations may be rewritten in the form

$$D * \vec{f} = \vec{r} \tag{48}$$

where D is a matrix over W_n, $\vec{f}$ is a vector of unknowns from $K[x_1, \ldots, x_n]$, $\vec{r}$ is a vector of polynomials from $K[x_1, \ldots, x_n]$.

Therefore we can reformulate the Differential Equations Problem as follows:

[36] This action became famous after it was used in quantum mechanics.

Given a matrix D over W_n, is there an algorithm which for every vector $\vec{r}$ over $K[x_1,\dots,x_n]$ determines whether or not the system (48) has a solution in $K[x_1,\dots,x_n]$?

To illustrate the proof of the undecidability of this problem, let us notice that

$$d_i x_i * x_1^{m_1}\dots x_n^{m_n} = (m_i+1)x_1^{m_1}\dots x_n^{m_n}$$

Hence

$$p(d_1x_1,\dots,d_nx_n) * x_1^{m_1}\dots x_n^{m_n} = p(m_1+1,\dots m_n+1)x_1^{m_1}\dots x_n^{m_n}.$$

Now let $p(y_1,\dots y_n)$ be a polynomial with integer coefficients. Consider the following equation:

$$p(d_1x_1\dots d_nx_n) * f(x_1,\dots,x_n) = 0. \tag{49}$$

This equation has a nonzero solution if and only if the equation $p(x_1+1,\dots,x_n+1)=0$ has a nonnegative integer solution. Indeed, if $(m_1,\dots,m_n)$ is a nonnegative integer solution of the equation $p(x_1+1,\dots,x_n+1)=0$, then $x_1^{m_1}\dots x_n^{m_n}$ is a nonzero solution of the equation (49). Conversely, if $p(x_1+1,\dots,x_n+1)=0$ does not have a nonnegative integer solution and $f(x_1,\dots,x_n)$ is a nonzero polynomial then $p(d_1x_1\dots d_nx_n) * f(x_1,\dots,x_n)$ cannot be zero because its leading term will be of the same degree as the polynomial $f(x_1,\dots,x_n)$.

Recall that by the famous theorem of Matiyasevich [252] there exists a polynomial p with 10 variables and integer coefficients such that the problem of whether the equation

$$p(x_1,\dots,x_{10}) - \ell = 0 \quad (\ell = 1,2,\dots) \tag{50}$$

has a positive integer solution is undecidable.

Now if we take this polynomial p, then the problem of whether the differential equation

$$(p(d_1x_1,\dots,d_{10}x_{10}) - \ell) * f = 0 \tag{51}$$

has a non-zero solution in $K[x_1,\dots,x_{10}]$ is undecidable.

This differential equation has two deficiencies. First of all it always has a solution (zero). Secondly, its "differential" part is not fixed, and its "free" part is fixed - just the opposite of what we need. So we have not yet proved the proposition.

To make our idea more productive we crossbreed it with the following idea. First, consider polynomials of two variables, say, x and y. For every k we have

$$x^k - (x-1)(x^{k-1}+x^{k-2}+\dots+1) = 1. \tag{52}$$

Let us multiply this equality by y^ℓ. We will have

$$x^k y^\ell - (x-1)(x^{k-1}+x^{k-2}+\dots+1)y^\ell = y^\ell.$$

Now let us take a polynomial of two variables, $p(x,y)$, apply the operator $p(d_x x, d_y y)$ to both sides of this equality, and use the above observations:

$$p(k+1,\ell+1)x^k y^\ell - p(d_x x, d_y y) * ((x-1)(x^{k-1}+x^{k-2}+\ldots+1)y^\ell) = p(1,\ell+1)y^\ell.$$

Now suppose that $p(1,\ell+1)$ is never equal to 0 (we can guarantee this by replacing $p(x,y)$ by $p(x,y+c)$ for a big enough constant c: a polynomial of one variable cannot have too many roots). Then the differential equation

$$p(d_x x, d_y y) * ((x-1)f(x,y)) = y^\ell \tag{53}$$

will have a solution if the Diophantine equation

$$p(x+1,\ell+1) = 0 \tag{54}$$

has a solution. Indeed, $f(x,y) = \frac{(x^{k-1}+x^{k-2}+\ldots+1)y^\ell}{p(1,\ell+1)}$ will be a solution.

The important thing is that the converse implication holds too: The Diophantine equation (54) is solvable if the differential equation (53) is solvable. Indeed, suppose $f(x,y)$ is a solution of the differential equation. Since the operator $p(d_x x, d_y y)$ does not change the exponents of monomials, the monomial y^ℓ must occur among the monomials of $(x-1)f(x,y)$. Therefore it must occur among the monomials of $f(x,y)$. Then the monomial xy^ℓ occurs among the monomials of $(x-1)f(x,y)$. Let us take the lexicographically leading monomial $x^k y^\ell$ of $(x-1)f(x,y)$. We know that $k>0$. So this monomial cannot cancel with the monomial y^ℓ from the right hand side of equality (53). Therefore the operator $p(d_x x, d_y y)$ must kill this monomial. Therefore $p(k+1,\ell+1)=0$ and $(k+1,\ell)$ is a solution of (54).

Thus the solvability of the differential equation (53) is equivalent to the solvability of the Diophantine equation (54). This is almost exactly what we need. In truth, we need polynomials of more than 2 variables (because the problem of solvability of Diophantine equations with 1 unknown is obviously decidable). Fortunately, we can extend our arguments to polynomials with an arbitrary number of variables. Instead of equality (52) one can use the fact that every monomial $x_1^{k_1}\cdots x_n^{k_n}$ may be uniquely represented in the following form

$$x_1^{k_1}\cdots x_n^{k_n} = (x_1-1)(x_2-1)\cdots(x_n-1)s + \sum_{1\le i\ne j\le n} x_i t_{i,j} + t_0 \tag{55}$$

where $s, t_{i,j}$ are polynomials, $t_{i,j}$ does not contain variable x_j, t_0 is an element from the field K. It is possible to prove that $t_0 = (-1)^{n+1}$ (this is a nice exercise in proofs by induction for high school students; the base of the induction, $n=1$, has been considered above).

Now we can multiply this equality by y^ℓ, where y is a new variable, and apply the operator $p(d_1 x_1, \ldots, d_n x_n, d_y y)$. As above, we can assume that the polynomial

$p(1, 1, \ldots, 1, \ell + 1)$ does not have integer roots. Notice that the sum

$$\sum_{1 \leq i \neq j \leq n} p(d_1 x_1, \ldots, d_n x_n, d_y y) * x_i t_{i,j}$$

may be expressed in the form

$$\sum_{1 \leq i \neq j \leq n} x_i g_{i,j},$$

where each polynomial $g_{i,j}$ does not contain the variable x_j. Notice also that the fact that $g_{i,j}$ does not contain the variable x_j may be expressed by the equality $d_j * g_{i,j} = 0$.

Therefore, for every natural number ℓ, the solvability of the following Diophantine equation

$$p(k_1 + 1, \ldots, k_n + 1, \ell + 1) = 0 \tag{56}$$

implies the solvability of the system of differential equations:

$$\begin{cases} p(d_1 x_1, \ldots, d_n x_n, d_y y) * (x_1 - 1)(x_2 - 1) \cdots (x_n - 1)s + \sum_{1 \leq i \neq j \leq n} x_i g_{i,j} = y^\ell \\ d_i * g_{i,j} = 0 (1 \leq i \neq j \leq n). \end{cases} \tag{57}$$

The converse implication also holds. This may be proved by almost the same argument as above. Therefore, if we take the Matiyasevich polynomial $p(x_1, \ldots, x_{10}) - \ell$ with non-recursive set $\{\ell \mid \text{ Equation (50) is solvable}\}$, then the corresponding system (57) will give us the desired undecidability of the Differential Equations Problem.

7.6.3 An Interpretation of Systems of Differential Equations

Let us return to sums of modules over different subrings of a "big" ring (see Section 7.6.1), and show how, given a system of differential equations, we can construct such a sum with undecidable membership problem.

As we have mentioned before, it is not clear that we will get every system of linear differential equations with polynomial coefficients, when we consider membership problems for sums of modules. Thus, before we construct the modules, we will simplify our system of differential equations (57).

We can do with system (57) what specialists in differential equations usually do when they want to reduce the degree of a system of differential equations by increasing the number of variables. For example, if we have an equation $d_x d_y * f = r$ we can introduce a new variable f_1 and replace this equation by the following system:

$$\begin{cases} d_y * f - f_1 = 0 \\ d_x * f_1 = r. \end{cases}$$

Of course, we won't change the solvability/unsolvability of the system by these transformations. After these and similar transformations we can get a system $D * \vec{f} = \vec{r}$ where the matrix D satisfies the following conditions.

1. every entry of D is of the form d_i, x_i (variable) or an element of the field $\mathcal{F}$ (constant);

2. every row of D contains at most one entry of type d_i;

3. every column of D contains at most one entry of type d_i;

4. no d_i appears in the top row.

Recall that we are dealing with a commutative algebra K over the field $\mathcal{F}$, given by generators $\{a_1, \ldots, a_e, b_1, \ldots, b_j\}$ and relations $(a_i - b_i)(a_j - b_j) = 0$. We know that this algebra is spanned by monomials in the b's, and monomials in the b's multiplied by $(a_i - b_i)$, $i = 0, 1, \ldots, N$.

Given the matrix D over W_n, satisfying conditions 1, 2, 3 above, we construct two subspaces M_1 and M_2 of a free K-module M such that M_1 is a $< a_1, \ldots, a_e >$-submodule of M, M_2 is a $< b_1, \ldots, b_e >$-submodule of $M =< q_1, q_2, \ldots >$ and the membership problem for $M_1 + M_2$ is equivalent to the system of differential equations $D * \vec{f} = \vec{r}$.

To show that it can be done (and to avoid routine calculations) consider the following special case: $N = 2$; so K is generated by $\{a_1, a_2, b_1, b_2\}$, and the matrix D is the following

$$D = \begin{pmatrix} 1 & x_2 \\ d_1 & 2 \end{pmatrix}$$

Clearly, D satisfies all three conditions formulated above.

The module M will be generated by 2 elements q_1, q_2. The submodule M_1 will be generated by two elements corresponding to the columns of matrix D: $q_1 + q_2, q_1 b_2 + 2q_2(a_1 - b_1)$

The submodule M_2 will be generated by four elements corresponding to the rows of D, two elements for each row: $q_1(a_1 - b_1), q_1(a_2 - b_2), q_2, q_2(a_2 - b_2)$.

You can see that different rows are treated differently. Everything depends on whether or not there is a d_i in the row.

Now for every vector $\vec{r} = (r_1, r_2)$ let us consider the element $m = q_1 r_1(b_1, b_2) + q_2(a_1 - b_1) r_2(b_1, b_2)$. We claim that m belongs to $M_1 + M_2$ if and only if the system $D * \vec{f} = \vec{r}$ has a solution.

Indeed, the fact that m belongs to $M_1 + M_2$ is equivalent to the existence of six polynomials $f_1(a_1, a_2), f_2(a_1, a_2), g_1(b_1, b_2), g_2(b_1, b_2), g_3(b_1, b_2), g_4(b_1, b_2)$ such that

$$\begin{aligned}
&(q_1 + q_2) f_1 + (q_1 b_2 + 2q_2(a_1 - b_1)) f_2 + \\
&q_1(a_1 - b_1) g_1 + q_1(a_2 - b_2) g_2 + q_2 g_3 + q_2(a_2 - b_2) g_4 = \\
&q_1 r_1(b_1, b_2) + q_2(a_1 - b_1) r_2(b_1, b_2).
\end{aligned}$$

Now, using the "Taylor expansion" (46) and the fact that the elements $q_i, q_i(a_j - b_j)$ form a basis of M considered as a $< b_1, b_2 >$-module, we can rewrite this equality in the following coordinate form:

$$\begin{cases} f_1 + b_2 f_2 = r_1 \\ \frac{\partial f_1}{\partial b_1} + 2 f_2 = r_2 \\ u_1 + g_1 = 0 \\ u_2 + g_2 = 0 \\ u_3 + g_3 = 0 \\ u_4 + g_4 = 0 \end{cases}$$

where u_i is an expression built up of polynomials f_j and their derivatives.

Since u_i does not depend on g_j, this system has a solution if and only if the subsystem of the first two equations has a solution. Finally, notice that this subsystem differs from our system $D * \vec{f} = \vec{r}$ only in the names of its variables (b instead of x).

One can see that the submodules M_1 and M_2 play different roles in this construction. The first module simulates the system of differential equations while the second module is a "garbage collector": it neutralizes the by-products of the work of M_1.

7.6.4 A Representation of Sums of Two Submodules in a Finitely Presented Associative Algebra

Sums of two modules over different rings may be interpreted in groups [185], Lie algebras [353], and in associative algebras [343]. All the interpretations are based on the same ideas. We choose associative algebras because, first of all, in this case the interpretation is more apparent, and second, it gives an immediate strong result — a minimal variety of associative algebras with undecidable word problem.

As in the previous subsections, let K be a commutative ring generated by elements $\{a_1, \ldots, a_e, b_1, \ldots, b_e\}$ subject to relations $(a_i - b_i)(a_j - b_j) = 0$. Let $M =< q_1, \ldots, q_N >$ be a free K-module, $M_1 =< t_1, \ldots, t_k >$ be a $< a_1, \ldots, a_e >$-submodule of M, $M_2 =< s_1, \ldots, s_n >$ be a $< b_1, \ldots, b_e >$-submodule of M.

We will build the associative algebra $S = S(M, M_1, M_2)$ step-by-step. First let S be generated by $\{q_1, \ldots, q_N, a_1, \ldots, a_e\}$. As usual we will have relations simulating the object we are interpreting (the sum $M_1 + M_2$) and auxiliary relations.

The auxiliary relations will make the subalgebra $K_1 =< a_1, \ldots, a_e >$ commutative, and the subspace V, spanned by words uq_iv where $u, v \in K_1$, a module over K. The first of these two tasks is easy: the relations

$$a_i a_j = a_j a_i$$

will do the job.

To make V into a K-module we need the relations

$$[q_i, a_j, a_\ell] = 0$$

for every triple i, j, ℓ.

We can define a K-module structure on the subspace V. For every word uq_iv from V and every $j = 1, 2, \dots, e$ let

$$uq_iv \circ a_i = ua_jq_iv,$$
$$uq_iv \circ b_i = uq_ib_jv.$$

The relations $[q_i, a_j, a_\ell] = 0$ guarantee us that, $uq_iv \circ (a_j - b_j)(a_\ell - b_\ell) = 0$. Therefore V is a (free) K-module. It is generated by the elements q_i, $i = 1, \dots, N$. It is easy to see that V is K-isomorphic to M.

Now we can easily define subspaces V_1 and V_2 of V which correspond to submodules M_1 and M_2.

Let $t_i = \sum_j q_jf_{ij}$, $s_i = \sum_j q_jg_{i,j}$. Then the subspace V_1 is spanned by all elements $u(\sum_j q_j \circ f_{ij})$, $u \in < a_1, \dots, a_e >$ and the subspace V_2 is spanned by all elements $(\sum_j q_j \circ g_{ij})u$.

Now, if V_1 and V_2 were finitely generated ideals we could take their generators as defining relations of S, and then (rightfully) claim that the word problem in S is equivalent to the membership problem for $M_1 + M_2$ in M.

Of course, neither V_1 nor V_2 is an ideal. We can make V_1 into a left ideal and V_2 into a right ideal by adding some relations which kill extra words. As in Section 7.2.4 let us kill all 2-letter words which are not subwords of words uq_jv, where $u, v \in < a_1, \dots, a_e >$. This means that for every such 2-letter word w we add a relation

$$w = 0.$$

From now on let us assume that S satisfies these relations. Then V_1 is a left ideal, V_2 is a right ideal of S. Moreover, it is easy to see that both of them are finitely generated. But we need two-sided ideals!

To make these one-sided ideals into two-sided ideals let us add two new (but very familiar looking!) generators A and B.

Using these generators we will change the definitions of V, V_1, V_2 as follows.

V is spanned by all words Auq_ivB,

V_1 is spanned by $u(\sum_j q_j \circ f_{ij})B$ and $Au(\sum_j q_j \circ f_{ij})B$,

V_2 is spanned by $A(\sum_j q_j \circ g_{ij})u$ and $A(\sum_j q_j \circ g_{ij})u$ where $u, v \in < a_1, \dots, a_e >$.

We also add new killing relations

$$xA = Bx = 0$$

where x is any generator.

Now V_1 and V_2 are both two-sided finitely generated ideals. New generators A and B play the role of locks. The letter A protects a word from the left, the letter B protects it from the right.

In truth, V_1 and V_2 are no longer subspaces of V. This is the bad news. But we have some good news also:

1. V is isomorphic to M as an Abelian group;

2. $V_1 \cap V$ is isomorphic to M_1, $V_2 \cap V$ is isomorphic to M_2, and $(V_1 + V_2) \cap V$ is isomorphic to $M_1 + M_2$ under the same isomorphism.

As a result, if the membership problem for $M_1 + M_2$ is undecidable, then the membership problem for $(V_1 + V_2) \cap V$ is undecidable, whence the (even harder) membership problem for the ideal $V_1 + V_2$ is undecidable. Therefore, if we take the finite set of generators of $V_1 + V_2$ as relators of S, we get an algebra with an undecidable word problem.

The auxiliary relations (killing relations and commutativity relations) make this algebra satisfy the identities from Theorem 4.7 (see [343]). Therefore it generates a minimal variety of associative algebras with an undecidable word problem. Since S is (absolutely) finitely presented, this variety has a strongly undecidable word problem.

7.6.5 Why Not Minsky Machines?

So, why can't we get as deep in the lattice of varieties of associative algebras (Lie algebras or groups) with Minsky machines, as we can with Systems of Differential Equations? Of course, it is difficult to give a precise answer to such a question. The concept of an interpretation of a Minsky machine (or a general Turing machine for that matter) is not rigorous enough to let us even raise the problem of finding a proof that Minsky machines are not interpretable in some variety of algebras. Nevertheless, we can try to explain why we feel that Minsky machines cannot be simulated in some varieties in which the Differential Equations problem can be interpreted.

Consider, for example the variety of associative algebras given by the following identity:

$$[x_1, x_2][y_1, y_2, y_3][z_1, z_2] = 0$$

The algebra constructed in Section 7.6.4 satisfies this identity, so the word problem is not solvable in this variety.

Now let us return to Sections 7.2.2 and 7.2.3, and recall that any interpretation of a Minsky machine begins with choosing canonical words. A canonical word $w(m, q_i, n)$ must contain a letter q_i which represents the head, subwords u_m and v_n, which simulate numbers m and n, and locks A and B which represent the ends of the tapes. The letters of u_m cannot seep through q_i or A, letters of v_n cannot seep through q_i or B, otherwise the whole interpretation collapses (see Section 7.2.3).

Recall the following equalities: $xy = [x,y]+yx$, $xyz = [x,y]z + yxz = [x,y,z] + z[x,y] + yxz = [x,y,z] + zxy - zyx + yxz$. Therefore if $[x,y] = 0$ then y can seep through x, if $[x,y,z] = 0$ then y or z can seep through x.

But we want to keep q_i, A, B, u_m, v_n in the same word and to avoid the leaks altogether[37]. This means that a product of three commutators, which can have arbi-

[37] This is reminiscent of the problem of keeping water in a can with many holes.

trary lengths (because u_m, v_n can have arbitrary lengths), is not 0. This contradicts the identity $[x_1, x_2][y_1, y_2, y_3][z_1, z_2] = 0$.

Don't ask us for more satisfactory explanations! We don't have them anyway.

7.7 The Undecidability of the Identity Problem

Recall that the identity problem for a class K of algebras is the following:

> *Find an algorithm which, given a finite set of identities $\Sigma = \{u_i = v_i \mid i \in I\}$ and an identity $u = v$, determines whether or not Σ implies $u = v$ in K.*

If K is a finitely based variety then the decidability of the identity problem is equivalent to the uniform decidability of the word problem for every relatively free algebra in every subvariety of K, which, in turn, is equivalent to the decidability of the problem of whether two finitely based subvarieties of K coincide.

We present here the proofs of the undecidability of the identity problem in the classes of all semigroups (Murskii [276]), all finite semigroups (Albert, Baldinger, Rhodes [9]), and all groups (Kleiman [197]). As was mentioned in the Introduction, the question of whether the identity problem is decidable for the class of all Lie algebras (over a good field), or for the class of all finite groups, is still open.

7.7.1 Semigroups

We have already mentioned the theorem of Murskii that the identity problem in the class of all semigroups is undecidable (see Murskii [276]). Here we are going to present the ideas of the proof of this result.

Notice first of all that the word problem in a variety $\mathcal{K}$ may be formulated in the following form:

> *Given a set of relations $\Sigma = \{u_i = v_i \mid i \in I\}$, find an algorithm which, given a relation $u = v$, determines whether or not Σ implies $u = v$ in $\mathcal{K}$.*

This looks very similar to the identity problem, though the relations are not identities, of course.

But, as a first step we can naively take the set of relations Σ of a finitely presented semigroup with an undecidable word problem, say S_2 from Section 7.2.4, and consider these relations as identities. We would win if we had the following property: For any canonical word $w(m, q_i, n)$ an identity $u = v$ is applicable to this word if and only if the relation $u = v$ is applicable to this word, and there is at most one way to apply this identity to the canonical word.

Of course, this is not the case. Indeed suppose that we have a relation $Aq_1B = Aq_2bB$. If we consider this formula as an identity then we can make substitutions (apply endomorphisms of the free semigroup). For example, we can make the following substitution: $A \to a, B \to b$ and apply this identity to the word Aaq_1bB.

So the problem is that we have too many endomorphisms and that the relations of S_2 are too flexible: they can be transformed into each other by some endomorphisms.

The solution suggested by V.L.Murskii is simple but very effective. Let us replace the letters in the interpretation of a Minsky machine[38] by words: $x \to \bar{x}$.

Then it would be harder to apply a "wrong" identity to a canonical word (where we also replaced all letters x by words $\bar{x}$).

For example, if $\bar{q}_1 = pq^9p^2,\ \bar{q}_2 = pq^{10}p^2,\ \bar{a} = pq^{11}p^2,\ \bar{b} = pq^{12}p^2,\ \bar{A} = pq^{13}p^2,\ \bar{B} = pq^{14}p^2$ then the identity $\bar{A}\bar{q}_1\bar{B} = \bar{A}\bar{q}_1\bar{b}\bar{B}$ is not applicable to the word $\bar{w}(1, q_1, 1) = \bar{A}\bar{a}\bar{q}_1\bar{b}\bar{B}$.

Indeed, the words $W_{k,i} = pq^{k+i}p^2,\ k \geq 3, 1 \leq i \leq k$ have the following two important properties (see Lemma 1 in Murskii [276]):

M1. No word of the form

$$W = W_{k,i_1} W_{k,i_2} \dots W_{k,i_n}, \quad n \geq 1 \tag{58}$$

contains occurrences of words $W_{k,i}$ distinct from the occurrences explicitly designated in (58);

M2. If a word W of the form (58) contains $\phi(W_{k,i}W_{k,j})$ as a subword, where ϕ is any substitution, then either $\phi(p) = p, \phi(q) = q$ or the sequence $i_1, i_2, \dots, i_n$ is not cube free.

The second condition in the property M2 is essential. Indeed if a word W of the form (58) contains a power, say $W_{k,1}^m$, and $m > |W_{k,i}W_{k,j}|$ then $\phi(W_{k,i}W_{k,j})$, where $\phi(p) = \phi(q) = W_{k,1}$, is a subword of W, and $\phi(p) \neq p,\ \phi(q) \neq q$.

This argument, by the way, shows that we cannot use interpretations of Minsky machines in semigroups S_1 or S_2, because the canonical words in these interpretations contain arbitrary big powers.

But we can use semigroups $S(M, \phi)$ from Section 7.2.5. Indeed, the interpretation presented there is cube free.

Finally, let us put the details of the construction together.

Take the interpretation of a Minsky machine M in the semigroup $S(M, \phi)$ (see Section 7.2.5). Let $t_1, \dots, t_k$ be the set of generators of $S(M, \phi)$, and let Σ be the set of defining relations of $S(M, \phi)$. Replace every letter t_i in these relations by the word $W_{k,i}$. Denote the resulting set of equalities by $\bar{\Sigma}$, and consider these equalities as (semigroup) identities. Let $w(m, q_i, n)$ be a canonical word in $S(M, \phi)$. Apply the above substitution to this word and denote the resulting word by $\bar{w}(m, q_i, n)$. Then properties M1 and M2 guarantee that

[38] Murskii uses an arbitrary Turing machine. We will use a Minsky machine for the sake of simplicity.

The identity $\bar{w}(m, q_i, n) = \bar{w}(m', q_{i'}, n')$ follows from the identities of $\bar{\Sigma}$ if and only if the relation $w(m, q_i, n) = w(m', q_{i'}, n')$ follows from the relations of Σ.

Now the undecidability of the identity problem follows from the undecidability of the word problem in $S(M, \phi)$.

To prove the more general Theorem 3.16, one has to consider any finitely based variety of semigroups $\mathcal{V}$ with non-locally finite nil-semigroups (Theorem 3.7 contains a description of such varieties) instead of the variety of all semigroups. Then one has to find words $W_{i,k}(\mathcal{V})$ which satisfy the property M1 and the property M2, where the words "square free" are replaced by "isoterm for identities of $\mathcal{V}$" (recall that a word w is an isoterm for identity $u = v$ if w cannot be changed by applying this identity). Then, instead of the semigroup $S(M, \phi)$, one has to take the semigroups from Section 3 of the paper Sapir [335]. The words $W_{k,i}(\mathcal{V})$ were found in Sapir [340].

7.7.2 Finite Semigroups

To prove the undecidability of the identity problem in the class of finite semigroups (see Albert, Baldinger, and Rhodes [9]), one needs only to combine the ideas of Sections 7.7.1 and 7.4.2. Let us define a "graded" analogue of the semigroup $S(M, \phi)$ by adding special elements c_1, c_2. For the grading parameter, instead of the power of c as in Section 7.4.2, we will use the power of an endomorphism ϕ or, more precisely, the word $\phi^n(c_1)$. The interpretation of each step of the Minsky machine will be accompanied by increasing the power of ϕ in $\phi^n(c_1)$. Let, as in Section 7.4.2, the Minsky machine calculate a function g_f where the function f has two values, 1 and 2. Using our "graded" interpretation and the ideas of Section 7.4.2, one can construct a system of relations Σ and canonical words $w(m, q_i, n)$ with the following properties:

1. If $f(n) = 1$ then the relation $w(2^n, q_1, 0) = w(2, q_0, 0)$ follows from Σ;

2. If $f(n) = 2$ then the relations $w(2^n, q_1, 0) = w(2, q_0, 0)$ fail in some finite homomorphic image of the semigroup given by the relations Σ;

3. All words $w(m, q_i, n)\phi^n(c_1)$ are cube free.

Now if we define the substitution $x \to \bar{x}$ as in Section 7.7.1 (using the same words $W_{k,i}$), then we will get a system of identities $\bar{\Sigma}$ with the following two properties:

1. If $f(n) = 1$ then the identity $\bar{w}(2^n, q_1, 0) = \bar{w}(2, q_0, 0)$ follows from $\bar{\Sigma}$;

2. If $f(n) = 2$ then the identity $\bar{w}(2^n, q_1, 0) = \bar{w}(2, q_0, 0)$ fails in some finite homomorphic image of the relatively free semigroup given by the identities from $\bar{\Sigma}$.

This implies the undecidability of the identity problem in the class of finite semigroups.

7.7.3 Groups

Here we present the main ideas of Kleiman's proof of the undecidability of the identity problem for groups (see Kleiman [197]).

First we can reformulate the identity problem for a variety of groups. Let $\mathcal{V}$ be a group variety. The identity problem reads as follows:

> *Find an algorithm which, given a finite set of identities* $\Sigma = \{u_i = 1 \mid i \in I\}$ *and an identity* $t = 1$*, determines whether or not* Σ *implies* $t = 1$ *in* $\mathcal{V}$.

Since every finite set of group identities is equivalent to one identity, the set Σ in this definition may be replaced by one identity $s = 1$. Therefore we can rewrite our problem as follows:

> *Find an algorithm which, given two identities* $s = 1$ *and* $t = 1$*, determines whether or not* $s = 1$ *implies* $t = 1$ *in* $\mathcal{V}$.

Now let R be a relatively free group of $\mathcal{V}$ with countably many generators. Then every word may be considered as an element of R. If t is a word then $t(R)$ denotes the verbal subgroup generated by t, i.e. the minimal normal subgroup such that $R/t(R)$ satisfies the identity $t = 1$. Then every countable group G from $\mathcal{V}$ which satisfies this identity is a factor group R/H with $H > t(R)$. Therefore $s = 1$ implies $t = 1$ in $\mathcal{V}$ if and only if $t(R) \leq s(R)$. Hence we can reformulate our problem for a variety $\mathcal{V}$ as follows:

> *Find an algorithm which, given two identities* $s = 1$ *and* $t = 1$*, determines whether or not* $s(R)$ *is contained in* $t(R)$*, where* R *is the relatively free group in countably many generators of* $\mathcal{V}$.

Now we will show that there exists a finitely based variety of groups $\mathcal{V}$ where this problem is undecidable. From this, it is easy to deduce that the problem is undecidable for the variety of all groups. Indeed, suppose that the identity problem is undecidable in a finitely based variety $\mathcal{V}$. Take an identity $\bar{s} = 1$ which defines the intersection of $\mathcal{V}$ and the variety given by $s = 1$. Then the problem "Given an identity $t = 1$, find out if $\bar{s} = 1$ implies $t = 1$ in the class of all groups" is equivalent to the problem "Given an identity $t = 1$, find out if $s = 1$ implies $t = 1$ in $\mathcal{V}$". Since the last problem is undecidable by the assumption, the first problem is undecidable also, which implies the undecidability of the identity problem in the class of all groups.

For any relatively free group R, the verbal subgroup $s(R)$ may be constructed in two steps [279]. First we construct the set of all images of s under endomorphisms of R. (This set is usually called *a verbal subset* and is denoted by $s[R]$. For example, if $s = (x, y)$ then $s[R]$ is the set of all commutators of elements of R.) Then the verbal subgroup $s(R)$ coincides with the subgroup generated by $s[R]$.

Thus if we replace round brackets by square brackets, i.e. if we consider verbal subsets instead of verbal subgroups, we will simplify the problem.

It is clear that $t[R] < s[R]$ if and only if t is an endomorphic image of s. We will say that s *covers* t in R if t is an endomorphic image of s in R. Thus we have the following problem:

Given elements s and t in R find out if s covers t.

It turns out that it is easier to solve a harder problem. Namely we shall find a relatively free group R, given by finitely many identities, and an element s in R such that the following problem is undecidable:

Given an element t in R find out if s covers t.

It is easy to see why this problem is harder: it has fewer degrees of freedom, in fact only one. In general, systems with more degrees of freedom are more likely to be undecidable.

This problem for arbitrary (not necessarily relatively free) groups is known as the *problem of endomorphic reducibility.* Roman'kov [323] proved that this problem is undecidable even for a nilpotent group.

But Kleiman cannot use this result because he needs a relatively free group, and Roman'kov's group is not relatively free. So he modifies Roman'kov's method.

Just as Roman'kov, he interprets Diophantine equations and uses the undecidability of Hilbert's 10th problem (Matiyasevich [252]). Following Roman'kov [323], Kleiman formulates Hilbert's 10th problem in the following way.

Let $p(x_1, \ldots, x_n), q(x_1, \ldots, x_n)$ be polynomials with integer coefficients. We say that p *covers* q if $f(p) = q$ for some linear transformation

$$f : x_i \to \sum \alpha_{ij} x_j + \beta_j \tag{59}$$

where α_{ij}, β_j are integers. It is easy to see that p covers an integer k if and only if the equation $p(x_1, \ldots, x_n) = k$ has an integer solution. Indeed, if $f(p) = k$ then $p(\beta_1, \ldots, \beta_n) = k$. Therefore, the Theorem of Matiyasevich may be formulated as follows:

There exists a polynomial p such that the set of integers covered by p is not recursive.

Now let us understand what it means "to translate Hilbert's 10th problem into the problem of endomorphic reducibility in a group R". An intuition gained from working with interpretations of various other things (Minsky machines, recursive functions, etc.), tells us that we need two correspondences:

- a mapping $p \to g_p$ between polynomials of given degree and given number of variables [39], and elements of R,
- a mapping $f \to \phi_f$ between linear transformations (59) f and endomorphisms of R.

These two mappings must satisfy the following "functorial" conditions:

1. If $f(p) = q$ then $\phi_f(g_p) = g_q$, and
2. if $\phi(g_p) = g_q$ for some endomorphism ϕ then for some linear transformation f we have $f(p) = q$.

To illustrate the idea of such an interpretation, let us consider polynomials of the form $ax^2 + bx + c$.

To simulate polynomials, we basically need to simulate the addition and the multiplication of numbers by two derived (defined by some words) group operations. Of course, these operations must satisfy the laws of the usual addition and multiplication of numbers. In particular a) addition must be commutative and b) multiplication must be distributive with respect to addition. In order to solve problem a) we will interpret addition by the usual group multiplication, but we will try to use this operation only inside a special Abelian subgroup.

In order to solve problem b) we can use the following well known fact: in any nilpotent group of degree k the k-ary operation $(x_1, x_2, \ldots, x_k)$ is distributive in every variable (see [136]).

Consider, for example, the following construction. Let R be the free nilpotent group of degree 4 with 3 generators x, y, z. For every four elements u, v, w, t in R the commutator (u, v, w, t) belongs to the center $Z(R)$ (this will be the special Abelian subgroup we were talking about) and this 4-ary operation is distributive in every variable, i.e., for example, $(u, v, ww', z) = (u, v, w, z)(u, v, w', z)$.

For every polynomial $p = ax^2 + bx + c$ let us consider the following word

$$g_p = (z, y, x, x)^a (z, y, y, x)^b (z, y, y, y)^c.$$

For every linear transformation $f : x \to \alpha x + \beta$ we can define an endomorphism ϕ_f of R by

$$\phi_f(x) = x^\alpha y^\beta, \phi_f(y) = y, \phi_f(z) = z.$$

Then it is easy to verify that if $f(p) = q$ then $\phi_f(g_p) = g_q$. So we have property 1.

Unfortunately we do not have property 2: we cannot say that if g_p covers g_q then p covers q. Indeed, the free nilpotent group R has too many endomorphisms and our

[39] Actually we need to simulate only one particular polynomial — the Matijasevich polynomial, but it is easier to interpret the whole set of polynomials of given degree and given number of variables.

construction is too unprotected against "uninvited" endomorphisms. For example, consider an endomorphism ψ given by the following rules:

$$\psi : x \to x, y \to y^{-1}, z \to z.$$

Let $p(x) = x^2$. Then $g_p = (z, y, x, x)$. Hence $\psi(g_p) = (z, y^{-1}, x, x) = (z, y, x, x)^{-1}$. Notice that $(z, y, x, x)^{-1} = g_q$ where $q(x) = -x^2$. But it is obvious that the polynomial $p(x) = x^2$ does not cover the polynomial $q(x) = -x^2$. So the word g_p covers the word g_q while the polynomial p does not cover the polynomial q.

To get rid of "uninvited" endomorphisms we need a security system. A simple idea is to multiply the words which we want to protect by a special factor ℓ (a "lock") such that if an endomorphism takes ℓ to ℓ then it fixes z and y. Using elementary calculations with commutators (see H. Neumann [279]) and a little bit of linear algebra it is easy to find such "locks". For example, the word $(z, y, z, z)(z, y, y, z)^3$ will work just fine. So it is enough to change the definition of g_p as follows:

$$g_p = (z, y, x, x)^a (z, y, y, x)^b (z, y, y, y)^c (z, y, z, z)(z, y, y, z)^3.$$

(Recall that $p(x) = ax^2 + bx + c$.)

Of course, Kleiman has to interpret polynomials of greater degrees with many variables, so his construction is much more complicated, but the main ideas are the same. The concrete form of this interpretation is not significant for us. So from now on it would be enough to assume that there exists a relatively free nilpotent group R and an interpretation $(p \to g_p, f \to \phi_f)$ of polynomials of sufficiently big degree and sufficiently many variables in R which has properties 1 and 2 indicated above.

Now we can return to round brackets, i.e. to verbal subgroups. To proceed from the square brackets to round brackets Kleiman invents the following very neat group-theoretic construction. Later, the same construction allowed him to solve many other important problems about group varieties.

Assume that A is a relatively free group, $A > N$ and N is an Abelian verbal (that is stable under endomorphisms) subgroup of exponent 2. Assume also that the wreath product of the cyclic group of order 2 and A/N belongs to the variety generated by A. Select a nonempty set of nonunit elements $M \subset N$ such that $M^A = M$ (that is M is stable under conjugations by elements of A). Denote the image of an element a in the factor group A/N by $\bar{a}$. Using the group A, let us construct the group B whose generators are all symbols of the form b^g and c^s, for arbitrary $g \in A$, $s \in A/N$, and whose defining relations have the form

$$\begin{aligned}
&(b^g)^2 = (c^s)^2 = 1\\
&(b^g, b^h) = 1 \text{ if } gh^{-1} \notin M,\\
&(b^g, b^h) = c^{\bar{g}} \text{ if } gh^{-1} \in M,\\
&(b^g, c^s) = 1
\end{aligned}$$

for all $g, h \in A$, and $s \in A/N$.

The third relation may make you suspicious, because it seems that we prefer the left factor of the commutator to the right one. This is strange because the group commutator is an anti-commutative operation in general, and a commutative operation in our case since every value of this operation is an involution (by virtue of the first relation). So it looks like we will identify different elements c^s. But in fact everything is all right. Indeed, note that if $gh^{-1} \in M$ then $\bar{g} = \bar{h}$ because $M \subset N$. Therefore

$$(b^g, b^h) = c^{\bar{g}} = c^{\bar{h}} = (c^{\bar{h}})^{-1} = (b^h, b^g).$$

It is easy to see that B belongs to the variety of nilpotent groups of class 2 and exponent 4, i.e. $B \in \mathcal{A}_2^2 \cap \mathcal{N}_2$. There is a natural action of A on B:

$$(b^g)^h = b^{gh}, (c^s)^h = c^{s\bar{h}},$$

where $g, h \in A$, $s \in A/N$. Let us denote the semidirect product of B by A, relative to this action, by $C(A, N, M)$.[40]

The following lemma contains the two main properties of the group $C(A, N, M)$.

Lemma 7.7 *1. The following equivalence holds for an arbitrary word $w \in N$: $w^4 = 1$ is an identity in the group $C(A, N, M)$ if and only if $w[A] \cap M = \emptyset$.*

2. If the identity $u = v$ holds in A then the identity $u^4 = v^4$ holds in $C(A, N, M)$.

Suppose now that the problem of endomorphic reducibility is not decidable in the group A for a word s from N, i.e. that there is no algorithm which tells us if an element t from N is covered by s. Then for every nonunit element t from N we can take $M_t = \{t^g \mid g \in A\}$. Now $s[A] \cap M_t = \emptyset$ if and only if $t \notin s[A]$, that is if and only if s does not cover t.

Therefore, by the first statement of Lemma 7.7, we have that $s^4 = 1$ is an identity in $C(A, N, M_t)$ if and only if s *does not cover* t in A.

Now consider the variety $\mathcal{C}$ generated by all groups $C(A, N, M_t)$.

Suppose that s does not cover t in A. Then $s^4 = 1$ is an identity in $C(A, N, M_t)$, but $t^4 = 1$ is not an identity of this group (since t obviously covers t). Therefore if s does not cover t in A then the identity $s^4 = 1$ does not imply the identity $t^4 = 1$ in $\mathcal{C}$.

The converse statement is also true. Indeed, suppose s covers t in A. Then $\phi(s) = t$ in A for some endomorphism ϕ of A. Since A is relatively free, we can deduce that $\phi(s) = t$ is an identity in A. Then by the second statement of Lemma 7.7, $\phi(s)^4 = t^4$ is an identity in every group $C(A, N, M)$. Therefore $\phi(s^4) = t^4$ is an identity in $\mathcal{C}$. Hence the identity $s^4 = 1$ implies the identity $t^4 = 1$ in $\mathcal{C}$.

[40] One can notice a connection between this construction and the construction of group $\hat{G}$ in Section 7.2.8. There we used a free Abelian group with generators indexed by elements of a semigroup, and then defined a semidirect product using a "natural" action. In some sense, both constructions are variations on the theme of the wreath product. Another variation was used in Ol'shanskii's work on the finite basis property in group varieties [287].

Thus there is no algorithm which can decide if the identity $s^4 = 1$ implies the identity $t^4 = 1$ in $\mathcal{C}$. This means that the variety $\mathcal{C}$ has undecidable identity problem.

We could apply these arguments for $A = R$ and get the undecidability of the identity problem for groups if

1) R had a verbal Abelian subgroup N of exponent 2, the wreath product of the cyclic group of order 2 and R/N belonged to $varR$, the problem of endomorphic reducibility was undecidable for elements of this normal subgroup, and

2) the variety $\mathcal{C}$ was finitely based.

Unfortunately the first of these statements is false and the second one is difficult to prove. Even the fact that $\mathcal{C}$ is actually generated by a single group $C(A, N, M_t)$ does not help much: in general the question of whether the identities of a group are finitely based is a very non-trivial one.

Therefore we need to replace R by a somewhat different relatively free (non-nilpotent) group and we need to replace $\mathcal{C}$ by a finitely based variety.

The solution to the first problem is almost straightforward. We can consider the Mal'cev product $\mathcal{V}$ of the Abelian variety of exponent 2 and the variety generated by R. Relatively free groups of this variety are semidirect products of Abelian groups of exponent 2 and relatively free groups from $varR$. Let A be the relatively free group from $\mathcal{V}$ with the same number of generators as R, let N be the verbal subgroup of A corresponding to the variety $varR$. Then A is a semidirect product of N and R. One can pick one element u_0 in N and for every element w consider the commutator (u_0, w), which also belongs to N. Using the Shmel'kin embedding of A into the wreath product of N and R [374], Kleiman shows that it is possible to choose u_0 in such a way that the following property holds in A:

(u_0, g_q) is an endomorphic image of (u_0, g_p) in A if and only if g_q is an endomorphic image of g_p in R.

Recall that g_p and g_q are words corresponding to polynomials p and q.

Now that we are given the relatively free group A and its Abelian subgroup N, we can construct groups $C(A, N, M_t)$ for every element t in N, and consider the variety $\mathcal{C}$ generated by these groups. As we showed above, *the identity $s^4 = 1$ implies the identity $t^4 = 1$ in the variety $\mathcal{C}$ if and only if t is an endomorphic image of s in A.*

The last difficulty that we need to overcome is that $\mathcal{C}$ is possibly infinitely based. Kleiman finds a finitely based variety $\mathcal{D}$ which is greater than $\mathcal{C}$ yet still satisfies the statement in the previous paragraph. Let $\mathcal{D}$ be any variety greater than $\mathcal{C}$. If one identity does not imply another one in $\mathcal{C}$, then the same holds in the bigger variety $\mathcal{D}$. Therefore $s^4 = 1$ does not imply $t^4 = 1$ in $\mathcal{D}$ if t is not an endomorphic image of s in A. Thus we do not have to care much about whether $\mathcal{D}$ preserves the "if" part of the statement of the previous paragraph. It is more difficult to preserve the "only if" part.

Let $\mathcal{P}$ be the Mal'cev product of $\mathcal{N}_2 \cap \mathcal{A}_2^2$ and $varA$, and let P be the free group of $\mathcal{P}$ with the same number of generators as A. Then P is an extension of a nilpotent class two group V by A. It is easy to see that all groups $C(A, N, M)$ belong to $\mathcal{P}$.

By a theorem of Higman (H. Neumann [279], 34.24) $\mathcal{P}$ is finitely based.

Now suppose that $s, t \in N$ (N is an elementary Abelian normal subgroup in A) and t is covered by s in A. It is enough to prove that t^4 is an endomorphic image of s^4 in P (then $s^4[P]$ will contain $t^4[P]$ and so $s^4(P)$ will contain $t^4(P)$).

For every endomorphism ψ of A we can define an endomorphism $\bar{\psi}$ of P which acts on generators in the same way as ψ does. We have $\bar{\psi}(s) = tv$ for some $v \in V$ and $\bar{\psi}(s^4) = (tv)^4$. Everything would be OK if we had $(tv)^4 = t^4$. A smart reader might guess that this is not the case. And (s)he is right as usual.

But we can add one identity to the equations of $\mathcal{P}$ to make this property true. Indeed, we want to have $(tv)^4 = t^4$ for every $t \in N, v \in V$. To this end it is enough to find a word f such that $f(A) = N$ and a word v such that $v(P) = V$. We can assume that f and v are written on different sets of variables. Then the identity $(fv)^4 = v^4$ will do the trick. Kleiman finds such words using a certain technique of Rhemtulla [319]. This finishes the proof.

References

[1] S. Aanderaa. A proof of Higman's embedding theorem using Britton extensions of groups. In *Word Problems, Decision Problems and the Burnside problem in group theory*, pages 1–18, North-Holland, Publ. Corp., Amsterdam, London, 1973. MR 54 #412.

[2] S. Aanderaa and D. E. Cohen. Modular machines and the Higman-Clapham-Valiev embedding theorem. In *Word problems II*, pages 17–28, North-Holland, Amsterdam, 1980. MR 81m:20049.

[3] S. I. Adian. Algorithmic unsolvability of the problem of recognizing certain properties of groups. *Dokl. Akad. Nauk. SSSR*, 133:533–535, 1955. MR 18-455.

[4] S. I. Adian. *The Burnside problem and identities in groups*. Nauka, Moscow, 1975. translated by John Lennox and J. Wiegold, Ergebnisse der Math and ihrer Grenzgebiete, 95, Springer-Verlag, 1979; MR 80d:20035.

[5] S. I. Adian. Normal subgroups of free periodic groups. *Izv. Akad. Nauk. SSSR. Ser. Mat.*, 45(5):931–947, 1981. MR 84g:20050.

[6] S. I. Adian and G. S. Makanin. Studies in algorithmic questions of algebra. *Trudy Steklov Institute of Mathematics*, 168:197–217, 1984. transl. 1986, issue 3 of 4, 207-226; MR 85k:20109.

[7] A. Agalakov. Remarks on non-residually finite algebras. preprint, VINITI, # 435-V89, 1989.

[8] S. V. Aivazyan. A problem of A. I. Mal'cev. *Sibirsk. Mat. Zh.*, 29(6):3–11, 1988. MR 90c:20041.

[9] D. Albert, R. Baldinger, and J. Rhodes. Undecidability of the identity problem for finite semigroups. *J. Symbolic Logic*, 57(1):179–192, 1992. MR 93f:03030.

[10] J. Almeida. Equations for pseudovarieties. In J. E. Pin, editor, *Formal Properties of Finite Automata and Applications*, pages 148–164, Springer-Verlag, 1989. MR 91c:08015.

[11] A. Z. Anan'in. Locally residually finite and locally representable varieties of algebras. *Algebra i Logika*, 16(1):3–23, 1977. MR 58 # 21895.

[12] J. Apel and W. Lassner. Computation and simlification in Lie fields. In J.H. Davenport, editor, *Eurocal '87. Lecture Notes in Computer Science. 378.*, pages 468–478, 1987. MR 90k:17026.

[13] S. E. Arshon. A proof of the existence of n-valued infinite assymmetric sequences. *Mat. Sb.*, 2(44):769–777, 1937.

[14] C. Ash. Finite semigroups with commuting idempotents. *J. Austr. Math. Soc. (Series A)*, 43:81–90, 1987. MR 88f:20075.

[15] C. Ash. Inevitable graphs: A proof of the type II conjecture and some related decision procedures. *Inter. J. Algebra Comput.*, 1(1):127–146, 1991. MR 92k:20114.

[16] A. Baker. Linear forms in the logarithms of algebraic numbers, I. *Mathematika*, 13:204–216, 1966. MR 36 3732.

[17] Yu. A. Bakhturin. The approximation of Lie algebras. *Mat. Zametki*, 12(6):713 – 716, 1972. MR 47 #3476.

[18] Yu. A. Bakhturin. *Identitical relations in Lie algebras.* Nauka, Moskow, 1985. translation Identical relations in Lie algebras, YNU Sci. Press, Utrecht, 1987; MR 86k:17015, 88f:17032.

[19] Yu. A. Bakhturin and A. Yu. Ol'shanski. Identities. In *Itogi Nauki i Techniki. Current problems in mathematics. Fundamental directions. 18*, pages 117–242, VINITI, Moscow, 1988. MR 89j:08001.

[20] Yu. A. Bakhturin and A. Yu. Ol'shanski. Identity relations in finite Lie algebras. *Mat. Sb.*, 96(4):543–559, 1975. MR 89j:08001.

[21] Yu. A. Bakhturin and K. N. Semenov. On the residual finiteness of solvable varieties of Lie algebras. *Vestnik MGY. Mat. Mekh.*, (6):59–61, 1986. transl. Moscow Univ. Math. Bull. 41, 1986; MR 88b:17016.

[22] A. M. Ballantyne and D. S. Lankford. New decision algorithms for finitely presented commutative semigroups. *Computers and Math. with Appl.*, 7:159–165, 1981. MR 83a:03038.

[23] G. Baumslag. Subalgebras of finitely presented solvable Lie algebras. *J. Algebra*, 45(2):295–305, 1977. MR 55 # 8128.

[24] G. Baumslag. Subgroups of finitely presented metabelian groups. *J. Austr. Math. Soc. (Series A)*, 16(1):98–110, 1973. MR 48 #11324.

[25] G. Baumslag, F. B. Cannonito, and C. F. Miller III. Some recognizable properties of solvable groups. *Math. Z.*, 178:289–295, 1981. MR 82k:20061.

[26] G. Baumslag, F. B. Cannonito, and D.J.S. Robinson. The algorithmic theory of finitely generated metabelian groups. *Trans. Amer. Math. Soc.*, 344(2):629–648, 1994.

[27] G. Baumslag, D. Gildenhuys, and R. Strebel. Algorithmically insoluble problems about finitely presented solvable groups, Lie and associative algebras. I. *J. of Pure and Applied Algebra*, 39:53–94, 1986. MR 88a:20046.

[28] G. Baumslag, D. Gildenhuys, and R. Strebel. Algorithmically insolvable problems about finitely presented solvable groups, Lie and associative algebras, III. *J. of Pure and Applied Algebra*, 54:1–35, 1988. MR 90a:20066.

[29] D. B. Bean, A. Ehrenfeucht, , and G. McNulty. Avoidable patterns in strings of symbols. *Pacific J. of Math.*, 85(2):261–294, 1979. MR81i:20075.

[30] T. Becker and V. Weispfenning. *Gröbner bases.* Springer-Verlag, New York, 1993.

[31] V. P. Belkin. Quasi-identities of finite rings and lattices. *Algebra i Logika*, 17(3):247–259, 1978. MR 80g:08015.

[32] V. Ya. Belyaev. Imbeddability of recursively defined inverse semigroups in finitely presented semigroups. *Sibirsk. Mat. Zh.*, 25(2):50–54, 1984. MR 85m:20083.

[33] V. Ya. Belyaev. Subrings of finitely presented associative rings. *Algebra i Logika*, 17:627–638, 1978. MR 81i:16026.

[34] G. M. Bergman. The diamond lemma for ring theory. *Advances in Math.*, 29:178–218, 1978. MR 81b:16001.

[35] G. M. Bergman. Some examples in PI ring theory. *Israel J. Math.*, 18(3):257–277, 1974. MR 50 #9956.

[36] R. Bieri and R. Strebel. Almost finitely presented soluble groups. *Comment. Math. Helv.*, 53:258–278, 1978. MR 58:16890.

[37] R. Bieri and R. Strebel. A geometric invariant for nilpotent-by-abelian-by-finite groups. *J. of Pure and Applied Algebra*, 25:1–20, 1982. MR 83i:20031.

[38] R. Bieri and R. Strebel. Valuations and finitely presented metabelian groups. *Proc. London Math. Soc. (ser. 3)*, 41(3):439–464, 1980. MR 81j:20080.

[39] J. C. Birget, S. Margolis, and J. Rhodes. Finite semigroups whose idempotents form a subsemigroup. *Bull. Austr. Math. Soc.*, 41:161–184, 1990. MR 91k:20067.

[40] A. P. Biryukov. Personal communication, 1981.

[41] A. P. Biryukov. Certain algorithmic problems for finitely presented commutative semigroups. *Sibirsk. Mat. Zh.*, 8:384–391, 1967. MR 36#285.

[42] L. A. Bokut'. Algorithmic problems and embedding theorems: some open problems for rings, groups and semigroups. *Izv. Vyssh. Uchebn. Zaved. Mat.*, (11):3–11, 1982. MR 84j:20001.

[43] L. A. Bokut'. Undecidability of certain algorithmic problems for Lie algebras. *Algebra i Logika*, 13(2):145–152, 1974. MR 51 #106.

[44] L. A. Bokut'. Undecidability of the word problem and subalgebras of finitely presented Lie algebras. *Izv. Akad. Nauk. SSSR. Ser. Mat.*, 36(6):1173–1219, 1972. MR 48 # 8588.

[45] L. A. Bokut'. Unsolvability of some algorithmic problems in a class of associative rings. *Algebra i Logika*, 9(2):137–144, 1970. MR 44 6492.

[46] L. A. Bokut' and G. P. Kukin. Undecidable algorithmic problems for semigroups, groups and rings. In *Itogi Nauki i Techniki. Algebra, Topology, Geometry*, pages 3–66, VINITI, 1987. MR 89a:20036.

[47] R. V. Book and F. Otto. *String-rewriting systems*. Springer-Verlag, New York, 1993. MR 94f:68108.

[48] W. W. Boone. Certain simple unsolvable problems in group theory I. *Proc. Kon. Ned. akad. Wetensch. A*, 57:231–237, 1954. MR 17-252.

[49] W. W. Boone. Certain simple unsolvable problems in group theory II. *Proc. Kon. Ned. akad. Wetensch. A*, 57:492–497, 1954. MR 17-253.

[50] W. W. Boone. Certain simple unsolvable problems in group theory III. *Proc. Kon. Ned. akad. Wetensch. A*, 58:252–256, 1955. MR 17-254.

[51] W. W. Boone. Certain simple unsolvable problems in group theory IV. *Proc. Kon. Ned. akad. Wetensch. A*, 58:571–577, 1955. MR 20 #5230.

[52] W. W. Boone. Certain simple unsolvable problems in group theory V. *Proc. Kon. Ned. akad. Wetensch. A*, 60:22–27, 1957. MR 20 #5231.

[53] W. W. Boone. Certain simple unsolvable problems in group theory VI. *Proc. Kon. Ned. akad. Wetensch. A*, 60:222–232, 1957. MR 20 #5231.

[54] W. W. Boone. The word problem. *Ann. Math.*, 70(2):207–265, 1959. MR 21 #80.

[55] A. Borel and Harish-Chandra. Arithmetic subgroups of algebraic groups. *Ann. Math.*, 75:485–535, 1962. MR 26 #5081.

[56] A. Braun. The radical in finitely generated PI-algebra. *Bull. Amer. Math. Soc.*, 7(2):385–386, 1982. MR 83h:16022.

[57] T. C. Brown. An interesting combinatorial method in the theory of locally finite semigroups. *Pacific J. of Math.*, 36:285–289, 1971. MR 43 # 6346.

[58] T. C. Brown. On locally finite semigroups. *Ukr. Matem. Zh.*, 20(6):732–738, 1968. MR 38 #5964.

[59] T. C. Brown. A semigroup union of disjoint locally finite subsemigroups which is not locally finite. *Pacific J. of Math.*, 22:11–14, 1967. MR 35 # 4292.

[60] J. Brzozowski. Open problems about regular languages. In R. V. Book, editor, *Formal Language Theory, Perspectives and Open Problems*, pages 23–47, Akademic Press, London, New York, 1980. MR 84j:68001.

[61] B. Buchberger. *Ein Algorithmus zum Auffindedder Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal.* PhD thesis, Inst. University of Innsbruck, Austria, 1965.

[62] B. Buchberger. *Gröbner bases: algorithmic method in polynomial ideal theory.* Technical Report 83-29., CAMP, November 1983.

[63] B. Buchberger. A survey on the method of Gröbner bases for solving problems in connection with systems of multivariate variables. In *The second RIKEN international symposium on symbolic and and algebraic computation by computer, 69-83, Ser. comp. sci. 2*, World Sci., Philadelphia, 1985. MR 87j:68061.

[64] S. Burris and R. McKenzie. *Decidability and Boolean Representations.* AMS, Providence, Rhode Island, 1981. MR 83j:03024.

[65] A. H. Rhemtulla C. K. Gupta, N. D. Gupta. Dichotomies in certain finitely generated soluble groups. *J. London Math. Soc.*, 3:517–525, 1971. MR 44 6799.

[66] F. Cannonito. On some algorithmic problems for finitely presented groups and Lie algebras. In *Proceedings of "Groups-Korea 1983"*, Kyoungju, August 1983. MR 86h:20042.

[67] F.B. Cannonito and Gupta N.D. On centre-by-free solvable groups. *Arch. Math.*, 41(6):493–497, 1983. MR 86a:20031.

[68] Edward W. Cardoza. *Computational complexity of the word problem for commutative semigroups.* MAC technical memorandum 67, MIT, October 1975.

[69] L. H. Chin and A. Tarski. *Distributive and modular laws in the arithmetic of relation algebras.* University of California, 1951. MR 13-312f.

[70] A. Church and J. Rosser. Some properties of conversion. *Trans. Amer. Math. Soc.*, 39:472–482, 1936.

[71] A. H. Clifford. The free completely regular semigroup on a set. *J. Algebra*, 59:434–451, 1979. MR 80h:20082a.

[72] A. H. Clifford. Semigroups admitting relative inverses. *Ann. Math.*, 42(2):1037–1049, 1941. MR 3,199c.

[73] A. H. Clifford and G. B. Preston. *The algebraic theory of semigroups.* Volume 1, 2, Amer. Math. Soc., Providence, 1961, 1967. MR 24 #A2627.

[74] P.M. Cohn. A non-nilpotent Lie ring satisfying the Engel condition and a non-nilpotent Engel group. *Proc. Cambridge Phil. Soc*, 51(3):401–405, 1955. MR 17,175b.

[75] M. Coudrain and M. P. Schützenberger. Une condition de finitude des monoides finiment engendrés. *C. R. Acad. Sci. Paris. Serie A*, 262:1149–1151, 1966. MR 33 #2742.

[76] A. de Luca and S. Varricchio. Combinatorial properties of uniformly recurrent words and an application to semigroups. *Inter. J. Algebra Comput.*, 1(2):227–246, 1991. MR 92h:20084.

[77] A. de Luca and S. Varricchio. On a conjecture of Brown. *Semigroup Forum*, 46(1):116–119, 1993. MR 93i:20070.

[78] A. de Luca and S. Varricchio. On non-counting regular classes. In M. S. Peterson, editor, *Automata, Languages and Programming. Lecture Notes Comp. Sci. 443*, pages 74–87, Springer-Verlag, Berlin, 1990. MR 91e:68004.

[79] A. de Luca and S. Varricchio. On non-counting regular classes. *Theoretical Computer Science*, 100:67–104, 1992. MR 93i:68136.

[80] M. Dehn. Über unendliche diskontinuerliche gruppen. *Math. Ann.*, 71:116–144, 1912.

[81] Nachum Dershowitz and Jean-Pierre Jouannaud. Rewrite systems. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science*, chapter 6, pages 244–320, Elsevier Science Publishers B.V., 1990. MR 92d:68002.

[82] W. Dicks and E. Formanek. Poincare series and a problem by S. Montgomery. *Linear and Multilinear Algebra*, 12:21–30, 1982. MR 84c:15032.

[83] L. E. Dickson. Finiteness of the odd perfect and primitive abundant numbers with n distinct prime factors. *Amer. J. Math.*, 35:413–422, 1913.

[84] Dnestrovskaya Tetrad'. Unsolved problems in the theory of rings and modules, Second edition. 1976.

[85] Alair Pereira do Lago. On the Burnside semigroups $x^n = x^{n+m}$. In I. Simon, editor, *1st Latin American Symposium on Theoretical Informatics. São Paulo, Brazil, April 1992. Lecture Notes Computer Science. 583*, pages 329–343, Springer-Verlag, 1992.

[86] V. S. Drensky. Finite generation of invariants of finite linear groups on relatively free algebras. *Linear and Multilinear Algebra*, 35(1):1–10, 1993.

[87] V. S. Drensky. Identities in Lie algebras. *Algebra i Logika*, 13(3):265–290, 1974. MR 51 #10420.

[88] V. S. Drensky. Polynomial identities of finite-dimensional algebras. *Serdica*, 12:209–216, 1986. MR 88e:16024.

[89] S. Eilenberg. *Automata, languages and machines.* Volume B of *Pure and Applied Mathematics*, Academic Press, 1976. MR 58 #26604 b.

[90] S. Eilenberg and M. P. Schützenberger. On pseudovarieties. *Advances in Math.*, 19:413–418, 1976. MR 53 #5431.

[91] V. A. Emelichev. Commutative semigroups with one defining relation. *Uchenye zap. Shuiskogo ped. in-ta*, 6:227–242, 1958.

[92] V. I. Epanchintsev and G. P. Kukin. The word problem in a variety of groups that contain $\mathcal{N}_2\mathcal{A}$. *Algebra i Logika*, 18(3):259–285, 1979. transl. in Algebra and Logic, 1980, 157-174; MR 81h:20043.

[93] Ju. L. Ershov. On elementary theories of groups. *Dokl. Akad. Nauk. SSSR*, 203(6):1240–1243, 1972. MR 45 #6892.

[94] T. Ershova and M. V. Volkov. The lattice of varieties with completely regular square. to appear.

[95] T. Evans. On multiplicative systems defined by generators and relations. I. Normal form theorems. *Proc. Cambridge Phil. Soc*, 47:637–649, 1951. MR 13-312g.

[96] T. Evans. The word problem for abstract algebras. *J. London Math. Soc.*, 26:64–71, 1951. MR 12-475c.

[97] Ronald Fagin. Finite model theory – a personal perspective. *Theoret. Comput. Sci.*, 116:3–31, 1993.

[98] Ronald Fagin. Generalized first-order spectra and polynomial-time recognizable sets. In *Complexity of Computation. SIAM-AMS Proceedings. 7*, pages 43–73, 1974. MR 51 #7840.

[99] P. C. Fisher, A. R. Meyer, and A. L. Rosenberg. Counter machines and counter languages. *Math. Systems Theory*, 2(3):265–283, 1968. MR 38#4233.

[100] E. Formanek. Noncommutative invariant theory. *Contemporary Math.*, 43:87–119, 1985. MR 87d:16046.

[101] R. Freese. Free modular lattices. *Trans. Amer. Math. Soc.*, 261:81–91, 1980. MR 81k:06010.

[102] R. Freese and R. McKenzie. Residually small varieties with modular congruence lattices. *Trans. Amer. Math. Soc.*, 264:419– 430, 1981. MR 83d:08012a.

[103] H. Furstenberg. Poincaré recurrence and number theory. *Bull. Amer. Math. Soc.*, 5:211–234, 1981. MR 83d:10067.

[104] M. R. Garey and D. S. Johnson. *Computers and intractibility. A guide to the theory of NP-completeness.* W. H. Freeman and Company, San Francisco, 1979. MR 80g:68056.

[105] J. A. Gerhard. Free completely regular semigroups. I. Representation. *J. Algebra*, 59:135–142, 1983. MR 84m:20063a.

[106] J. A. Gerhard. Free completely regular semigroups. II. Word problem. *J. Algebra*, 59:143–156, 1983. MR 84m:20063b.

[107] J. A. Gerhard. Subdirectly irreducible idempotent semigroups. *Pacific J. of Math.*, 39(3):669–676, 1971. MR 47 #1980.

[108] J. A. Gerhard and M. Petrich. Word problems for free objects in certain varieties of completely regular semigroups. *Pacific J. of Math.*, 104:351–359, 1983. MR 84f:20062.

[109] J. A. Gerhard and M. Petrich. The word problems for orthogroups. *Can. J. Math.*, 33(4):893–900, 1981. MR 82m:20063.

[110] Robert H. Gilman. Presentations of groups and monoids. *J. Algebra*, 57:544–554, 1979. MR 80j:20055.

[111] S. Ginsburg and E. H. Spanier. Semigroups, Presburger formulas, and languages. *Pacific J. of Math.*, 16(2):285–296, 1965. MR 32 #9172.

[112] E. A. Golubov and M. Sapir. Varieties of residually finite semigroups. *Izv. Vyssh. Uchebn. Zaved. Mat.*, (11):21–29, 1982. MR 85f:20051.

[113] E. A. Golubov and M. V. Sapir. All varieties of residually finite semigroups. In *Proceedings of the 2nd All-Union Symposium on the Semigroup Theory*, page 21, Ural State University, Sverdlovsk, 1978.

[114] E. A. Golubov and M. V. Sapir. Varieties of residually finite semigroups. *Dokl. Akad. Nauk. SSSR*, 247(5):1037–1041, 1979. MR 81d:20050.

[115] J. A. Green and D. Rees. On semigroups in which $x^r = x$. *Proc. Cambridge Phil. Soc*, 48(1):35–40, 1952. MR 13,720c.

[116] J. R. J. Groves. An extension of dichotomy of P. Hall to some varieties of soluble groups. *Arch. Math.*, 23:573–580, 1972.

[117] J. R. J. Groves. Finitely presented centre-by-metabelian groups. *J. London Math. Soc.*, 18(1):65–69, 1978. MR 58 #28187.

[118] J. R. J. Groves. On some finiteness conditions for varieties of metanilpotent groups. *Arch. Math.*, 24:252–268, 1973. MR 48 #6253.

[119] J. R. J. Groves. Varieties of soluble groups and a dichotomy of P.Hall. *Bull. Austr. Math. Soc.*, 5:391–410, 1971. MR 47#5114.

[120] J. R. J. Groves and J. S. Wilson. Finitely presented metanilpotent groups. *J. London Math. Soc.*, 50(1):87–104, 1994.

[121] J.R.J Groves and C.F. Miller III. Recognizing free metabelian groups. *Illinois Journal of Mathematics*, 30(2):246–254, 1986. MR 87i:20057.

[122] F. Grunewald and D. Segal. Some general algorithms. 1. Arithmetic groups. *Ann. Math.*, 112:531–583, 1980. MR 82d:20048a.

[123] F. Grunewald and D. Segal. Some general algorithms. 2. Nilpotent groups. *Ann. Math.*, 112(3):585–617, 1980. MR 82d:20048b.

[124] F. J. Grunewald, P. F. Pickel, and D. Segal. Polycyclic groups with isomorphic finite quotients. *Ann. Math.*, 111:155–195, 1980. MR 81i:20045.

[125] V. Guba. Word problem for free Burnside semigroup satisfying $x^n = x^{n+m}$, $n \geq 4$. preprint; November, 1991.

[126] V. Guba. The word problem for relatively free Burnside semigroup satisfying $t^m = t^{m+n}$, with $m \geq 4$ or $m = 3, n = 1$. *Inter. J. Algebra Comput.*, 3(2), 1993.

[127] V. Guba. The word problem for the relatively free semigroup satisfying $t^m = t^{m+n}$, $m \geq 3$. *Inter. J. Algebra Comput.*, 3(3):335–347, 1993.

[128] C. K. Gupta. The free centre-by-metabelian groups. *J. Austr. Math. Soc. (Series A)*, 16:294–299, 1973. MR49 419.

[129] C. K. Gupta. Conjugacy in certain solvable groups. In *Algebraic structures and number theory (Hong Kong, 1988)*, pages 124–131, World Sci Publishing, Teaneck, NJ, 1990. MR 92a:20033.

[130] C. K. Gupta, W. Herfort, and F. Levin. The conjugacy problem for centre-by-metabelian groups. *Arch. Math.*, 52(2):117–121, 1989. MR 90e:20031.

[131] C. K. Gupta and N. S. Romanovskii. A generalized Freiheitssatz for centre-by-metabelian groups. *Bull. London Math. Soc.*, 24(1):71–75, 1992. MR 92m:20023.

[132] C.K. Gupta, N.D. Gupta, and F. Levin. On conjugacy p-separability of free centre-by-metabelian groups. *J. Austr. Math. Soc. (Series A)*, 47(2):334–342, 1989. MR 90h:20036.

[133] Yu. Gurevich. The problem of equality of words for certain classes of semigroups. *Algebra i Logika*, 5(5):25–35, 1966. MR 34 #5904.

[134] Yu. Gurevich and H. R. Lewis. The inference problem for template dependencies. *Information and Control*, 55(1-3):69–79, 1982. MR 85e:68013.

[135] W. Haken. Connections between topological and group theoretical decision problems. In *Word Problems*, pages 27–451, North-Holland, Publ. Corp., Amsterdam, London, 1973. MR 53 #1594.

[136] M. Hall. *The theory of groups.* The Macmillan Company, New York, 1959. MR 21 #1996.

[137] P. Hall. Finiteness conditions for soluble groups. *Proc. London Math. Soc. (ser. 3)*, 4:419–436, 1954. MR 17-344c.

[138] P. Hall. On the finiteness of certain soluble groups. *Proc. London Math. Soc. (ser. 3)*, 3(9):595–622, 1959. MR 22 #1618.

[139] P. Hall and G. Higman. On the p-length of p-soluble groups and reduction teorems for Burnside's problem. *Proc. London Math. Soc. (ser. 3)*, 6(3):1–42, 1956. MR 17-344.

[140] T. Hall, S. Kublanovsky, S. Margolis, M. Sapir, and P. Trotter. Decidable and undecidable problems related to finite 0-simple semigroups. To appear.

[141] K. Henckell, S. Margolis, J-E. Pin, and J. Rhodes. Ash's type II theorem, profinite topology and Mal'cev products. Part I. *Inter. J. Algebra Comput.*, 1(4):411–436, 1991. MR 93h:20064.

[142] G. Hermann. Die Frage der enlich vielen Schritte in der Theorie der Polynomideale. *Math. Ann.*, 95:736–788, 1926.

[143] Christian Herrmann. On the undecidability of implications between embedded multivalued database dependencies. preprint # 1397. Technische Hochschule Darmstadt; July, 1991.

[144] G. Higman. Subgroups of finitely presented groups. *Proc. Royal Soc. London Ser. A*, 262:455–475, 1961. MR 24 #A152.

[145] H. Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero: I, II. *Ann. Math.*, 79:109–326, 1964. MR 33 #7333.

[146] Derek F. Holt and Sarah Rees. Software for automatic groups and isomorphism testing and finitely presented groups. In G. A. Niblo and M. A. Roller, editors, *Geometric group theory*, pages 120–125, Cambridge University Press, 1993.

[147] G. Huet. Confluent reductions: abstract properties and applications to term rewriting systems. *J. Assoc. Comput. Mach.*, 27(4):797–821, 1980. MR 82a:68090.

[148] R. D. Hurwitz. A survey of the conjugacy problem. *Contemp. Math.*, 33, 1984. 86f:20036.

[149] Pawel M. Idziak. A characterization of finitely decidable congruence modular varieties. 1994. to appear.

[150] C. F. Miller III. Decision problems for groups — survey and reflections. In G. Baumslag and C. F. Miller III, editors, *Algorithms and Classification in Combinatorial Group Theory*, pages 1–59, Springer-Verlag, 1992.

[151] A. V. Il'tyakov. On finite basis of identities of Lie algebra representations. *Nova J. of Algebra and Geometry*, 1(3):207–259, 1992. MR94e:17008.

[152] N. Immerman. Relational queries computable in polynomial time. *Information and Control*, 68:76–98, 1986. MR 88a:68014.

[153] S. V. Ivanov. The Burnside problem for all sufficiently large exponents. *Inter. J. Algebra Comput.*, 4(1, 2):1–300, 1994.

[154] S. V. Ivanov. On the Burnside problem on periodic groups. *Bull. Amer. Math. Soc.*, 27(2):257–260, 1992. MR 93a:20061.

[155] S. V. Ivanov and A. Yu. Ol'shanskii. Some applications of graded diagrams in combinatorial group theory. *London Math. Sci. Lecture Notes series*, 160:258–308, 1991. 92j:20022.

[156] N. Jacobson. *Lie algebras.* Wiley, New York, 1962. MR 26 #1345.

[157] N. Jacobson. *Structure of rings.* AMS, Providence, Rhode Island, 1956. MR 36 5158.

[158] N. D. Jones and A. L. Selman. Turing machines and the spectra of first-order formulas with equality. *J. Symbolic Logic*, 28:139–150, 1974. MR 58 #5164.

[159] P. R. Jones. Completely simple semigroups: free products, free semigroups and varieties. *Proc. Royal Soc. Edinburgh*, 88A:293–313, 1981. MR 82f:20088.

[160] A. Kandri-Rody and V. Weispfenning. Non-commutative Gröbner bases in algebras of solvable type. *J. Symbolic Computation*, 9(1):1–26, 1990. MR 91e:13025.

[161] I. Kaplansky. On a problem of kurosh and jacobson. *Bull. Amer. Math. Soc.*, 52:496–500, 1946. MR 8-63b.

[162] I. Kaplansky. Problems in the theory of rings. 1957. Publ. No. 502, Nat. Acad. Sci. and Nat. Res. Council, Washington, D. C. ; MR 20 # 3179.

[163] M. I. Kargapolov and Ju. I. Merzliakov. *Fundamentals of the theory of groups.* Nauka, Moskva, 3d edition, 1982. "Fundamentals of the theory of groups", transl. from the second Russian edition by G. Burns. Graduate Texts in Mathematics, 62, Springer-Verlag, New-York, Berlin, 1979, MR 83k:20001, 80k:20002.

[164] M.I. Kargapolov and Remeslennikov V.N. The conjugacy problem for free solvable groups. *Algebra i Logika*, 5(6):15–25, 1966. MR 34 #5905.

[165] J. Kaďourek. On the word problem for free bands of groups and for free objects in some other varieties of completely regular semigroups. *Semigroup Forum*, 38(1):1–55, 1989. MR 89k:20085.

[166] J. Kaďourek. On varieties of combinatorial inverse semigroups I. *Semigroup Forum*, 43(3):305–330, 1991. MR 92j:20060.

[167] J. Kaďourek. On varieties of combinatorial inverse semigroups II. *Semigroup Forum*, 44(1):53–78, 1992. MR 92k:20126.

[168] J. Kaďourek and L. Polák. On free semigroups satisfying $x^r = x$. *Simon Stevin*, 64(1):3–19, 1990. MR 91j:20134.

[169] J. Kaďourek and L. Polák. On the word problem for free completely regular semigroup. *Semigroup Forum*, 34:127–138, 1986. MR 88a:20066.

[170] A. Kemer. Finite bases for identities of associative algebras. *Algebra i Logika*, 26:597–641, 1987. MR 90b:08008.

[171] A. Kemer. *Ideals of identities of associative algebras.* Ann. Math. Stud., Providence, Rhode Island, 1991. MR 92f:16031.

[172] A. Kemer. Identities of algebras over a field of characteristic p. In *Recent developments in the theory of algebras with polynomial identities. Palermo, 15-18 June 1992, Supplemento ai Rendiconti del Circolo Matematico di Palermo, Serie 2 – numero 31 – anno 1993*, pages 133–148, 1993.

[173] A. Kemer. Representability of relatively free algebras. *Algebra i Logika*, 27:274–294, 1988. transl. in Algebra and Logic, 27 (1988), no.3, 167-184; MR 90e:16027.

[174] V. K. Kharchenko. Noncommutative invariants of finite groups and Noetherian varieties. *J. of Pure and Applied Algebra*, 31:83–90, 1984. MR 85j:16052.

[175] O. G. Kharlampovich. A finitely presented solvable group with unsolvable word problem. *Izv. Akad. Nauk. SSSR. Ser. Mat.*, 45(4):852–873, 1981. MR 82m:20036.

[176] O. G. Kharlampovich. Finitely presented solvable groups and Lie algebras with undecidable word problem. *Mat. Zametki*, 46(3-4):731–738, 1990. MR 91b:20040.

[177] O. G. Kharlampovich. Lyndon's condition for solvable Lie algebras. *Izv. Vyssh. Uchebn. Zaved. Mat.*, (9):50–59, 1984. MR 86c:17016.

[178] O. G. Kharlampovich. Minimal varieties of Lie algebras with unsolvable word problem. to appear.

[179] O. G. Kharlampovich. The universal theory of certain classes of Lie rings. In *Investigations of Algebraic Systems*, pages 156–164, Ural State University, Sverdlovsk, 1984. 87d:03117.

[180] O. G. Kharlampovich. The universal theory of the class of finite nilpotent groups is undecidable. *Mat. Zametki*, 33(4):499–516, 1983. MR 85b:20046.

[181] O. G. Kharlampovich. The word problem for groups and Lie algebras, doctor's thesis (russian). 1990.

[182] O. G. Kharlampovich. The word problem for solvable groups and Lie algebras. In *Algorithms and Classification in Combinatorial Group Theory*, pages 61–67, MSRI, 1992.

[183] O. G. Kharlampovich. The word problem for solvable groups and Lie algebras. A boundary between solvability and unsolvability. *Contemporary Math.*, 131(2):53–57, 1992. MR 93e:20044.

[184] O. G. Kharlampovich. The word problem for solvable Lie algebras and groups. *Mat. Sb.*, 180(8):1033–1066, 1989. translation in Math USSR-Sb, Vol. 67, No. 2, 489-525 (1990); MR 90k:17008.

[185] O. G. Kharlampovich. The word problem for subvarieties of the variety $\mathcal{N}_2\mathcal{A}$. *Algebra i Logika*, 26:481–501, 1987. translation in Algebra and Logic, Vol. 26 (1987); MR 89m:20036.

[186] O. G. Kharlampovich. The word problem in the variety B_n. Journal of Algebra, to appear.

[187] O. G. Kharlampovich. The word problem in the variety $Z\mathcal{N}_2\mathcal{A}$. *Izv. Vyssh. Uchebn. Zaved. Mat.*, (11):82–84, 1988. translation in Soviet Math. (Iz. Vuz) Vol. 32, No 11, 136-140, (1988); MR 90j:20073.

[188] O. G. Kharlampovich and D. Gildenhuys. Identities in the variety of center-by-$\mathcal{N}_2\mathcal{A}$ Lie algebras. *Inter. J. Algebra Comput.*, 1(4):493–521, 1991. MR 93c:17008.

[189] O. G. Kharlampovich and D. Gildenhuys. Locally residually finite metanilpotent varieties of groups. preprint, 1992.

[190] O. G. Kharlampovich and D. Gildenhuys. Varieties of Lie algebras with solvable word problem. *Communications in Algebra*, 21(10):3571–3609, 1993.

[191] O. G. Kharlampovich and D. Gildenhuys. The word problem for some varieties of solvable Lie algebras. *Inter. J. Algebra Comput.*, 4, 1994.

[192] O. G. Kharlampovich and M. Sapir. Non-residually finite relatively finitely presented groups in the variety $\mathcal{N}_2\mathcal{A}$. preprint, 1993.

[193] Kirkinskii and V.N. Remeslennikov. The isomorphism problem for solvable groups. *Mat. Zametki*, 18(3):437–443, 1975. 53 #628.

[194] E. I. Kleiman. Bases of identities of varieties of inverse semigroups. *Sibirsk. Mat. Zh.*, 20(4):760–777, 1979. MR 81c:20045.

[195] E. I. Kleiman. On basis of identities of Brandt semigroups. *Semigroup Forum*, 13(3):209–218, 1977. MR 56 #509.

[196] Yu. G. Kleiman. Generation of varieties of groups by certain classes of groups. *Sibirsk. Mat. Zh.*, 23(6):117–132, 1982. MR. 84d:20026.

[197] Yu. G. Kleiman. On identities in groups. *Trudy Moskov. Mat. Obsh.*, 44:62–108, 1982. transl. in Trans. Moscow Math. Soc. , Vol. 44, No. 2, 3-28 (1983); MR 84e:20040.

[198] D. E. Knuth and P. B. Bendix. Simple word problems in universal algebras. In J. Leech, editor, *Computational problems in abstract algebra*, pages 263–297, Pergamon Press, 1970. MR 41 #134.

[199] *Kourovskaja tetrad' Unsolved problems in group theory, 5-th eddition.* Novosibirsk, 1976.

[200] *Kourovskaja tetrad. Unsolved problems in group theory, 11-th eddition.* Novosibirsk, 1990. MR 93e:20002.

[201] A. I. Kostrikin. *Around Burnside.* Nauka, Moscow, 1986. MR 89d:20032.

[202] A. I. Kostrikin. The Burnside problem. *Izv. Akad. Nauk. SSSR. Ser. Mat.*, 23:3–34, 1959. MR 24 #A1947.

[203] A. I. Kostrikin. Sandwiches in Lie algebras. *Mat. Sb.*, 110:3–12, 1979. MR 81e:17004.

[204] A. N. Krasilnikov. On the finiteness of the basis of identities of groups with a nilpotent commutator group. *Izv. Akad. Nauk. SSSR. Ser. Mat.*, 54(6):1181–1195, 1990. translation in Math. USSR-Izv. 37 (1991), No. 3, 539-553; MR 92d:20045.

[205] A. N. Krasilnikov. On the property of having a finite basis of some varieties of Lie algebras. *Vestnik MGY. Mat. Mekh.*, (2):34–38, 1982. MR 83i:17019.

[206] Heinz Kredel. *Solvable polynomial rings.* Verlag Shaker, Aachen, 1993.

[207] K. Krohn, J. Rhodes, and B. Tilson. Lectures on the algebraic theory of finite semigroups and finite-state machines. In M. A. Arbib, editor, *Algebraic theory of machines, languages and semigroups*, Academic Press, New York and London, 1968. MR 38 #1198.

[208] S. I. Kublanovskii. *Approximation of algebraic systems with respect to predicates.* PhD thesis, Mathematics Institute, Moldova Acad. of Sci., Kishinev, 1983.

[209] S. I. Kublanovskii. Finite approximability and algorithmic questions. In *Algebraic actions and orderings*, pages 59–78, Leningrad Gos. Ped. Inst., Leningrad, 1983. MR 85m:20088.

[210] S. I. Kublanovskii. Finite approximability of prevarieties of semigroups with respect to predicates. In *Modern Algebra*, pages 58–88, Leningrad Gos. Ped. Inst., Leningrad, 1980. MR 82g:20084.

[211] S. I. Kublanovskii. Locally residually finite and locally representable varieties of associative rings and algebras. preprint, VINITI, # 6143-82DEP; 1982.

[212] S. I. Kublanovskii. On varieties of associative algebras with finiteness conditions. In *17 All-Union Algebra Conference, Proceedings*, pages 118–119, Minsk State University, Minsk, 1983.

[213] G. P. Kukin. Algorithmic problems for solvable Lie algebras. *Algebra i Logika*, 17(4):402–415, 1978. transl. in Algebra and Logic, 1979, 270-278. MR 80h:17017.

[214] G. P. Kukin. Embedding of recursively presented Lie algebras and groups. *Dokl. Akad. Nauk. SSSR*, 251(1):37–39, 1980. MR 82c:20066.

[215] G. P. Kukin. Letter to the editors. "algorithmic problems for solvable Lie algebras, algebra i logika, 17 (1978),4". *Algebra i Logika*, 26(4):522, 1987. MR 89i:17009.

[216] G. P. Kukin. Subalgebras of finitely presented Lie algebras. *Algebra i Logika*, 18(3):311–327, 1979. transl. in Algebra and Logic, 1980, 190-201; MR 81k:17010.

[217] G. P. Kukin. The word problem and free products of Lie and associative algebras. *Sibirsk. Mat. Zh.*, 24(2):85–96, 1983. MR 84m:17012.

[218] G. Lallement. *Semigroups and combinatorial applications.* Wiley & Sons, New York, 1979. MR 81j:20082.

[219] J. Lambek. How to program an infinite abacus. *Canad. Math. Bull.*, 4(3):295–302, 1961. MR 24 #A2532.

[220] V. N. Latyshev. *Combinatorial ring theory. Standard bases.* Moscow State University, Moscow, 1988.

[221] V. N. Latyshev. A generalization of Hilbert's theorem on the finiteness of bases. *Sibirsk. Mat. Zh.*, 7:1422–1424, 1966. transl. in Siberian Math. J. 7 (1966); MR 34 #2606.

[222] V. N. Latyshev. On the complexity of non-matrix varieties. 1. *Algebra i Logika*, 16(2):149–182, 1977. MR 58 #27695.

[223] Philippe Le Chenadec. *Canonical forms in finitely presented algebras*. Pitnam, London, 1986. MR 88b:68117.

[224] J. Lewin. A matrix representation for associative algebras. *Trans. Amer. Math. Soc.*, 188:293–308, 1974. MR 49 #2828.

[225] I. V. L'vov. Locally weak Noetherian varieties of algebras over a Noetherian ring. In *3 All-Union conference on algebras and modules. Abstracts of talks*, Tartu, 1976.

[226] I. V. L'vov. The maximality conditions in algebras with identity relations. *Algebra i Logika*, 8(4):449–459, 1969. MR 43 #4853.

[227] I. V. L'vov. Varieties of associative rings I. *Algebra i Logika*, 12(3):269–297, 1973. MR 52 #10802.

[228] I. V. L'vov. Varieties of finitely approximable rings. *Algebra i Logika*, 17(4):488, 1978.

[229] Roger C. Lyndon and Paul E. Schupp. *Combinatorial group theory*. Springer-Verlag, 1977. MR 58 #28182.

[230] I. G. Lysenok. Infinity of Burnside groups of period 2^k for $k \geq 13$. *Russian Math. Surveys*, 47(2):229–230, 1992. MR 93g:20077.

[231] W. Magnus. Das identitätsproblem für gruppen mit einer definierenden relation. *Math. Ann*, 106:295–307, 1932.

[232] W. Magnus. Über diskontinuierliche gruppen mit einer definierenden relation (der freiheitssatz). *J. Reine Angew. Math*, 163:141–165, 1930.

[233] A. I. Mal'cev. *Algorithms and recursive functions*. Nauka, Moscow, 1965. MR 34 #2453.

[234] A. I. Mal'cev. Identity relations in varieties of quasigroups. *Mat. Sb.*, 69(1):3–12, 1966. MR 34 #61.

[235] A. I. Mal'cev. On isomorphic matrix representations of infinite groups. *Mat. Sb.*, 8(3):405–422, 1940. MR 2-216d.

[236] A. I. Mal'cev. On the homomorphisms onto finite groups. *Uchen. Zapiski Ivanovsk. ped. instituta*, 18(5):49–60, 1958. also in "Selected works", Vol. 1, Algebra, (1976), 450-461; MR 54:12465.

[237] A. I. Mal'cev. On the representations of infinite algebras. *Mat. Sb.*, 13:263–286, 1943. MR 6-116.

[238] A. I. Mal'cev. Two remarks on nilpotent groups. *Mat. Sb.*, 37(3):567–572, 1955. MR 17-345b.

[239] Yu. N. Mal'cev. On varieties of associative algebras. *Algebra i Logika*, 15(5):579–584, 1976. MR 58 #5457.

[240] Yu. N. Mal'cev and A. A. Nechaev. Critical rings and varieties of algebras. *Algebra i Logika*, 18(3):341–347, 1979. MR 81g:16024.

[241] S. G. Mamikonian. Varieties of finitely approximable semigroups. *Mat. Sb.*, 88(3):353–359, 1972. MR 47 # 379.

[242] S. Margolis. Kernels and expansions: an historical and technical perspective. In J. Rhodes, editor, *Monoids and Semigroups with Applications*, pages 3–30, World Scientific, 1991. MR 93a:20101.

[243] S. Margolis and J. Meakin. E-unitary inverse monoids and the Cayley graph of a group presentation. *J. of Pure and Applied Algebra*, 58:45–76, 1989. MR 90f:20096.

[244] S. Margolis, J. Meakin, and J. Stephen. Free objects in certain varieties of inverse semigroups. *Can. J. Math.*, 42(6):1084–1097, 1990. MR 92j:20057.

[245] S. Margolis and J. E. Pin. New results on the conjecture of rhodes and on the topological conjecture. *J. of Pure and Applied Algebra*, 80(3):305–313, 1992. 94d:20075.

[246] S. Margolis and M. Sapir. Quasi-identities of finite semigroups, Burnside-type problems, and symbolic dynamics. to appear in Isr. J. Math.

[247] S. W. Margolis and J. C. Meakin. Inverse monoids, trees ans context-free languages. *Trans. Amer. Math. Soc.*, 335(1):259–276, 1993. 93h:20062.

[248] A. Markov. The insolubility of the problem of homeomorphy. *Doklady AN SSSR*, 121(2):218–220, 1958. MR 20 #4260.

[249] A. Markov. On the impossibility of certain algorithms in the theory of associative systems. *Dokl. Akad. Nauk. SSSR*, 55:583–586, 1947. MR 8-558c.

[250] A. Markov. On the impossibility of certain algorithms in the theory of associative systems. II. *Dokl. Akad. Nauk. SSSR*, 58:353–356, 1947. MR 9-321b.

[251] V. T. Markov. Systems of generators of T-ideals of finitely generated free algebras. *Algebra i Logika*, 18(5):587–598, 1979. MR 81k:16004.

[252] Yu. V. Matiyasevich. Studies in certain algorithmic problems of algebra and number theory. *Trudy Steklov Institute of Mathematics*, 168:218–237, 1984. translation (1986), issue 3 of 4, 227-252; MR 85k:01040.

[253] Ernst W. Mayr and Albert R. Meyer. The complexity of the word problem for commutative semigroups and polynomial ideals. *Advances in Math.*, 46(3):305–329, 1982. MR 84g:20099.

[254] J. McCammond. The solution of the word problem for the relatively free semigroups satisfying $x^a = x^{a+b}$ with $a \geq 6$. *Inter. J. Algebra Comput.*, 1(1):1–32, 1991. MR 92f:20060.

[255] R. McKenzie. A new product of algebras and a type reduction theorem. *Algebra Universalis*, 19:29–69, 1984. MR 86h:08011.

[256] R. McKenzie. A note on residually small varieties of semigroups. *Algebra Universalis*, 17(2):143–149, 1983. MR 85e:20048.

[257] R. McKenzie. Residually small varieties of K-algebras. preprint, 1979.

[258] R. McKenzie. Residually small varieties of K-algebras. *Algebra Universalis*, 14:181–196, 1981. MR 83d:08012b.

[259] R. McKenzie. Residually small varieties of semigroups. *Algebra Universalis*, 13(2):171–201, 1981. MR 82k:20097.

[260] R. McKenzie and M. Valeriote. *The structure of decidable locally finite varieties*. Birkhäuser, Boston, 1989. MR 92j:08001.

[261] J. C. C. McKinsey. The decision problem for some classes of sentences without quantifiers. *J. Symbolic Logic*, 8(3):61–76, 1943. MR 5-85.

[262] G. McNulty and C. Shallon. Inherently non-finitely based finite algebras. *Lecture Notes Math.*, 1004:206–231, 1982. MR 85h:08011.

[263] J. Meakin and M. Sapir. Congruences on free monoids and submonoids of polycyclic monoids. *J. Austr. Math. Soc. (Series A)*, 54(2):236–253, 1993. MR 93k:20094.

[264] J. Meakin and M. Sapir. On the word problem in varieties of inverse semigroups. J. London Math. Soc., to appear.

[265] Alan H. Mekler, Evelyn Nelson, and Saharon Shelah. A variety with solvable, but not uniformly solvable, word problem. *Proc. London Math. Soc. (ser. 3)*, 66(2):225–256, 1993. MR 93m:03018.

[266] I. L. Mel'nichuk. Solvability of problems of equality and divisibility in certain finitely based varieties of semigroups. In *Properties of semigroups*, pages 91–103, Leningrad State Pedagogical Institute, Leningrad, 1984. MR 86j:20053.

[267] I. L. Mel'nichuk. Undecidability of problems of equality and divisibility in certain varieties of semigroups. *Algebra i Logika*, 23(4):430–438, 1984. MR 86d:20065.

[268] I. L. Mel'nichuk, M. Sapir, and O. G. Kharlampovich. The word problem in varieties of semigroups, rings and Lie algebras. *Sibirsk. Mat. Zh.*, 27(6):144–156, 1986. translated from Russian into English by Plenum Corp. in Sib. Math. J. ; MR 88d:08004.

[269] M. Mignotte, T. N. Shorey, and R. Tijdeman. The distance between terms of an algebraic recurrence sequence. *J. Reine Angew. Math.*, 349:63–76, 1984. MR 85f:11007.

[270] M. L. Minsky. An e-mail message. May 23 1993.

[271] M. L. Minsky. *Computation: finite and infinite machines*. Prentice-Hall, Englewood Cliffs, N. J., 1967. MR 50 #9050.

[272] M. L. Minsky. Recursive unsolvability of Post's problem of "tag" and other topics in the theory of Turing Machnes. *Ann. Math.*, 74:437–455, 1961. MR 25 #3825.

[273] T. Mora. Gröbner bases for non-commutative polynomial rings. In J. Calmet, editor, *AAECC-3. Lecture Notes in Computer Science. 229.*, pages 353–362, Springer-Verlag, 1986. MR87m:94005.

[274] M. Morse and G. Hedlund. Unending chess, symbolic dynamics and a problem in semigroups. *Duke Math. J.*, 11:1–7, 1944. MR 5-202e.

[275] V. L. Murskii. Isomorphic embeddability of a semigroup with an enumerable set of defining relations into a finitely presented semigroup. *Mat. Zametki*, 1(2):217–224, 1967. MR 35 #1686.

[276] V. L. Murskii. Some examples of varieties of semigroups. *Mat. Zametki*, 3:663–670, 1968. MR 38 #256.

[277] M. Muzalewski. Restricted decision problems in some classes of algebraic systems. *Z. Math. Logik Grundlag. Math.*, 24(3):279–287, 1978. MR 81g:03010.

[278] B. H. Neumann. Ascending derived series. *Compositio Math.*, 13:47–64, 1956. MR 19-632b.

[279] H. Neumann. *Varieties of groups.* Springer-Verlag, Berlin, Heidelberg, 1967. MR 35 #6734.

[280] M. H. A. Newman. On theories with a combinatorial definition of equivalence. *Ann. Math.*, 43:223–243, 1942. MR 4-126c.

[281] T. E. Nordahl. On the join of the variety of all bands and the variety of all commutative semigroups via conditional rewrite rules. In Ito, editor, *Proceedings of the International Colloquium of Words, Semigroups and Combinatorics, Kyoto*, pages 365–372, 1990.

[282] G.A. Noskov. On conjugacy in metabelian groups. *Mat. Zametki*, 31(4):495–507, 1982. MR 83i:20029.

[283] G.A. Noskov. On the genus of a free metabelian group. 1990. Preprint of the Comp. Center of AN SSSR, Novosibirsk.

[284] G.A. Noskov, V.N. Remeslennikov, and V. A. Roman'kov. *Infinite groups*, pages 65–157. VINITI, 1979. MR 83c:20001.

[285] P. S. Novikov. On algorithmic unsolvability of the problem of identity. *Dokl. Akad. Nauk. SSSR*, 85(4):709–719, 1952. MR 14-618.

[286] P. S. Novikov and S. I. Adian. On infinite periodic groups, I, II, III. *Izv. Akad. Nauk. SSSR. Ser. Mat.*, 32(1, 2, 3):212–244, 251–524, 709–731, 1968. MR 39 #1532 a,b,c.

[287] A. Yu. Ol'shanskii. The finite basis problem for identities in groups. *Izv. Akad. Nauk. SSSR. Ser. Mat.*, 34(2):376–384, 1970. MR 44 #4079.

[288] A. Yu. Ol'shanskii. *The geometry of defining relations in groups.* Nauka, Moscow, 1989. MR 91i:20035.

[289] A. Yu. Ol'shanskii. Residually finite varieties of groups. *Izv. Akad. Nauk. SSSR. Ser. Mat.*, 33(4):915–927, 1969. MR 41 #3572.

[290] F. Pastijn and P. G. Trotter. Free bands of abelian groups. *Proc. London Math. Soc. (ser. 3)*, 63:344–370, 1991. MR 92g:20094.

[291] F. Pastijn and P. G. Trotter. Residial finiteness in completely regular semigroup varieties. *Semigroup Forum*, 37(2):127–147, 1988. MR 89d:20046.

[292] John Pedersen. Efficient solutions of the word problem in slim varieties. To appear in Algebra Universalis.

[293] P. Perkins. Basic questions for general algebras. *Algebra Universalis*, 19:16–23, 1984. MR 85k:08010.

[294] C. E. Peterson and M. E. Stickel. Complete sets of reductions for some equational theories. *J. Assoc. Comput. Mach.*, 28(2):233–264, 1981. MR 83d:03037.

[295] M. Petrich. *Inverse semigroups.* Wiley, New York, 1984. MR 85k:20001.

[296] P. F. Pickel. Finitely generated nilpotent groups with isomorphic finite quotiens. *Bull. Amer. Math. Soc.*, 77(2):216–219, 1971. MR 42 4634.

[297] P. F. Pickel. Metabelian groups with the same finite quotients. *Bull. Austr. Math. Soc.*, 11:115–120, 1974. MR 51 #709.

[298] J. E. Pin and C. Reutenauer. A conjecture on the Hall topology for the free group. *Bull. London Math. Soc.*, 23:556–562, 1991. MR 92g:20035.

[299] V. P. Platonov. Linear groups with identical relations. *Dokl. Akad. Nauk. SSSR*, (11):581–582, 1967. MR 35 #4314.

[300] L. Polák. On varieties of completely regular semigroups. I. *Semigroup Forum*, 32(1):97–123, 1985. MR 86i:20082.

[301] L. Polák. On varieties of completely regular semigroups. II. *Semigroup Forum*, 36(2):253–284, 1987. MR 89a:20063.

[302] L. Polák. On varieties of completely regular semigroups. III. *Semigroup Forum*, 37(1):1–30, 1988. MR 89c:20086.

[303] E. L. Post. Recursive unsolvability of a problem of Thue. *J. Symbolic Logic*, 12(1):1–11, 1947. MR 8-558.

[304] A. A. Premet and K. N. Semenov. Varieties of residually finite Lie algebras. *Mat. Sb.*, 137(1):103–113, 1988. translation in Math. U.S.S.R., Sbornik, 65 (1990) , 1; MR 90b:17023.

[305] M. Presburger. über die Vollstandigkeit eines gewissen Systems der Arithmetic ganzer Zahlen, in welchem die Addition als einzige Operation hervortritt. In *Sprawozdanie z I Kongresu Matematykow Krajow Slowianskich*, pages 92–101, Warsaw, 1929.

[306] C. Procezi. *Rings with polynomial identity.* Marcel Dekker, New-York, 1973. MR 51 #3214.

[307] M. E. Prouhet. Mémoire sur quelques relations entre les puissances dees nombres. *C. R. Acad. Sci. Paris. Serie A*, 33(8):225, 1851.

[308] M. O. Rabin. Recursive unsolvability of group theoretic problems. *Ann. Math.*, 67:172–174, 1958. MR 22 #1611.

[309] M. O. Rabin and D. Scott. Finite automata and their decision problems. *IBM Journal of Res. and Devel.*, 3(2):114–125, 1969. MR 21 #2559.

[310] V. V. Rasin. Free completely simple semigroups. In *Issledov. po sovremennoj algebre*, pages 140–151, Ural State University, Sverdlovsk, 1979. MR 81i:20079a.

[311] V. V. Rasin. Varieties of orthodox Clifford semigroups. *Izv. Vyssh. Uchebn. Zaved. Mat.*, (11):82–85, 1982. MR 84i:20062.

[312] Yu. P. Razmyslov. Lie algebras satisfying Engel conditions. *Algebra i Logika*, 10:33–44, 1971. MR 45 #3498.

[313] N. R. Reilly and P. G. Trotter. Properties of relatively free inverse semigroups. *Trans. Amer. Math. Soc.*, 294:243–262, 1986. MR 87b:20080.

[314] W. Reisig. *Petri Nets. An Introduction.* Springer-Verlag, New York, 1985. MR 86d:68056.

[315] V. N. Remeslennikov. An example of a group finitely presented in the variety A^5 with an undecidable word problem. *Algebra i Logika*, 12(5):577–602, 1973. MR 51 #8266.

[316] V. N. Remeslennikov. Studies on infinite solvable and finitely approximable groups. Abstracts of the doctor's thesis. *Mat. Zametki*, 17(5):819–824, 1975. MR 52 #14051.

[317] V. N. Remeslennikov and V. A. Roman'kov. Model-theoretic and algorithmic questions in group theory. In *Algebra, Topology, Geometry*, pages 3–79, VINITI, 1983. MR 85f:20002.

[318] V. N. Remeslennokov and N. S. Romanovskii. Algorithmic problems for solvable groups. In *Word problems, II (Conf. on Decision Problems in Algebra, Oxford, 1976). Stud. Logic Found. Math. 95*, pages 337–346, North-Holland, Amsterdam-New York, 1980. MR 81h:20044.

[319] A. H. Rhemtulla. Commutators of certain finitely generated soluble groups. *Can. J. Math.*, 21:1160–1164, 1969. MR 40 #1467.

[320] L. Ribes and P. Zalesskii. On the profinite topology on a free group. *Bull. London Math. Soc.*, 25(1):37–43, 1993. MR 93j:120062.

[321] Lorenzo Robbiano, editor. *Computational aspects of commutative algebra.* Academic Press, London, 1989. MR 89j:68004.

[322] H. Rogers. *Theory of recursive functions and effective computability.* McGraw-Hill Book Company, 1967. MR 37 #61.

[323] V. A. Roman'kov. Unsolvability of the problem of endomorphic reducibility in free nilpotent groups and in free rings. *Algebra i Logika*, 16(4):457–471, 1977. MR 81b:20027.

[324] N. S. Romanovskii. Certain algorithmic problems for solvable groups. *Algebra i Logika*, 13(1):26–34, 1974. MR50 #10092.

[325] N. S. Romanovskii. The embedding problem for abelian-by-nilpotent groups. *Sibirsk. Mat. Zh.*, 21(2):170–174, 1980. MR 82c:20069.

[326] N. S. Romanovskii. Free subgroups in finitely generated groups. *Algebra i Logika*, 16(1):88–97, 1977. MR 58 #22300.

[327] N. S. Romanovskii. A generalized theorem of freedom for pro-p-groups. *Sibirsk. Mat. Zh.*, 27(2):154–170, 1986. MR 88e:20027.

[328] N. S. Romanovskii. The word problem for centre-by-metabelian groups. *Sibirsk. Mat. Zh.*, 23(4):201–205, 1982. MR 83m:20050.

[329] Lajos Rónyai. Computing the structure of finite algebras. *J. Symbolic Computation*, 9(3):355–373, 1990. MR 91h:68093.

[330] J. Rotman. *An Introduction to the Theory of Groups.* Allyn & Bacon, third edition, 1984. MR 85f:20001.

[331] L. H. Rowen. *Polynomial identities in ring theory.* Academic Press, New York, 1980. MR 82a:16021.

[332] B. V. Rozenblat. Elementary theories of relatively free semigroups. *Izv. Vyssh. Uchebn. Zaved. Mat.*, (2):41–43, 1986. MR 87f:20085.

[333] B. V. Rozenblat. Permutation semigroup varieties. In *17 All-Union Algebr. Conf. part 2*, pages 195–196, Minsk State University, Minsk, 1983.

[334] M. Sapir. Algorithmic problems in varieties of semigroups. *Izv. Vyssh. Uchebn. Zaved. Mat.*, (12):71–74, 1985. MR 87g:20100.

[335] M. Sapir. Algorithmic problems in varieties of semigroups. *Algebra i Logika*, 27(4):440–463, 1988. transl. in Algebra and Logic, 27 (1988), No 4, 275-292; MR 90h:20087.

[336] M. Sapir. Algorithmic problems in varieties of semigroups, groups and associative algebras. Doctoral dissertation. 1990.

[337] M. Sapir. Burnside-type problems and identities of finite semigroups. In *18 All-Union Algebr. Conf. Part. 2*, page 151, Kishinev, 1985.

[338] M. Sapir. Church-Rosser varieties of semigroups and associative algebras. to appear.

[339] M. Sapir. Identities of finite inverse semigroups. *Inter. J. Algebra Comput.*, 3(1):115–124, 1993. MR 94c:20106.

[340] M. Sapir. The identity problem in varieties and pseudo-varieties of semigroups. to appear.

[341] M. Sapir. Inherently nonfinitely based finite semigroups. *Mat. Sb.*, 133(2):154–166, 1987. MR 88m:20116.

[342] M. Sapir. Locally residually finite and locally representable varieties of semigroups. *Contemporary Math.*, 131 (3):263–282, 1992. MR 93g:20109.

[343] M. Sapir. A minimal variety of associative algebras with an unsolvable word problem. *Mat. Sb.*, 180(12):1691–1708, 1989. transl. in Math USSR-Sb, (1990); MR 91c:16023.

[344] M. Sapir. On cross semigroup varieties and related questions. *Semigroup Forum*, 20(4):120–132, 1990. MR 92c:20106.

[345] M. Sapir. On the word problem in periodic group varieties. *Inter. J. Algebra Comput.*, 1(1):115–126, 1991. MR 92c:20063.

[346] M. Sapir. Problems of Burnside type and the finite basis property in varieties of semigroups. *Izv. Akad. Nauk. SSSR. Ser. Mat.*, 51(2):319–340, 1987. transl. in Math USSR-Izv, Vol. 30 (1988), No 2, 295-314; MR 88h:20078.

[347] M. Sapir. The restricted Burnside problem for varieties of semigroups. *Izv. Akad. Nauk. SSSR. Ser. Mat.*, 55(3):670–679, 1991. MR 93a:20092.

[348] M. Sapir. Sur la propriete de base finie pour les pseudovarietes de semigroupes finis. *C. R. Acad. Sci. Paris. Serie A*, 306:705–708, 1988. MR 89d:08008.

[349] M. Sapir. Varieties with a countable number of subquasi-varieties. preprint, VINITI, # 4738-81 DEP; 1981.

[350] M. Sapir. Varieties with a countable number of subquasi-varieties. *Sibirsk. Mat. Zh.*, 25(3):148–163, 1984. MR 85i:20066.

[351] M. Sapir. Varieties with finitely many subquasi-varieties. *Sibirsk. Mat. Zh.*, 22(6):168–187, 1981. MR 83e:08025.

[352] M. Sapir. Weak word problem for finite semigroups. In J. Rhodes, editor, *Monoids and semigroups with applications. Proceedings of the Berkeley Workshop in Monoids. Berkeley, 1989*, pages 206–219, World Scientific, 1991. MR 92h:20085.

[353] M. Sapir and O. G. Kharlampovich. The word problem in varieties of associative algebras and Lie algebras. *Izv. Vyssh. Uchebn. Zaved. Mat.*, (6):76–84, 1989. translation in Soviet Math. (Iz. Vuz); MR 91c:08006.

[354] M. Sapir and L. N. Shevrin. Residually small varieties of semigroups. *Izv. Vyssh. Uchebn. Zaved. Mat.*, (10):41–49, 1988. transl. in Soviet. Math. (Iz. Vuz.), 32 (1988), no.10, 57–73; MR 90e:20054.

[355] M. Sapir and E. V. Sukhanov. Varieties of periodic semigroups. *Izv. Vyssh. Uchebn. Zaved. Mat.*, (4):48–55, 1981. MR 82j:20108.

[356] R. A. Sarkisian. A problem of equality for Galois cohomologies. *Algebra i Logika*, 19(6):707–725, 1980. MR 83i:12012.

[357] R.A. Sarkisyan. The conjugacy problem for collections of integral matrices. *Mat. Zametki*, 25(6):811–824, 1979. MR 80k:20046.

[358] L. N. Savina. Extremal residual properties of semigroup varieties. preprint, VINITI, # 1259-B86, 1986.

[359] L. N. Savina. Semigroups with recognizable ideals. preprint, VINITI, # 1258-B86; 1986.

[360] L. N. Savina. Varieties of semigroup which are residually finite with respect to the idempotency. In *Semigroups with additional structures*, pages 125–132, Leningrad State Pedagogical Inst., Leningrad, 1986. MR 88b:20092b.

[361] L. N. Savina. Varieties of semigroups which are residually finite with respect to the regularity. In *Semigroups with additional structures*, pages 116–124, Leningrad State Pedagogical Inst., Leningrad, 1986. MR 88b:20092a.

[362] W. Savitch. Relationships between non-deterministic and deterministic tape complexities. *J. Comput. System Sci.*, 4:177–192, 1970. MR 42 #1605.

[363] V. Sazonov. A logical approach to the problem P=NP. *Lect. Notes Comp. Sci.*, 88:562–575, 1980. MR 83j:68055.

[364] V. Sazonov. Polynomial computability and recursivity in finite domains. *Electronische Informationverarbeitung und Kybernetik*, 16:319–323, 1980. MR 83c:68053.

[365] D. Segal. Decidable properties of polycyclic groups. *Proc. London Math. Soc. (ser. 3)*, 61:497–528, 1990. MR 91h:20050.

[366] D. Segal. *Polycyclic groups.* Cambridge University Press, Cambridge, 1983. MR 85h:20003.

[367] L. N. Shevrin. On locally finite semigroups. *Dokl. Akad. Nauk. SSSR*, 162(4):770–773, 1965. MR 31 #3529.

[368] L. N. Shevrin and E. V. Sukhanov. Structural aspects of the theory of varieties of semigroups. *Izv. Vyssh. Uchebn. Zaved. Mat.*, (6):3–39, 1989. MR 91a:20071.

[369] L. N. Shevrin and M. V. Volkov. Identities of semigroups. *Izv. Vyssh. Uchebn. Zaved. Mat.*, (11):3–47, 1985. MR 87f:20094.

[370] A. I. Shirshov. Algorithmic problems for solvable Lie algebras. *Uspekhi Mat. Nauk*, 17(6):228, 1962.

[371] A. I. Shirshov. On rings with identical relations. *Mat. Sb.*, 43:277–283, 1957. MR 20 #1698.

[372] A. I. Shirshov. Some algorithmic problems for Lie algebras. *Sibirsk. Mat. Zh.*, 3(2):292–296, 1962. MR 32 #1231.

[373] A. L. Shmel'kin. The note to the Shmel'kin's article "Wreath products and varieties of groups". *Izv. Akad. Nauk. SSSR. Ser. Mat.*, 31:443–444, 1967. MR 35 #1655.

[374] A. L. Shmel'kin. Wreath products and varieties of groups. *Dokl. Akad. Nauk. SSSR*, 157:1063–1065, 1964. English transl. in Soviet Math. Dokl. 5 (1964); MR 33 #1352.

[375] Lev Shneerson. Conditions under which finitely presented algebras are free. *Izv. Vyssh. Uchebn. Zaved. Mat.*, (7):66–68, 1979. MR 81g:16002.

[376] Lev Shneerson. The free subsemigroups of finitely presented semigroups. *Sibirsk. Mat. Zh.*, 15:450–454, 1974. MR 48 11362.

[377] I. Simon. Notes on non-counting languages of order 2. manuscript, 1970.

[378] A. M. Slobodskoii. Undecidability of the universal theory of finite groups. *Algebra i Logika*, 20(2):207–230, 1981. MR 83h:03062.

[379] V. G. Sprindzhuk. *Classical Diophantine equations in two unknowns.* Nauka, Moscow, 1982. MR 85d:11022.

[380] C. C. Squier. Word problems and a homological finiteness condition for monoids. *J. of Pure and Applied Algebra*, 49:201–217, 1987. MR 89a:20059.

[381] J. B. Stephen. *Application of automata theory to presentations of monoids and inverse monoids.* PhD thesis, Univ. of Nebr. at Lincoln, Lincoln (NE), 1987.

[382] A. Storozhev. On abelian subgroups of relatively free groups. *Comm. Algebra*, 22(7):2677–2701, 1994.

[383] R. Strebel. Finitely presented soluble groups. In *Group theory: essays for Philip*, pages 257–314, London Math. Soc., 1984. MR 86g:20050.

[384] Sverdlovsk notebook. Unsolved problems in semigroup theory. 1989.

[385] Huỳnh Dũng T. Complexity of the word problem for commutative semigroups of fixed dimension. *Acta Inf.*, 22(4):421–432, 1985. MR 87f:03109.

[386] M. A. Taiclin. Algorithmic problems for commutative semigroups. *Dokl. Akad. Nauk. SSSR*, (9):201–204, 1968. MR 37 #2878.

[387] M. A. Taiclin. The isomorphism problem for commutative semigroups. *Mat. Sb.*, 93:103–128, 1974. MR 48:8663.

[388] M. A. Taiclin. The isomorphism problem for commutative semigroups solved affirmatively. In *Theory of models and its applications*, pages 75–81, Kazah. Gos. Univ., Alma-Ata, 1980. MR 82h:20079.

[389] M. A. Taiclin. On elementary theories of commutative semigroups with cancellation. *Algebra i Logika*, 5(1), 1966. MR 33 # 5486.

[390] M.A. Taiclin. On elementary theories of commutative semigroups. *Algebra i Logika*, 5(4), 1966. MR 35 # 2985.

[391] V.V. Talapov. Polynilpotent Lie algebras that are defined by one defining relation. *Sibirsk. Mat. Zh.*, 23(5):192–204, 1982. MR 84c:17008.

[392] A. Tarski. Equational logic and equational theories of algebras. In *Contribution to Math. Logic.*, pages 278–288, North-Holland, Amsterdam, 1968. MR 38 #5692.

[393] A. Tarski. Some metalogical results concerning the calculus of relations. *J. Symbolic Logic*, 18:188–189, 1953.

[394] A. Tarski and S. Givant. *A formalization of set theory without variables.* AMS, Providence, Rhode Island, 1987. MR 89g:03012.

[395] A. Tarski, A. Mostowski, and R.M. Robinson. *Undecidable Theories.* North-Holland, 1953. MR 15-384h.

[396] W. Taylor. Residually small varieties. *Algebra Universalis*, 2(1):33–53, 1972. MR 47 #3278.

[397] Axel Thue. Probleme über Veränderungen von Zeichenreihen nach gegebenen Regeln. *Norske Vid. Selsk. Skr. , I. Mat. Nat. Kl. , Christiana*, 10:, 1914.

[398] Axel Thue. Über unendliche Zeichenreihen. *Norske Vid. Selsk. Skr. , I. Mat. Nat. Kl. , Christiana*, 7:1–22, 1906.

[399] H. Tietze. Über die topologischen Invarianten mehrdimensionaler Mannigfaltigkeiten. *Monatsh. Math. Phys.*, 19:1–118, 1912.

[400] Ivan Tonov. Just-nonweakly Noetherian varieties of associative algebras with identity. *Pliska Studia Mathematica Bulgarica*, 2:162–165, 1981. MR 83d:16019.

[401] P. G. Trotter. Free completely regular semigroups. *Glasgow Math. J.*, 25:241–254, 1984. MR 85m:20093.

[402] P. G. Trotter. Relatively free bands of groups. *Acta Sci. Math.*, 53:19–31, 1989. MR 91d:20067.

[403] P. G. Trotter. The word problem for relatively free completely regular semigroups. In J. Rhodes, editor, *Monoids and semigroups with applications. Proceedings of the Berkeley Workshop in Monoids. Berkeley, 1989*, pages 198–205, World Scientific, 1991. 92m:20048.

[404] V. A. Ufnarovsky. Combinatorial and asymptotic methods in algebra. In *Itogi Nauki i Techniki. Contemporary mathematical problems. Fundamental directions. 57*, pages 5–177, VINITI, Moscow, 1990. MR 92h:16024.

[405] U. U. Umirbaev. The word problem for center-by-metabelian Lie algebras. *Algebra i Logika*, 23:305–318, 1984. transl. Algebra and Logic 23 (1984), 209–219.

[406] M. K. Valiev. A theorem of G. Higman. *Algebra i Logika*, 7(3):9–22, 1968. MR 41 #1849.

[407] M. Y. Vardi. The complexity of relational query languages. In *Proc. 14th ACM Symp. on Theory of Computing*, pages 137–146, 1982. 87g:68003.

[408] M. R. Vaughan-Lee. Abelian by nilpotent varieties. *Quart. J. Math. Oxford*, 21(2):193–202, 1970. MR 41 #6946.

[409] M. R. Vaughan-Lee. Uncountably many varieties of groups. *Bull. London Math. Soc.*, 2(6):280–286, 1970. MR 43#2054.

[410] M. R. Vaughan-Lee and E. Zel'manov. Upper bounds in the restricted Burnside problem. *J. Algebra*, 162(1):107–146, 1993.

[411] I. B. Volichenko. On one variety of Lie algebras connected with standard identities II. *Izv. Akad. Nauk. SSSR. Ser. Mat.*, (2):22–27, 1980. MR 82e:17011b.

[412] I. B. Volichenko. *On varieties of central-by-metabelian Lie algebras.* Technical Report 16 (96), Math. Inst. of Belorussian Academy of Sciences, Minsk, 1980.

[413] M. V. Volkov. On a class of semigroup pseudovarieties without finite pseudoidentity basis. to appear.

[414] M. V. Volkov. Varieties of associative rings with the amalgamation property. *Mat. Zametki*, 33(1):3–13, 1983. transl. in Math. Notes (1983) 33, no.1, 3-7; MR 84b:16024.

[415] J. Wilson. Finite presentations of pro-p groups and descrete groups. *Invent. Math.*, 105:177–183, 1991.

[416] C. K. Yap. A new lower bound construction for Commutative Thue systems with applications. *J. Symbolic Computation*, 12(1):1–27, 1991. MR 92i:03046.

[417] M. V. Zaitsev. Almost locally representable varieties of Lie algebras. *Mat. Zametki*, 51(3):9–15, 1992. MR 93d:17008.

[418] M. V. Zaitsev. Locally Hopfian varieties of Lie algebras. *Mat. Zametki*, 45(6):56–60, 1989. transl. in Math. Notes. 45 (1989), no 5,6; 469-472; MR 90i:17031.

[419] M. V. Zaitsev. Locally representable varieties of Lie algebras. *Mat. Sb.*, 180(6):249–259, 1989. MR 90k:17009.

[420] M. V. Zaitsev. Locally residually finite varieties of Lie algebras. *Mat. Zametki*, 44(3):352–361, 1988. MR 90b:17040.

[421] A. P. Zamjatin. Decidability of the elementary theories of certain varieties of rings. In *Algebraic systems. Varieties. Lattices of subsystems*, pages 52–74, Ural State University, Sverdlovsk, 1983. MR 85h:03016.

[422] A. P. Zamjatin. A nonabelian group variety has an undecidable elementary theory. *Algebra i Logika*, 17(1):20–27, 1978. MR 80c:03045.

[423] A. P. Zamjatin. Prevarieties of associative rings whose elementary theory is decidable. *Sibirsk. Mat. Zh.*, 19(6):1266–1282, 1978. transl. in Sib. Math. J. , July, 1979; MR 80i:16049".

[424] A. P. Zamjatin. Prevarieties of semigroups whose elementary theory is decidable. *Algebra i Logika*, 12:417–432, 1973. MR52:10401.

[425] E. I. Zel'manov. Engel Lie algebras. *Dokl. Akad. Nauk. SSSR*, 292(2):265–268, 1987. MR 88d:17009.

[426] E. I. Zel'manov. On additional laws in the burnside problem for periodic groups. *Inter. J. Algebra Comput.*, 3(4):583–600, 1993.

[427] E. I. Zel'manov. Solution of the restricted Burnside problem for 2-groups. *Mat. Sb.*, 182(4):568–592, 1991. MR 93a:20063.

[428] E. I. Zel'manov. Solution of the restricted Burnside problem for groups of odd exponent. *Izv. Akad. Nauk. SSSR. Ser. Mat.*, 54(1):42–59, 1990. transl. in Math. USSR-Izv. 36 (1991), no.1, 41-60; MR 91i:20037.

[429] A. I. Zhukov. Reduced systems of defining relations in non-associative algebras. *Mat. Sb.*, 27(5):267–280, 1950. MR 12-314d.

[430] B. I. Zilber. An example of two elementary equivalent but non-isomorphic finitely generated metabelian groups. *Algebra i Logika*, 10(3):309–315, 1971. MR 45 #49.

[431] A. I. Zimin. Blocking sets of terms. *Mat. Sb.*, 119(3):363–375, 1982. MR 84d:20072.

[432] A. I. Zimin. On semigroup varieties with some finiteness conditions. preprint, VINITI, # 4519-80 DEP; 1980.

[433] A. I. Zimin. On varieties defined by one balanced identity. In *Proc. 2 All-Union Symposium*, Ural State University, Sverdlovsk, 1978.

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