Decidable and undecidable problems related to completely 0-simple semigroups

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Abstract

The undecidable problems of the title are concerned with the question :- is a given finite semigroup embeddable in a given type of completely 0-simple semigroups? It is shown, for example, that the embeddability of a (finite) 3-nilpotent semigroup in a finite completely 0-simple semigroup is decidable yet such embeddability is undecidable for a (finite) 4-nilpotent semigroup. As well the membership of the pseudovariety generated by finite completely 0-simple semigroups (or alternatively by finite Brandt semigroups) over groups from a pseudovariety of groups with decidable membership is shown to be decidable.

1 Introduction

One of the most important classes of semigroups is the class of 0-simple finite semigroups. Recall that a semigroup is called 0-simple if it does not have ideals except itself and possibly $\{0\}$. Every finite semigroup may be obtained from 0-simple semigroups by a sequence of ideal extensions. The classic theorem of Sushkevich [**3**] (which was arguably the first theorem in the algebraic theory of semigroups) shows that finite 0-simple semigroups have the following structure. Let G be a group, let L and R be two sets and let $P = (p_{r\ell})$ be an $|R| \times |L|$-matrix over the group G with 0 adjoined such that every row and every column of P contains a non-zero element. Let $M^0(G; L, R, P)$ be the set $(L \times G \times R) \cup \{0\}$ with the following binary operation:

$$(\ell, g, r)(\ell', g', r') = \begin{cases} (\ell, gp_{r\ell'}g', r') & \text{if } p_{r\ell'} \neq 0; \\ 0 & \text{if } p_{r\ell'} = 0. \end{cases}$$

Then $M^0(G; L, R, P)$ is a 0-simple semigroup, and every finite 0-simple semigroup is isomorphic to $M^0(G; L, R, P)$ for some G, L, R, P (necessarily finite). Not every infinite 0-simple semigroup can be described in this way. By the theorem of Rees-Sushkevich a (not necessarily finite) semigroup is isomorphic to a semigroup of the form $M^0(G; L, R, P)$ if and only if it is 0-simple and has a minimal non-zero idempotent. Such semigroups also play a very important role in semigroup theory; they are called *completely 0-simple*.

simple semigroup over the group G with sandwich matrix P. If $L = R$ and P is the identity matrix then $M^0(G; L, R, P)$ is called the $|L| \times |L|$ *Brandt semigroup over the group G*. It is denoted by $B_L(G)$. Brandt semigroups are precisely the completely 0-simple inverse semigroups. As one can see the structure of finite 0-simple semigroups and finite 0-simple inverse semigroups is extremely clear. This is why the class of finite 0-simple semigroups is considered to be one of the most transparent classes of semigroups. If $|L| = m, |R| = n$ we shall write $M^0(G; m, n, P)$ and $B_m(G)$ instead of $M^0(G; L, R, P)$ and $B_L(G)$.

Thus the following results of Kublanovsky were very unexpected.

Theorem 1.1 *(Kublanovsky, 1994). The set of all subsemigroups of (finite) completely 0-simple semigroups is not recursive. The set of subsemigroups of direct products of (finite) completely 0-simple semigroups is also not recursive.*

Kublanovsky uses the unsolvability of the uniform word problem for finite groups (Slobodskoii [**14**]). Recall that if $\mathcal{K}$ is a class of universal algebras of some type then the *uniform word problem* for this class asks whether there exists an algorithm which, given a system of relations $u_i = v_i, i = 1, 2, ..., n$ and a relation $u = v$, where all u_i, v_i, u, v are words over some alphabet X, decides if u is equal to v in every $X-$generated algebra from $\mathcal{K}$ in which all relations $u_i = v_i$ hold.

A class of universal algebras is called a *pseudovariety* if it is closed under taking subsemigroups, homomorphic images and finite direct products.

There exists an important connection between the uniform word problem in a pseudovariety and finite partial algebras. This connection was found by Evans (see [**5**] or [**10**; Connection 2.2]). Recall that a *partial universal algebra* is a set with partial operations. If A is a partial universal algebra, B is a universal algebra of the same type, $A \subseteq B$ and every operation of A is a restriction of the corresponding operation of B then we say that the partial algebra A is *embedded* in the algebra B.

Theorem 1.2 *(Evans, [**5**]). Let $\mathcal{V}$ be a pseudovariety of universal algebras. The uniform word problem is solvable in $\mathcal{V}$ if and only if the set of finite partial algebras embeddable in algebras from $\mathcal{V}$ is recursive.*

Using the ideas of the proof of Theorem 1.1 we prove here the following result which is stronger than Theorem 1.1.

Recall that a semigroup S with a zero element is called *n-nilpotent* if the product of any n elements of S is equal to zero.

Theorem 1.3 *For every pseudovariety of groups $\mathcal{V}$ and for any natural numbers n, m the following conditions are equivalent.*

1. *The uniform word problem in $\mathcal{V}$ is solvable.*
2. *The set of finite subsemigroups of completely 0-simple semigroups over groups in $\mathcal{V}$ is recursive.*

4. *The set of finite 4-nilpotent subsemigroups of finite completely 0-simple semigroups over groups in $\mathcal{V}$ is recursive.*

5. *The set of finite 3-nilpotent subsemigroups of finite Brandt semigroups over groups in $\mathcal{V}$ is recursive.*

6. *The set of finite 4-nilpotent subsemigroups of direct products of finite completely 0-simple semigroups over groups in $\mathcal{V}$ is recursive.*

7. *The set of finite 3-nilpotent subsemigroups of direct products of finite Brandt semigroups over groups in $\mathcal{V}$ is recursive.*

8. *The set of finite subsemigroups of $m \times n$ completely 0-simple semigroups over groups in $\mathcal{V}$ is recursive provided $m \geq 3, n \geq 3$.*

9. *The set of finite subsemigroups of $m \times m$ Brandt semigroups over groups in $\mathcal{V}$ is recursive provided $m \geq 3$.*

10. *The set of finite 4-nilpotent subsemigroups of $m \times n$ completely 0-simple semigroups over groups in $\mathcal{V}$ is recursive provided $m \geq 4, n \geq 4$.*

11. *The set of finite 3-nilpotent subsemigroups of $m \times m$ Brandt semigroups over groups in $\mathcal{V}$ is recursive provided $m \geq 3$.*

12. *The set of finite 4-nilpotent subsemigroups of direct products of $m \times n$ completely 0-simple semigroups over groups in $\mathcal{V}$ is recursive provided $m \geq 4, n \geq 4$.*

13. *The set of finite 3-nilpotent subsemigroups of direct products of $m \times m$ Brandt semigroups over groups in $\mathcal{V}$ is recursive provided $n \geq 3$.*

A substantial amount of information is known about pseudovarieties of groups with undecidable uniform word problem (see [**10**]). The undecidability of the uniform word problem in the class of all groups was proved by Novikov [**12**]. The undecidability of the uniform word problem in the class of finite groups has been proved by Slobodskoii [**14**]. The following results of Kharlampovich imply that many other pseudovarieties of groups also have undecidable uniform word problem.

Let $\mathcal{G}$ be the class of all finite groups. Let $\mathbf{N}$ be the class of all nilpotent groups, $\mathbf{N}_k$ be the class of all nilpotent groups of step $\leq k$, let $\mathbf{N}_2\mathbf{A}$ be the class of all extensions of groups from $\mathbf{N}_2$ by Abelian groups and let $Z\mathbf{N}_2\mathbf{A}$ be the class of all groups G such that the factor of G over its centre belongs to $\mathbf{N}_2\mathbf{A}$. Let $\mathbf{X}$ be an arbitrary variety of groups such that $Z\mathbf{N}_2\mathbf{A} \subseteq \mathbf{X}$.

Theorem 1.4 *(Kharlampovich, [**8**], [**9**]). The uniform word problem is undecidable for the following classes of groups: $\mathcal{G}$; $\mathbf{N}$; $\mathcal{G} \cap \mathbf{N}$; $\mathcal{G} \cap \mathbf{X}$; $\mathbf{N} \cap \mathbf{X}$; $\mathbf{N} \cap \mathbf{X} \cap \mathcal{G}$.*

the sets is undecidable.

On the other hand the universal word problem is solvable, for example, in any variety of groups where every finitely generated group is residually finite and also in the set of finite groups from this variety (see [**10**]). In particular, this problem is solvable in the variety $\mathbf{N}_k$ and in the pseudovariety $\mathcal{G} \cap \mathbf{N}_k$. It is also solvable in the pseudovariety of all finite groups of any fixed exponent (this is a corollary of the Theorem of Zelmanov giving a positive solution of the Restricted Burnside Problem, see [**10**]). Thus, in particular, the set of all finite subsemigroups of finite completely 0-simple semigroups over groups in $\mathcal{G} \cap \mathbf{N}_k$ is recursive for any fixed $k \geq 1$.

Theorem 1.3 will be proved in the next section. This theorem together with Theorem 1.4 imply that the set of finite 4-nilpotent subsemigroups of finite 0-simple semigroups is undecidable. We will prove that embeddability of (finite) 3-nilpotent semigroups in (finite) completely 0-simple semigroups is decidable.

Theorems 1.3 and 1.4 show that the set of subsemigroups of finite direct products of finite 0-simple semigroups is undecidable. It turns out that if we close this set under homomorphic images then we obtain a decidable collection of finite semigroups. This collection is closed under subsemigroups, finite direct products and homomorphisms so it is a *pseudovariety* of finite semigroups [**4**] generated by the set of all finite 0-simple semigroups. Let $\mathcal{V}$ be a pseudovariety of groups. Define $\mathcal{CS}^0(\mathcal{V})$ and $\mathcal{B}(\mathcal{V})$ to be the pseudovarieties generated respectively by finite 0-simple semigroups over groups from $\mathcal{V}$ and finite Brandt semigroups over groups from $\mathcal{V}$. In the final section the following theorem will be proved.

Theorem 1.5 *Let $\mathcal{V}$ be a pseudovariety of groups with decidable membership. Then $\mathcal{CS}^0(\mathcal{V})$ and $\mathcal{B}(\mathcal{V})$ have decidable membership. Moreover the membership problems in $\mathcal{CS}^0(\mathcal{V})$ and $\mathcal{B}(\mathcal{V})$ are polynomial time reducible to the membership problem of $\mathcal{V}$.*

In section 3 we also determine a sequences of identities that ultimately define $\mathcal{CS}^0$, $\mathcal{B}$ and some of their sub-pseudovarieties. As well we show that $\mathcal{CS}^0(\mathcal{V})$ is the semidirect product of the pseudovariety of semilattices of groups over $\mathcal{V}$ by the pseudovariety of right zero semigroups.

2 Proof of Theorem 1.3

A set A with a partial binary operation $\cdot$ on it and a distinguished element 1 such that $1 \cdot a = a \cdot 1 = a$ for all $a \in A$ will be called a *partial group*.

Let us take a pseudovariety of groups $\mathcal{V}$ in which the uniform word problem is unsolvable. We have to prove that each of the sets of finite semigroups listed in conditions $2 - 13$ of Theorem 1.3 is not recursive. This will be done in a series of reductions. We shall prove even more.

Proposition 2.1 *Let $\mathcal{V}$ be a pseudovariety of groups in which the uniform word problem is not solvable.*

subsemigroups of direct products of arbitrary completely 0-simple semigroups over groups from $\mathcal{V}$ is not recursive.

(b) Every set of finite 3-nilpotent semigroups which contains the set of finite 3-nilpotent subsemigroups of 3×3 Brandt semigroups over groups from $\mathcal{V}$ and is contained in the set of finite subsemigroups of direct products of arbitrary Brandt semigroups over groups from $\mathcal{V}$ is not recursive.

(c) Every set of finite semigroups which contains the set of all finite 4-nilpotent subsemigroups of 4×4 Brandt semigroups over groups from $\mathcal{V}$ and is contained in the set of finite subsemigroups of direct products of arbitrary completely 0-simple semigroups over groups from $\mathcal{V}$ is not recursive.

First of all notice that by Theorem 1.2 the following problem is undecidable:

> Given a finite partial group A, decide whether or not A is embeddable in a group from $\mathcal{V}$.

Let us call a partial group A *symmetric* if for every $a \in A$ there exists a unique element $a' \in A$ such that $aa' = a'a = 1$. A partial group $B > A$ is called a *symmetric extension* of A if B is symmetric and for every element $b \in B$ either b or b' belongs to A. Thus the order of every symmetric extension of A does not exceed $2|A|$ so every finite partial group has only finitely many symmetric extensions and all of them may be effectively listed. It is clear that a partial group A is embeddable in a group if and only if one of its symmetric extensions is embeddable in this group. Therefore the following problem is undecidable:

> Given a finite symmetric partial group A, decide whether or not A is embeddable in a group from $\mathcal{V}$.

Let B be a partial group and let A be a subset of B. For every $i = 1, 2 \ldots$ let us define a subset A^i of B. Let $A^1 = A$ and for every $i > 1$ let $A^{i+1} = A^i \cdot A$. We shall call B an *extension* of A *of rank* k if:

- 1 is the identity element for B.
- for all numbers i and j between 1 and k such that $i + j \leq k$ and for every pair of elements $x \in A^i$ and $y \in A^j$ the product xy exists in B and belongs to A^{i+j}.
- all products $x \cdot y$ where $x \in A^i \setminus \bigcup_{s=1}^{i-1} A^s$, $y \in A^j \setminus \bigcup_{s=1}^{j-1} A^s$ and $i + j > k$ are undefined unless $x = 1$ or $y = 1$.
- For every $x \in A^i, y \in A^j, z \in A^m$ such that $i + j + m \leq k$ both $(xy)z$ and $x(yz)$ are defined and $(xy)z = x(yz)$.
- $B = \bigcup_{s=1}^{k} A^s$.

It is also clear that for any natural number k a partial group A is embeddable in a group G if and only if some extension of A of rank k is embeddable in G.

Let $E_n = (e_{r\ell})$ be the identity $n \times n$ matrix, let C be any partial group and let $B_n(C)$ be the set $\mathbf{n} \times C \times \mathbf{n}$ (where $\mathbf{n}$ stands for the set $\{1, 2, \ldots, n\}$) with the following partial operation:

$$(\ell, g, r)(\ell', g', r') = \begin{cases} (\ell, gg', r') & \text{if } e_{r\ell'} = 1 \text{ and } gg' \text{ is defined ;} \\ 0 & \text{if } e_{r\ell'} = 0. \end{cases}$$

Lemma 2.2 *If a partial group A is embeddable in a group G then for every $k > 1,\ n \geq 1$ there exists an extension C of A of rank k such that $B_n(C)$ is embeddable in the Brandt semigroup $B_n(G)$.*

Proof. It is enough to compare the definitions of $B_n(G)$ and $B_n(C)$.

Now let A be a finite symmetric partial group, let A' be extension of A of rank 2 and let A'' be any extension of A of rank 3. Let us define the following three sets $S_1 \subseteq B_3(A'), S_2 \subseteq B_3(A'), S_3 \subseteq B_4(A'')$.

$$\begin{aligned} S_1 = &(\{1\} \times A \times \{2\}) \cup (\{2\} \times A \times \{3\}) \cup (\{1\} \times A' \times \{3\}) \\ &\cup \{(1,1,1)\} \cup \{(2,1,2)\} \cup \{(3,1,3)\} \cup \{0\}; \end{aligned}$$

$$S_2 = (\{1\} \times A \times \{2\}) \cup (\{2\} \times A \times \{3\}) \cup (\{1\} \times A' \times \{3\}) \cup \{0\};$$

$$\begin{aligned} S_3 = &(\{1\} \times A \times \{2\}) \cup (\{2\} \times A \times \{3\}) \cup (\{3\} \times A \times \{4\}) \\ &\cup (\{1\} \times (A^2 \cup A) \times \{3\}) \cup (\{2\} \times (A^2 \cup A) \times \{4\}) \cup (\{1\} \times A'' \times \{4\}) \cup \{0\}. \end{aligned}$$

It is easy to check that each of these sets is a subsemigroup of $B_3(A')$ or $B_4(A'')$ respectively, that is the operation is everywhere defined and associative on each of these sets.

Notice that the semigroup S_2 is 3-nilpotent and the semigroup S_3 is 4-nilpotent. The semigroup S_1 is an extension of S_2 and $S_1 \backslash S_2$ consists of three idempotents $(1,1,1)$, $(2,1,2)$ and $(3,1,3)$ which we shall call e_1, e_2 and e_3 respectively. It is easy to check that e_1 is a left unit for $\{1\} \times A \times \{2\}$ and for $\{1\} \times A' \times \{3\}$, e_2 is a left unit for $\{2\} \times A \times \{3\}$ and a right unit for $\{1\} \times A \times \{2\}$ and e_3 is a right unit for $\{2\} \times A \times \{3\}$ and $\{1\} \times A' \times \{3\}$.

Lemma 2.3 *(a) If S_1 is a subsemigroup of a direct product of completely 0-simple semigroups over groups $G_i, i \in I$ then A is embeddable in the direct product $\prod_i G_i$.*

(b) If S_2 is a subsemigroup of a direct product of Brandt semigroups over groups $G_i, i \in I$ then A is embeddable in the direct product $\prod_i G_i$.

(c) If S_3 is a subsemigroup of a direct product of completely 0-simple semigroups over groups $G_i, i \in I$ then A is embeddable in the direct product $\prod_i G_i$.

$\{3\}) \neq \{0\}$. Then $\phi(e_i) \neq 0$. Indeed, if $\phi(e_1) = 0$ then $\phi(\{1\} \times A' \times \{3\}) = \{0\}$ because e_1 is a left unit for $\{1\} \times A' \times \{3\}$. If $\phi(e_3) = 0$ then $\phi(\{1\} \times A' \times \{3\}) = \{0\}$ because e_3 is a right unit for $\{1\} \times A' \times \{3\}$. If $\phi(e_2) = 0$ then $\phi(\{1\} \times A \times \{2\}) = \{0\} = \phi(\{2\} \times A \times \{3\})$ because e_2 is a right unit for $\{1\} \times A \times \{2\}$ and a left unit for $\{2\} \times A \times \{3\}$. Therefore $\phi(\{1\} \times A' \times \{3\}) = \phi(\{1\} \times A \times \{2\})\phi(\{2\} \times A \times \{3\}) = 0$.

Let $\phi(e_i) = (\ell_i, g_i, r_i)$. Since $e_1(1, a, 2)e_2 = (1, a, 2), e_2(2, a, 3)e_3 = (2, a, 3)$ and $e_1(1, a', 3)e_3 = (1, a', 3)$ for every $a \in A$ and every $a' \in A'$ we have that

$$(\ell_1, g_1, r_1)\phi(1, a, 2)(\ell_2, g_2, r_2) = \phi(1, a, 2),$$

$$(\ell_2, g_2, r_2)\phi(2, a, 3)(\ell_3, g_3, r_3) = \phi(2, a, 3),$$

$$(\ell_1, g_1, r_1)\phi(1, a', 3)(\ell_3, g_3, r_3) = \phi(1, a', 3).$$

Therefore

$$\phi(1, a, 2) \in \{(\ell_1, f(a), r_2), 0\},$$

$$\phi(2, a, 3) \in \{(\ell_2, g(a), r_3), 0\},$$

$$\phi(1, a', 3) \in \{(\ell_1, h(a'), r_3), 0\}$$

for some (possibly partial) functions $f, g : A \to G$ and $h : A' \to G$.

Notice that since $e_2e_2 = e_2 \neq 0$, $p_{r_2\ell_2} \neq 0$. It is known (see [**3**]) that if we multiply all entries of a row (a column) of the matrix P by any element $x \in G$ from the left (right) and obtain a new matrix P' then the semigroups $M^0(G; L, R, P)$ and $M^0(G; L, R, P')$ are isomorphic. Thus we can assume that the entry $p_{r_2\ell_2}$ of the matrix P is equal to 1.

Since $\phi(\{1\} \times A' \times \{3\}) \neq \{0\}$ there exists $z \in A'$ such that $\phi(1, z, 3) \neq 0$. Since $z = 1 \cdot z = z \cdot 1$ we have that

$$0 \neq \phi(1, z, 3) = \phi(1, 1, 2)\phi(2, z, 3) = \phi(1, z, 2)\phi(2, 1, 3). \tag{1}$$

Now suppose that there exists an element $c \in A'$ such that $\phi(1, c, 3) = 0$. By definition of an extension of rank 2, $c = xy$ for some $x, y \in A$. Then $0 = \phi(1, c, 3) = \phi(1, x, 2)\phi(2, y, 3)$. Since $p_{r_2\ell_2} \neq 0$, one of the elements $\phi(1, x, 2)$ or $\phi(2, y, 3)$ is equal to 0. Let us suppose, without loss of generality that $\phi(1, x, 2) = 0$.

Since A is symmetric there exists $x^{-1} \in A$ such that $xx^{-1} = 1$ (the identity element of A). Then $\phi(1, 1, 3) = \phi(1, x, 2)\phi(2, x^{-1}, 3) = 0$. Therefore $0 = \phi(1, 1, 3) = \phi(1, 1, 2)\phi(2, 1, 3)$. Hence one of the elements $\phi(1, 1, 2)$ or $\phi(2, 1, 3)$ is equal to 0. In both cases

$$\phi(1, z, 3) = \phi(1, 1, 2)\phi(2, z, 3) = \phi(1, z, 2)\phi(2, 1, 3) = 0$$

which contradicts (1).

Thus $\phi(1, c, 3) \neq 0$ for every $c \in A'$. From this, one can easily deduce that $\phi(1, a, 2) \neq 0$ and $\phi(2, a, 3) \neq 0$ for every $a \in A$. Therefore

$$\phi(1, a, 2) = (\ell_1, f(a), r_2), \phi(2, a, 3) = (\ell_2, g(a), r_3), \phi(1, b, 3) = (\ell_1, h(b), r_3)$$

$$(\ell_1, f(a), r_2)(\ell_2, g(b), r_3) = (\ell_1, f(a)g(b), r_3).$$

On the other hand since ϕ is a homomorphism we must have

$$(\ell_1, f(a), r_2)(\ell_2, g(b), r_3) = (\ell_1, h(ab), r_3).$$

Notice that here we used the fact that ab is defined in A'. Therefore for every a, b in A we have

$$f(a)g(b) = h(ab).$$

¿From this we can deduce that $f(1)g(a) = h(a) = f(a)g(1)$ for every $a \in A$. Therefore $f(a) = h(a)g(1)^{-1}, g(a) = f(1)^{-1}h(a)$ and $h(a)g(1)^{-1}f(1)^{-1}h(b) = h(ab)$. Thus

$$h(a)(g(1)^{-1}f(1)^{-1})h(b)(g(1)^{-1}f(1)^{-1}) = h(ab)(g(1)^{-1}f(1)^{-1}).$$

We conclude that the mapping $\psi : A \to G$ which takes a to $h(a)(g(1)^{-1}f(1)^{-1})$ is a homomorphism of A.

Now suppose that the homomorphism ϕ separates two elements $(1, a, 3)$ and $(1, b, 3)$ with $a, b \in A$. Therefore the image of one of these elements is not 0. Then as we have shown before the images of all elements from $\{1\} \times A \times \{3\}$ are non-zero. Therefore

$$\phi(1, a, 3) = (\ell_1, h(a), r_3) \neq (\ell_1, h(b), r_3) = \phi(1, b, 3).$$

Hence $h(a) \neq h(b)$ and therefore ψ is a homomorphism of A into G which separates a and b.

Suppose now that S_1 is a subsemigroup of the direct product $\prod_i M^0(G_i; L_i, R_i, P_i)$ with $G_i \in \mathcal{V}$. Then every two distinct elements $(1, a, 3)$ and $(1, b, 3)$ with $a, b \in A$ are separated by a projection ϕ_i into $M^0(G_i; L_i, R_i, P_i)$ for some i. Therefore a and b are separated by some homomorphism from A into G_i. Thus every two distinct elements $a, b \in A$ can be separated by a homomorphism into a group from $\mathcal{V}$. Hence A is embeddable in a direct product of groups from $\mathcal{V}$. Since A is finite, we can suppose that this direct product has finitely many factors. Since $\mathcal{V}$ is closed under taking finite direct products, A is embeddable in a group of $\mathcal{V}$.

(b) Let ϕ be a homomorphism from S_2 into a Brandt semigroup $B_n(G)$ such that $\phi(1, z, 3) \neq 0$ for some $z \in A$. The same argument as above shows that $\phi(1, b, 3) \neq 0$ for every $b \in A'$ (when we proved this fact in part (a), we did not use elements e_1, e_2 and e_3). This implies that $\phi(1, a, 2) \neq 0$, $\phi(2, a, 3) \neq 0$, $\phi(1, b, 3) \neq 0$ for every $a \in A, b \in A'$.

Then for every $a \in A$ we have

$$\phi(1, a, 2) = (\ell_1(a), f(a), r_1(a)),$$

$$\phi(2, a, 3) = (\ell_2(a), g(a), r_2(a)),$$

$$\phi(1, b, 3) = (\ell_3(b), h(b), r_3(b))$$

Since ϕ is a homomorphism and since every entry of the matrix of a Brandt semigroup is either 0 or 1, we have $f(a)g(b) = h(ab)$ for every $a, b \in A$ such that $ab \in A'$.

Notice also that since $1 \cdot a = a = a \cdot 1$ for every $a \in A$ we have:

$$\begin{aligned}(\ell_3(a), h(a), r_3(a)) &= (\ell_1(a), f(a), r_1(a))(\ell_2(1), g(1), r_2(1)) \\ &= (\ell_1(1), f(1), r_1(1))(\ell_2(a), g(a), r_2(a)).\end{aligned}$$

Therefore $r_3(a) = r_2(a) = r_2(1)$ and $\ell_3(a) = \ell_1(a) = \ell_1(1)$ for every $a \in A$. Next suppose $b \in A' \backslash A$, so $b = xy$ for some $x, y \in A$. Hence

$$(\ell_3(b), h(b), r_3(b)) = (\ell_1(x), f(x), r_1(x))(\ell_2(y), g(y), r_2(y))$$

and we have $r_3(b) = r_2(1), \ell_3(b) = \ell_1(1)$.

Now we can complete the proof in the same way as in part (a).

(c) Let ϕ be a homomorphism from S_3 into $M^0(G; L, R, P)$ such that $\phi(1, z, 4) \neq 0$ for some $z \in A$. Then in the same way as above we can prove that $\phi(1, a, 2)$, $\phi(2, a, 3)$, $\phi(3, a, 4)$, $\phi(1, b, 3)$, $\phi(2, b, 4)$, $\phi(1, c, 4)$ are not equal to 0 for every $a \in A$, $b \in A \cup A^2$, $c \in A''$ and that $\phi(2, x, 4) = (\ell(x), k(x), r)$ for all $x \in A \cup A^2$. It follows that

$$\phi(3, x, 4) = (\ell_1(x), g(x), r) \text{ and likewise } \phi(1, y, 3) = (\ell, f(y), r_1),\ \phi(1, t, 4) = (\ell, h(t), r)$$

where r, r_1 and ℓ do not depend on x, y or t, $g : A \to G$, $f : A \cup A^2 \to G$, $h : A'' \to G$ are some mappings.

Then for every $a, b \in A$ we have

$$f(a)p_{r_1\ell_1(b)}g(b) = h(ab). \tag{2}$$

As we mentioned above, we can multiply each column of the matrix P by an element of G without changing the completely 0-simple semigroup (the resulting semigroup will be isomorphic to the original one). Therefore we can assume that the r_1th row of P consists of zeroes and ones. Thus each of $p_{r_1\ell_1(b)}$ in (2) is equal to 1 (it cannot be equal to 0 because otherwise $\phi(1, ab, 4)$ would be equal to 0). Therefore we have $f(a)g(b) = h(ab)$ and we can complete the proof in the same way as above.

The lemma is proved.

Proof of Proposition 2.1. Let A be any finite symmetric partial group. If A is embeddable in a group G from $\mathcal{V}$ then some extension A' of rank 2 and some extension A'' of rank 3 is embedded in G. Then by Lemma 2.2, S_1 and S_2 are embeddable in $B_3(G)$ and S_3 is embeddable in $B_4(G)$.

On the other hand suppose that A is not embeddable in a group from $\mathcal{V}$. Then by Lemma 2.3, S_1 and S_3 are not embeddable in a direct product of completely 0-simple semigroups over groups from $\mathcal{V}$ and S_2 is not embeddable in a direct product of Brandt semigroups over groups from $\mathcal{V}$.

Since the problem of whether a finite symmetric partial group is embeddable in a group from $\mathcal{V}$ is undecidable, we conclude that every set of finite semigroups mentioned in Proposition 2.1 is not recursive.

form word problem is decidable in the pseudovariety $\mathcal{V}$ then each of the sets of finite semigroups mentioned in Theorem 1.3 is decidable.

We shall use the following connection between the uniform word problem and the universal theory of a pseudovariety (see [**10**], Connection 2.2). Recall that the *universal theory* of a class $\mathcal{V}$ is the set of all universal formulae that hold in $\mathcal{V}$.

Lemma 2.4 *Let $\mathcal{V}$ be a pseudovariety of universal algebras. The uniform word problem in $\mathcal{V}$ is solvable if and only if the universal theory of $\mathcal{V}$ is decidable.*

Now let $\mathcal{V}$ be a pseudovariety of groups with solvable uniform word problem. Then by Lemma 2.4 the universal theory of $\mathcal{V}$ is decidable.

Let S be a finite semigroup. If S does not have a zero, we can formally add zero 0 to S and obtain the semigroup S^0. It is clear that every homomorphism ϕ from S into a completely 0-simple semigroup such that $|\phi(S)| \neq 1$ can be uniquely extended to a homomorphism of S^0. Thus we shall always assume that S contains zero.

A pair of equivalence relations $\lambda, \rho \subseteq S \times S$ will be called *admissible* if

1. $(x, xy) \in \lambda$ for every $x, y \in S$ provided $xy \neq 0$;
2. $(x, yx) \in \rho$ for every $x, y \in S$ provided $yx \neq 0$;
3. for every $x, y, z, t \in S$ if $(x, z) \in \rho$, $(y, t) \in \lambda$ and $xy = 0$ then $zt = 0$.

For example, let ϕ be an embedding of S into a completely 0-simple semigroup $M^0(G; L, R, P)$. Then the preimage of 0 is 0. If $\phi(x) \neq 0$ for some $x \in S$ then $\phi(x) = (\ell(x), g(x), r(x))$ where $\ell(x) \in L$, $g(x) \in G$, $r(x) \in R$. Let us define two equivalence relations λ and ρ:

$$(x, y) \in \lambda \text{ if and only if } \ell(x) = \ell(y) \text{ or } x = y = 0;$$

$$(x, y) \in \rho \text{ if and only if } r(x) = r(y) \text{ or } x = y = 0.$$

Then it is easy to check that the pair (λ, ρ) is admissible.

It is also easy to check that the following formula $\theta(\lambda, \rho)$ holds in G:

$$\begin{aligned}\theta(\lambda, \rho) \rightleftharpoons\ & \exists_{x^\rho \in S/\rho, y^\lambda \in S/\lambda, xy \neq 0} p_{x^\rho y^\lambda} \quad \exists_{x \in S, x \neq 0} g(x) \\ & \textstyle\bigwedge_{x,y \in S, xy \neq 0} g(x) p_{x^\rho y^\lambda} g(y) = g(xy) \ \wedge\ \bigwedge_{x,y \in S, x^\lambda = y^\lambda, x^\rho = y^\rho, x \neq y} g(x) \neq g(y).\end{aligned}$$

Now suppose that the formula $\theta(\lambda, \rho)$ holds in a group G for some admissible pair (λ, ρ). Let α be a symbol which does not belong to S/λ or S/ρ. Let $L = S/\lambda \cup \{\alpha\}$ if S has a non-zero element x such that $xy = 0$ for every $y \in S$ otherwise let $L = S/\lambda$. Let $R = S/\rho \cup \{\alpha\}$ if S has a non-zero element x such that $yx = 0$ for every $y \in S$ otherwise let $R = S/\rho$. Consider the following $R \times L$-matrix Q:

$$q_{x^\rho y^\lambda} = \begin{cases} p_{x^\rho y^\lambda} & \text{if } xy \neq 0; \\ 0 & \text{if } xy = 0; \end{cases}$$
$$q_{\alpha y^\lambda} = q_{\alpha\alpha} = q_{x^\rho \alpha} = 1.$$

$y \in S$ $xy \neq 0$ then $q_{x^\rho y^\lambda} = p_{x^\rho y^\lambda} \neq 0$. Similarly every column of Q has a non-zero element. Thus the semigroup $M^0(G; L, R, Q)$ is completely 0-simple.

Now define the following map $\phi : S \to M^0(G; L, R, Q)$:

$$\phi(x) = \left\{ \ (x^\lambda, g(x), x^\rho) \text{ if } x \neq 0; 0 \text{ if } x = 0. \right.$$

Formula $\theta\,(\lambda, \rho)$ and the definition of Q show that for every $x, y \in S$, $xy \neq 0$

$$g(x) q_{x^\rho y^\lambda} g(y) = g(x) p_{x^\rho y^\lambda} g(y) = g(xy).$$

Since the pair (λ, ρ) is admissible $x^\lambda = (xy)^\lambda, (xy)^\rho = y^\rho$ for every $x, y \in S$ such that $xy \neq 0$. This implies that $\phi(xy) = \phi(x)\phi(y)$ for every $x, y \in S$, $xy \neq 0$. If $xy = 0$ then by the definition of Q we have that $q_{x^\rho y^\lambda} = 0$ so $\phi(x)\phi(y) = 0 = \phi(xy)$. Thus ϕ is a homomorphism. Since for every $x, y \in S$, where $x \neq y$ either $x^\lambda \neq y^\lambda$ or $x^\rho \neq y^\rho$ or $g(x) \neq g(y)$, this map is one-to-one. Therefore ϕ is an embedding of S into the completely 0-simple semigroup $M^0(G; L, R, Q)$.

Thus a finite semigroup S is embeddable into a completely 0-simple semigroup over a group from $\mathcal{V}$ if and only if for some admissible pair of equivalences λ, ρ the formula $\theta(\lambda, \rho)$ holds in some group of $\mathcal{V}$. Notice that the negation of formula $\theta(\lambda, \rho)$ is a universal formula, so we can algorithmically check whether or not $\theta(\lambda, \rho)$ holds in $\mathcal{V}$. This gives us an algorithm to check whether or not S is a subsemigroup of a completely 0-simple semigroup over a group from $\mathcal{V}$ and proves conditions 2 and 4 of Theorem 1.3.

In order to check whether or not a finite semigroup S is a subsemigroup of a direct product of completely 0-simple semigroups over groups from $\mathcal{V}$ it is enough to check for every pair of distinct elements a, b and every congruence σ on S such that $(a, b) \notin \sigma$ whether or not S/σ is embeddable in a completely 0-simple semigroup. Since S is finite there are only finitely many triples (a, b, σ) to check. This proves condition 6 of Theorem 1.3.

Now let us fix natural numbers m and n and consider an embedding $\phi : x \to (\ell(x), g(x), r(x))$ of S in a completely 0-simple semigroup $M^0(G; m, n, P)$. Then the relations λ and ρ defined as above form an admissible pair and $|S/\lambda| \leq m$, $|S/\rho| \leq n$. Suppose that S contains an element $x \neq 0$ such that $xy = 0$ for every $y \in S$. Then $p_{r(x)\ell(y)} = 0$ for every $y \neq 0$ from S. Therefore L contains an element which is not equal to $\ell(y)$ for any $y \in S$. Thus $|S/\lambda| \leq m - 1$. Similarly, if S contains an element x such that $yx = 0$ for every $y \in S$ then $|S/\rho| \leq n - 1$. As above the formula $\theta(\lambda, \rho)$ holds in the group G.

On the other hand let (λ, ρ) be an admissible pair of equivalence relations on S. Suppose that

(a) $|S/\lambda| \leq m$ if for every $x \in S$, $x \neq 0$ there exists $y \in S$ such that $xy \neq 0$ or $|S/\lambda| \leq m - 1$ otherwise;

(b) $|S/\rho| \leq n$ if for every $x \in S$, $x \neq 0$ there exists $y \in S$ such that $yx \neq 0$ or $|S/\rho| \leq n - 1$ otherwise;

Then construct the sets L and R and the $R \times L$-matrix Q as above. As before, the semigroup $M^0(G; L, R, Q)$ is completely 0-simple and S is embeddable in this semigroup. Notice that by definition $|L| \leq m$, $|R| \leq n$.

on S satisfying the conditions (a) and (b) and such that the formula $\theta(\lambda, \rho)$ holds in a group of $\mathcal{V}$.

A semigroup S is embeddable in a direct product of $m \times n$ completely 0-simple semigroups over groups from $\mathcal{V}$ if and only if for every pair of distinct elements (a, b) in S there exists a congruence σ on S such that S/σ is embeddable in an $m \times n$ completely 0-simple semigroup over groups from $\mathcal{V}$.

This proves conditions 8, 10 and 12 of Theorem 1.3.

In order to deal with Brandt semigroups, instead of arbitrary completely 0-simple semigroups we have to change the definition of an admissible pair of equivalence relations. We shall call a pair (λ, ρ) *admissible* if

1. $(x, xy) \in \lambda$ for every $x, y \in S$ provided $xy \neq 0$;
2. $(x, yx) \in \rho$ for every $x, y \in S$ provided $yx \neq 0$;
3. for every $x, y, z, t \in S$ if $(x, z) \in \rho$, $(y, t) \in \lambda$ and $xy = 0$ then $zt = 0$;
4. $xy \neq 0$ and $xz \neq 0$ implies $(y, z) \in \rho$;
5. $xy \neq 0$ and $zy \neq 0$ implies $(x, z) \in \lambda$.

The first three of these conditions are the same as above and the reason for the other two conditions is that a Brandt semigroup has an identity matrix: it has exactly one non-zero element in each column and in each row. We need to use the following new definition of the formula $\theta(\lambda, \rho)$:

$$\begin{aligned}\theta(\lambda, \rho) \rightleftharpoons \quad & \exists_{x \in S, x \neq 0} g(x) \\ & \bigwedge_{x,y \in S, xy \neq 0} g(x)g(y) = g(xy) \ \wedge \ \bigwedge_{x,y \in S, x^\lambda = y^\lambda, x^\rho = y^\rho, x \neq y} g(x) \neq g(y).\end{aligned}$$

We also need to change the definition of the sets L, R and the matrix Q.

$$L = S/\lambda \cup \{\alpha_x \mid xy = 0 \text{ for all } y \in S\},$$

$$R = S/\rho \cup \{\beta_x \mid yx = 0 \text{ for all } y \in S\},$$

$$q_{x^\rho y^\lambda} = \begin{cases} 1 & \text{if } xy \neq 0; \\ 0 & \text{if } xy = 0; \end{cases}$$

$$q_{\beta_x x} = 1 \text{ if } \beta_x \text{ exists}; \quad q_{\beta_x y} = 0 \text{ if } y \neq x;$$

$$q_{x\alpha_x} = 1 \text{ if } \alpha_x \text{ exists}; \quad q_{y\beta_x} = 0 \text{ if } y \neq x;$$

$$q_{\alpha_x \beta_y} = 0 \text{ for every } x, y.$$

In order to prove conditions 3, 5, 7, 9, 11 and 13, one has to almost literally repeat the above arguments replacing the old definitions by the new ones and we leave this to the reader as an exercise. This completes the proof of Theorem 1.3.

nilpotent semigroups.

A semigroup S with zero is *categorical at zero* if and only if for all $a, b, c \in S$, $abc = 0$ implies that $ab = 0$ or $bc = 0$.

Theorem 2.5 *A (finite) 3-nilpotent semigroup S is embeddable in a (finite) completely 0-simple semigroup if and only if S is categorical at zero.*

Proof. The condition is necessary since any completely 0-simple semigroup is categorical at zero.

Suppose S is categorical at zero. We must show that S embeds in a completely 0-simple semigroup. Let

$$A = \{a \in S;\ ax \neq 0 \text{ for some } x \in S\},$$

$$B = \{b \in S;\ yb \neq 0 \text{ for some } y \in S\}.$$

Then $A \cap B = \emptyset$; otherwise, since S is categorical at zero, $ybx \neq 0$ for some $b \in A \cap B$ and $x, y \in S$ which contradicts the requirement $S^3 = 0$. Now let

$$C = S^2 \setminus \{0\}, \qquad D = S \setminus \left(A \cup B \cup S^2\right).$$

It is clear that S is the disjoint union $A \cup B \cup C \cup D \cup \{0\}$. Notice that $D \cup \{0\}$ is a null semigroup and S is the 0-direct union of A$\cup B \cup C \cup \{0\}$ with $D \cup \{0\}$. Since a 0-direct union of completely 0-simple semigroups is completely 0-simple (see [**3**]) it suffices to embed $A \cup B \cup C \cup \{0\}$ in a completely 0-simple semigroup; of course $D \cup \{0\}$ embeds in a completely 0-simple semigroup.

Suppose $A = \{a_1, a_2, ..., a_h\}$, $B = \{b_1, b_2, ..., b_k\}$ and $C = \{c_1, c_2, ..., c_m\}$ for some ordinals h, k and m. Let G be any group containing at least m distinct elements $g_1, g_2, ..., g_m$ and with identity 1. Put $\Lambda = \{0, 1, ..., h\}$, $I = \{0, 1, ..., k\}$ and define a $\Lambda \times I$ matrix P over G^0 as follows: for all $i \in I \setminus \{0\}$ and $\lambda \in \Lambda \setminus \{0\}$,

$$p_{00} = p_{\lambda 0} = p_{0i} = 0, \qquad p_{\lambda i} = \begin{cases} 0 \text{ if } a_\lambda b_i = 0 \\ g_j \text{ if } a_\lambda b_i = c_j. \end{cases}$$

Let $M = M(G; I, \Lambda; P)$, a Rees matrix semigroup; of course M embeds in a regular Rees matrix semigroup (that is, in a completely 0-simple semigroup). So it suffices to embed A$\cup B \cup C \cup \{0\}$ in M.

Define an injective map $\phi : (A \cup B \cup C \cup \{0\}) \to M$ by

$$\phi(0) = 0, \phi(a_\lambda) = (0, 1, \lambda), \phi(b_i) = (i, 1, 0), \phi(c_j) = (0, g_j, 0).$$

Notice that

$$(0, p_{\lambda i}, 0) = \begin{cases} 0 \text{ if } a_\lambda b_i = 0 \\ (0, g_j, 0) \text{ if } a_\lambda b_i = c_j. \end{cases}$$

Hence $\phi(a_\lambda b_i) = (0, p_{\lambda i}, 0) = (0, 1, \lambda)(i, 1, 0) = \phi(a_\lambda)\phi(b_i)$. But $A^2 = B^2 = C^2 = BA = S^3 = 0$ while $AB = C$, so ϕ is an embedding.

Observe that in the above, if S is finite we may choose M and a completely 0-simple extension of M to be finite.

It is convenient to prove many of the following results for varieties of semigroups and to then derive the analagous pseudovariety results. A bold capital $\mathbf{V}$ will always denote a variety. A script capital $\mathcal{V}$ denotes a pseudovariety of semigroups.

Let us denote by $\mathcal{CS}^0, \mathcal{CS}, \mathcal{B}, \mathcal{SG}, \mathcal{LSG}, \mathcal{G}, \mathcal{S}\updownarrow, \mathcal{RZ}$ and $\mathcal{T}$, respectively the semigroup pseudovarieties generated by finite 0-simple semigroups, completely simple semigroups, Brandt semigroups, semilattices of groups, local semilattices of groups, groups, semilattices, right zero semigroups and trivial semigroups. Recall that a semigroup is a *semilattice of groups* if it is a union of groups and its idempotents commute. A semigroup S is a *local semilattice of groups* if eSe is a semilattice of groups for each idempotent e of S.

A variety of groups $\mathbf{H}$ is said to have *exponent* $n \geq 1$ if $x^n = 1$ is an identity for $\mathbf{H}$. If $\mathbf{H}$ has no finite exponent we say $\mathbf{H}$ has *exponent* 0. Likewise for pseudovarieties.

For $n \geq 1$ let $\mathbf{G}_n$ be the variety of all groups of exponent n and let $\mathcal{G}_n$ be the pseudovariety of all finite groups of exponent n. We write $\mathbf{G}_1 = \mathbf{T}$ and $\mathcal{G}_1 = \mathcal{T}$.

The aim now is to prove Theorem 1.5. We present two quite different proofs that $\mathcal{CS}^0(\mathcal{V})$ has decidable membership when the pseudovariety $\mathcal{V}$ has decidable membership. This is because from one of the proofs we obtain sequences of identities that ultimately define $\mathcal{CS}^0$ and $\mathcal{CS}^0(\mathcal{G}_n)$ (and $\mathcal{B}$ and $\mathcal{B}(\mathcal{G}_n)$) while from the other proof we get useful decompositions of $\mathcal{CS}^0(\mathcal{V})$.

In [**11**] G. Mashevitzky describes a basis of identities for the variety $\mathbf{CS}^0(\mathbf{H})$ generated by completely 0-simple semigroups over groups from the variety $\mathbf{H}$ of groups of exponent $n \geq 1$. Unfortunately there are gaps in the proof in [**11**]; we will fill the gaps here to obtain an equivalent result.

Lemma 3.1 *For $n \geq 1$ the identities*

$$x^{n+2} = x^2,\ (xy)^{n+1} x = xyx,\ xyx(zx)^n = x(zx)^n yx \tag{3}$$

and

$$x^{n+2} = x^2,\ (xy)^{n+1} x = xyx,\ xyz(xhz)^n = (xhz)^n xyz \tag{4}$$

determine the same variety of semigroups.

Proof. From (3) $z(xy)z((xh)z)^n = z((xh)z)^n (xy)z$. But then

$$xyz(xhz)^n x = x(yz)x(hzx)^n = x(hzx)^n (yz)x = (xhz)^n xyzx.$$

Now, by (3) and these identities

$$xyz(xhz)^n = xyz(xhz)^{2n} = (xhz)^n xyz(xhz)^n = (xhz)^{2n} xyz = (xhz)^n xyz.$$

Hence (4) is a consequence of (3).

$$
\begin{aligned}
xyx(zx) &= xyx(zx) = [x(yx)z][x(zx)z](xz) = x = [x(zx)z][x(yx)z](xz) = x \\
&= (xz)^{2n} xyx(zx)^n = x(zx)^{2n} yx(zx)^n = x(zx)^{2n} yx(zx)^{2n} \\
&= x\left[z\left((xz)^{2n-1} xy\right)x\right][z(xz)x]^n = x[z(xz)x]^n\left[z\left((xz)^{2n-1} xy\right)x\right] \\
&= x(zx)^{4n} yx = x(zx)^n yx.
\end{aligned}
$$

Lemma 3.2 . *Let* $\mathbf{V}$ *be the variety of semigroups determined by the identities (3). For any* $S \in \mathbf{V}$ *and distinct regular elements* $a, b \in S$ *there exists a completely 0-simple semigroup* K *and a surjective homomorphism* $\phi : S \to K$ *such that* $\phi(a) \neq \phi(b)$.

Proof. . For each regular element $z \in S$ let $I_z = \{u \in S; z \notin S^1 u S^1\}$; I_z is an ideal of S. Define equivalence relations ρ_z and λ_z on S by

$$
\begin{aligned}
\rho_z &= \{(x,y) \in S \times S;\ \text{ for all } t \in SzS,\ xt = yt \text{ (modulo } I_z)\}, \\
\lambda_z &= \{(x,y) \in S \times S;\ \text{ for all } t \in SzS,\ tx = ty \text{ (modulo } I_z)\}.
\end{aligned}
$$

Clearly ρ_z and λ_z are congruences on S; in fact, they are the kernels of the Schützenberger representations for S (see [**3**]).

Suppose $a \notin SbS$; since b is a regular element of S then $b \in I_a$ and since a has an inverse $a' \in S$ then $a'a \in SaS$, $a(a'a) = a \notin I_a$ and $b(a'a) \in I_a$, whence $(a,b) \notin \rho_a$. Alternatively suppose $a \in SbS$ and $b \in SaS$; that is, a and b are in the same $\mathcal{J}-$class $SaS \backslash I_a$ of S. Since a is regular and S is periodic by (3) then $K_a = SaS/(I_a \cap SaS)$ is a completely 0-simple semigroup. But then either $ba'a \neq a$ or $aa'b \neq a$ whence $(a,b) \notin \rho_a$ or $(a,b) \notin \lambda_a$ respectively.

It remains to prove that S/ρ_a and S/λ_a are completely 0-simple semigroups. We will check this for S/ρ_a where $a \neq 0$. By definition ρ_a contains the Rees congruence modulo I_a. Hence we may assume $I_a = \{0\}$ or I_a is empty. Then $SaS \cong K_a$ is a 0-minimal ideal of S and hence is a completely 0-simple semigroup. Therefore SaS/ρ_a is a completely 0-simple semigroup and in order to complete the proof we need only prove that for each $x \in S$ there exists $y \in SaS$ such that $(x,y) \in \rho_a$.

Now suppose $x \in S$; without loss of generality we may assume $x \notin SaS$. Since $x \notin I_a$ then $a = pxq$ for some $p, q \in S^1$ and therefore $a = pxqa'pxq$. Put $w = qa'p$, then $w \in SaS$ and $xwx \neq 0$. Let $t \in SaS$ and $t \neq 0$. Since SaS is completely 0-simple then $t = uxwxv$ for some $u, v \in SaS$. Now, applying the identities (3) we get

$$
xt = xuxwxv = xu(xw)^{n+1} xv = xux(wx)^n wxv = x(wx)^n uxwxv = (xw)^n xt.
$$

Thus $(x, (xw)^n x) \in \rho_a$. Since $(xw)^n x \in SaS$ the Lemma is proved.

We can now prove Mashevitzky's result.

Proposition 3.3 . *Let* $\mathbf{G}_n$ *be the variety of all groups of exponent* $n \geq 1$. *Then* $\mathbf{CS}^0(\mathbf{G}_n)$ *has (3) (or (4)) for a basis of identities.*

in a subword where the first and last letter are the same. An identity $u = v$ is covered by cycles if and only if u and v are covered by cycles. In the proof of [**11**; Lemma 7] it is shown that any identity $u = v$ of $\mathbf{CS}^0(\mathbf{G}_n)$ is a consequence of identities of $\mathbf{CS}^0(\mathbf{G}_n)$ that are covered by cycles. Hence $\mathbf{CS}^0(\mathbf{G}_n)$ is determined by identities covered by cycles.

Let $\mathbf{V}$ be the variety of semigroups determined by (3). It is easy to see that $\mathbf{V} \supseteq \mathbf{CS}^0(\mathbf{G}_n)$. By [**11**; Lemma 6] if $S \in \mathbf{V}$, $\phi : F(X) \to S$ is a homomorphism and $u \in F(X)$ is covered by cycles then $\phi(u)$ is regular in S.

Now suppose $u = v$ is an identity covered by cycles for $\mathbf{CS}^0(\mathbf{G}_n)$ and that there exists $S \in \mathbf{V}$ such that S does not satisfy the identity. So there is a homomorphism $\phi : F(X) \to S$ such that $\phi(u) \neq \phi(v)$, while $\phi(u)$ and $\phi(v)$ are regular elements of S. By Lemma 3.2 there is a surjective homomorphism of S onto a completely 0-simple semigroup that separates $\phi(u)$ and $\phi(v)$. But then $u = v$ is not an identity for $\mathbf{CS}^0(\mathbf{G}_n)$, which is a contradiction. Therefore $\mathbf{V} = \mathbf{CS}^0(\mathbf{G}_n)$.

Suppose $u = u(x_1, x_2, ..., x_m)$ is a word from the free semigroup $F(X)$ on a countably infinite set X, where $x_1, x_2, ..., x_n \in X$ are the variables that appear in u. Then define $u_n = u(x^n x_1 x^n, x^n x_2 x^n, ..., x^n x_m x^n)$ for some $x \in X \setminus \{x_1, x_2, ..., x_m\}$ and $n \geq 1$.

In a variety of groups $\mathbf{H}$ of exponent $n \geq 1$, identities of a basis of identities can be expressed in the form $v = 1$ where v is a word from $F(X)$. Mashevitzky, in [**11**], went on to prove the following extension of Proposition 3.3.

Theorem 3.4 *(Mashevitzky [**11**]). Let $\mathbf{H}$ be a variety of groups of exponent $n \geq 1$, with basis of identities $v_\gamma = 1$; $\gamma \in \Gamma$. Then $\mathbf{CS}^0(\mathbf{H})$ has a basis of identities*

$$\begin{gathered} x^{n+2} = x^2, x(yx)^{n+1} = xyx, xyx(zx)^n = x(zx)^n yx, \\ \left(v_\gamma^2\right)_n = (v_\gamma)_n ; \gamma \in \Gamma. \end{gathered} \tag{5}$$

Corollary 3.5 *Let $\mathbf{H}$ be a variety of groups of exponent $n \geq 1$ and let $\mathbf{V}$ be the largest subvariety of $\mathbf{CS}^0(\mathbf{G}_n)$ such that $\mathbf{V} \cap \mathbf{G} = \mathbf{H}$. Then $\mathbf{V} = \mathbf{CS}^0(\mathbf{H})$.*

Proof. Let $\mathbf{LSG}(\mathbf{H})$ be the variety of local semilattices of groups from $\mathbf{H}$ and assume $\mathbf{H}$ has a basis of identities $v_\gamma = 1$; $\gamma \in \Gamma$. Of course $\left(v_\gamma^2\right)_n = (v_\gamma)_n ; \gamma \in \Gamma$ are identities for $\mathbf{LSG}(\mathbf{H})$. Hence, by Theorem 3.4, $\mathbf{LSG}(\mathbf{H}) \cap \mathbf{CS}^0(\mathbf{G}_n) \subseteq \mathbf{CS}^0(\mathbf{H})$. But any completely 0-simple semigroup over a group from $\mathbf{H}$ is in $\mathbf{LSG}(\mathbf{H}) \cap \mathbf{CS}^0(\mathbf{G}_n)$ so $\mathbf{CS}^0(\mathbf{H}) = \mathbf{LSG}(\mathbf{H}) \cap \mathbf{CS}^0(\mathbf{G}_n)$. It follows that $\mathbf{CS}^0(\mathbf{H}) \cap \mathbf{G} \subseteq \mathbf{H}$ so $\mathbf{CS}^0(\mathbf{H}) \subseteq \mathbf{V} \subseteq \mathbf{LSG}(\mathbf{H}) \cap \mathbf{CS}^0(\mathbf{G}_n) = \mathbf{CS}^0(\mathbf{H})$. The Corollary is proved.

Let $F(X)$ denote the free semigroup on a countably infinite set X. A semigroup S *satisfies* the identity $u = v$ for some $u, v \in F(X)$ if $\phi(u) = \phi(v)$ in S for every homomorphism $\phi : F(X) \to S$. A pseudovariety $\mathcal{V}$ of semigroups is *equationally defined* by some set of identities if $\mathcal{V}$ consists precisely of those finite semigroups that satisfy the set of identities. There are many pseudovarieties that are not equationally defined. However, Eilenberg and Schützenberger, in [**4**], have shown that every pseudovariety $\mathcal{V}$ of semigroups is *ultimately defined* by identities in the sense that there is a sequence of

for any $u \in F(X)$. Then, for example, $\mathcal{G}_n$ is defined by $x^\omega y = y, yx^\omega = y$; that is, the sequence $\left(x^{n!}y = y, yx^{n!} = y\right)_{n\geq 1}$ ultimately defines $\mathcal{G}_n$.

A variety of algebras is called *locally finite* if and only if each of its finitely generated members is finite. A variety that is generated by a finite set of finite algebras is locally finite [**2**].

Theorem 3.6 *(i) $\mathcal{CS}^0$ is defined by*

$$x^{\omega+2} = x^2, (xy)^{\omega+1} x = xyx, xyx\,(zx)^\omega = x\,(zx)^\omega\, yx. \tag{6}$$

(ii) For any $n \geq 1$, $\mathcal{CS}^0(\mathcal{G}_n)$ is defined by

$$x^{n+2} = x^2, (xy)^{n+1} x = xyx, xyx\,(zx)^n = x\,(zx)^n\, yx. \tag{7}$$

Proof. (i) Let $\mathcal{V}$ be the pseudo-variety of semigroups that satisfy (6). It is easy to verify that any finite 0-simple semigroup $S \in \mathcal{CS}^0$ satisfies the pseudo-identities (6). Hence $\mathcal{V} \supseteq \mathcal{CS}^0$.

Now suppose $S \in \mathcal{V}$, so S is finite and satisfies (7) for some $n > 1$. Let $\mathbf{G}(S)$ be the variety of groups generated by the subgroups of S. Since $\mathbf{G}(S)$ is locally finite its free object of finite rank r has finite order, say $m(r)$. It follows from the universal property of the free object that any r-generated members of $\mathbf{G}(S)$ have order $\leq m(r)$. But then there is a natural number $m'(r) \leq r^2 m(r)$ such that any r-generated subsemigroup of a completely 0-simple semigroup over groups in $\mathbf{G}(S)$ has order $\leq m'(r)$; this is because any such semigroup embeds in an $r \times r$ completely 0-simple semigroup over a t-generated group from $\mathbf{G}(S)$, for some $t \leq r$. There are only finitely many non-isomorphic semigroups of orders $\leq m'(r)$, so any r-generated subdirect product of completely 0-simple semigroups, with its groups in $\mathbf{G}(S)$, is finite and is therefore in $\mathcal{CS}^0(\mathbf{G}(S) \cap \mathcal{G})$. Furthermore, since the rank r free object in $\mathbf{CS}^0(\mathbf{G}(S))$ is a homomorphic image of such a subdirect product, it is also in $\mathcal{CS}^0(\mathbf{G}(S) \cap \mathcal{G})$. It follows that the finite members, including S, of $\mathbf{CS}^0(\mathbf{G}(S))$ are in $\mathcal{CS}^0(\mathbf{G}(S) \cap \mathcal{G}) \subseteq \mathcal{CS}^0$, so $\mathcal{V} \subseteq \mathcal{CS}^0$ and the result follows.$\mathcal{C}$

(ii) Let $\mathcal{V}$ be the pseudo-variety of semigroups that satisfy (7). Clearly $\mathcal{V} \supseteq \mathcal{CS}^0(\mathcal{G}_n)$. By the same argument as used in (i), $\mathcal{V} \subseteq \mathcal{CS}^0(\mathcal{G}_n)$.

Corollary 3.7 *Let $\mathcal{H}$ be a pseudovariety of groups and let $\mathcal{V}$ be the largest subpseudovariety of $\mathcal{CS}^0$ such that $\mathcal{V} \cap \mathcal{G} = \mathcal{H}$. Then $\mathcal{V}$=$\mathcal{CS}^0(\mathcal{H})$.*

Proof. Let $\mathcal{SR}(\mathcal{H})$ be the pseudovariety of locally semilattices of groups from $\mathcal{H}$. Then $\mathcal{V} = \mathcal{SR}(\mathcal{H}) \cap \mathcal{CS}^0 \supseteq \mathcal{CS}^0(\mathcal{H})$. By the argument of the second paragraph of the last proof, $S \in \mathcal{V}$ only if $S \in \mathcal{CS}^0(\mathbf{G}(S) \cap \mathcal{G})$. But the finite members of $\mathbf{G}(S)$ are homomorphic images of finitely generated subdirect products of groups from $\mathcal{H}$ so $\mathbf{G}(S) \cap \mathcal{G} \subseteq \mathcal{H}$. Thus $\mathcal{V} \subseteq \mathcal{CS}^0(\mathcal{H})$. The Corollary is proved.

Corollary 3.8 *Membership of the pseudovariety $\mathcal{CS}^0(\mathcal{H})$ is decidable for any pseudovariety of groups $\mathcal{H}$ that has decidable membership.*

There is a similar result for the pseudovariety $\mathcal{B}(\mathcal{H})$.

Proposition 3.9 *Let* $\mathbf{H}$ *be a variety of groups of exponent* $n \geq 1$ *with basis of identities* $v_\gamma = 1; \gamma \in \Gamma$. *Then the variety* $\mathbf{B}(\mathbf{H})$ *generated by Brandt semigroups over groups from* $\mathbf{H}$ *has a basis of identities*

$$x^{n+2} = x^2, (xy)^{n+1} x = xyx, x^n y^n = y^n x^n, \left(v_\gamma^2\right)_n = (v_\gamma)_n ; \gamma \in \Gamma.$$

Proof. By M. V. Volkov (see [**13**; Theorem 20.7]) and A. N. Trakhtman (see [**13**; Theorem 20.4]), for $n > 1$ and $n = 1$ respectively, the first three of the identities determine $\mathbf{B}(\mathbf{G}_n)$. The proof of Theorem 3.4 that is used by Mashevitzky in [**11**] also proves this Proposition.

Theorem 3.10 *(i)* $\mathcal{B}$ *is defined by*

$$x^{\omega+2} = x^2, (xy)^{\omega+1} x = xyx, x^\omega y^\omega = y^\omega x^\omega. \tag{8}$$

(ii) For any $n \geq 1$, $\mathcal{B}(\mathcal{G}_n)$ *is defined by*

$$x^{n+2} = x^2, (xy)^{n+1} x = xyx, x^n y^n = y^n x^n. \tag{9}$$

Proof. This theorem can be proved by the proof of Theorem 3.6(i) and (ii) with suitable (minor) modifications.

The argument used to prove Corollary 3.5 also proves the next result.

Corollary 3.11 *Let* $\mathcal{H}$ *be a pseudovariety of groups and let* $\mathcal{V}$ *be the largest subpseudovariety of* $\mathcal{B}$ *such that* $\mathcal{V} \cap \mathcal{G} = \mathcal{H}$. *Then* $\mathcal{V} = \mathcal{B}(\mathcal{H})$.

Corollary 3.12 *Membership of the pseudovariety* $\mathcal{B}(\mathcal{H})$ *is decidable for any pseudovariety of groups* $\mathcal{H}$ *that has decidable membership.*

The decidability part of Theorem 1.5 is a combination of Corollaries 3.8 and 3.12 and is therefore proved. The complexity part of this theorem follows from the easy facts that one can check if a finite semigroup unltimately satisfies identities (5) or (8) in polynomial time and that the problem of finding maximal subgroups of a finite semigroup given by its multiplication table is solvable in polynomial time.

Let us now consider decompositions of $\mathbf{CS}^0(\mathbf{H})$ for a variety of groups $\mathbf{H}$ and the analogous pseudovariety decompositions via joins and semidirect products. $\mathbf{CS}(\mathbf{H})$ is the variety generated by completely simple semigroups over groups from $\mathbf{H}$.

Lemma 3.13 $\mathbf{CS}^0(\mathbf{H}) = \mathbf{CS}^0(\mathbf{T}) \vee \mathbf{CS}(\mathbf{H})$ *for any group variety* $\mathbf{H}$. *For any pseudovariety* $\mathcal{H}$ *of groups,* $\mathcal{CS}^0(\mathcal{H}) = \mathcal{CS}^0(\mathcal{T}) \vee \mathcal{CS}(\mathcal{H})$.

zero terms by 1 (the identity of G). Obtain $\hat{P}$ from P by replacing all zero terms by 1. Put $R = M^0\left(\{1\}; I, \Lambda; \bar{P}\right)$, $T = M\left(G; I, \Lambda; \hat{P}\right)$ and $I = \{(0,t) \in R \times T\}$. Then $(R \times T)/I$ is a semigroup in $\mathbf{CS}^0(\mathbf{T}) \vee \mathbf{CS}(\mathbf{H})$ that embeds S under the assignment $(i,a,\lambda) \longmapsto ((i,1,\lambda),(i,a,\lambda))$ modulo I. Thus $S \in \mathbf{CS}^0(\mathbf{T}) \vee \mathbf{CS}(\mathbf{H})$ and it follows that $\mathbf{CS}^0(\mathbf{H}) \subseteq \mathbf{CS}^0(\mathbf{T}) \vee \mathbf{CS}(\mathbf{H})$. The same proof can be used for the corresponding pseudovariety result.

There is a similar result for $\mathbf{B}(\mathbf{H})$.

Lemma 3.14 $\mathbf{B}(\mathbf{H}) = \mathbf{B}(\mathbf{T}) \vee \mathbf{H}$ *for any group variety* $\mathbf{H}$. *For any pseudovariety* $\mathcal{H}$ *of groups,* $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{T}) \vee \mathcal{H}$.

Proof. We have $\mathbf{B}(\mathbf{H}) \supseteq \mathbf{B}(\mathbf{T}) \vee \mathbf{H}$. Let $S = M^0(G, I, I; \Delta)$ be a Brandt semigroup in $\mathbf{B}(\mathbf{H})$, $R = M^0(\{1\}; I, I; \Delta)$ and $J = \{(0,g) \in R \times G\}$. Then S embeds in the semigroup $(R \times G)/J$ by the assignment $(i,g,\lambda) \longmapsto ((i,1,\lambda),g)$ modulo J. So $\mathbf{B}(\mathbf{H}) \subseteq \mathbf{B}(\mathbf{T}) \vee \mathbf{H}$. Similarly, the pseudovariety result holds.

In the following results, use is made of semidirect and wreath products of semigroups.

Let R and S be semigroups. Suppose that each element $s \in S$ acts by endomorphism on R, $r \longmapsto {}^s r$ such that ${}^s(r_1 r_2) = {}^s r_1\, {}^s r_2$, ${}^s({}^t r) = {}^{(st)} r$, and if S is a monoid then ${}^1 r = r$, for any $r, r_1, r_2 \in R$ and $s, t \in S$. The *semidirect product* $R * S$ is the semigroup consisting of the set $R \times S$ with binary operation given by $(r_1, s_1)(r_2, s_2) = (r_1^{s_1} r_2, s_1 s_2)$ for all $r_1, r_2 \in R$, $s_1, s_2 \in S$. The *wreath product* $R\, wr\, S$ is the semidirect product $R^S * S$ of the cartesian power R^S by S, with the action such that for any $s, t \in S$ and $\alpha \in R^S$ then ${}^t\alpha(s) = \alpha(st)$. Recall that $\alpha \in R^S$ means that $\alpha : S \to R$ is a map and if, as well, $\beta \in R^S$ and $s \in S$ then $(\alpha\beta)(s) = \alpha(s)\beta(s)$.

For varieties of semigroups $\mathbf{U}$ and $\mathbf{V}$ define $\mathbf{U} * \mathbf{V}$ to be the variety generated by $\{U * V;\ U \in \mathbf{U} \text{ and } V \in \mathbf{V}\}$. Of course $U\, wr\, V \in \mathbf{U} * \mathbf{V}$; in fact $\mathbf{U} * \mathbf{V}$ is generated by $U\, wr\, V$ for all $U \in \mathbf{U}$, $V \in \mathbf{V}$ (see [**1**]). We also use the analogous notions for pseudovarieties.

Theorem 3.15 $\mathbf{CS}^0(\mathbf{H}) = \mathbf{SG}(\mathbf{H}) * \mathbf{RZ}$ *for any variety of groups* $\mathbf{H}$. *For any pseudovariety* $\mathcal{H}$ *of groups,* $\mathcal{CS}^0(\mathcal{H}) = \mathcal{SG}(\mathcal{H}) * \mathcal{RZ}$

Proof. Notice that $\mathbf{SG}(\mathbf{H}) = \mathbf{Sl} \vee \mathbf{H}$. Suppose $E \in \mathbf{Sl}$, $G \in \mathbf{H}$ and $R \in \mathbf{RZ}$. There is a bijection

$$(E \times G)^R \to E^R \times G^R;\ \alpha \longmapsto (\alpha_E, \alpha_G) : \alpha(r) = (\alpha_E(r), \alpha_G(r)) \in E \times G \qquad \forall r \in R.$$

Define maps

$$\tau : (E \times G)\ wr\, R \to G\, wr\, R;\ (\alpha, r)\,\tau = (\alpha_G, r)\,.$$

Observe that for $\alpha, \beta \in (E \times G)^R$ and $r, t \in R$ we have $rt = t$ and

$$(\alpha^t \beta)(r) = \alpha(r)^t \beta(r) = \alpha(r)\beta(t) = (\alpha_E(r), \alpha_G(r))(\beta_E(t), \beta_G(t)) = (\alpha_E(r)\beta_E(t), \alpha_G(r)\beta_G(t)) = ((\alpha_E{}^t\beta_E)(r), (\alpha_G{}^t\beta_G)(r))\,.$$

It follows easily that π and τ are homomorphisms. As well, for distinct $(\alpha, r), (\beta, t) \in (E \times G)\ wr\, R$, either $\pi(\alpha, r) \neq \pi(\beta, t)$ or $\tau(\alpha, r) \neq \tau(\beta, t)$. Hence $(E \times G)\ wr\, R$ is a subdirect product of $E\, wr\, R$ by $G\, wr\, R$. We have shown that $(E \times G)\ wr\, R \in (\mathbf{Sl} * \mathbf{RZ}) \vee (\mathbf{H} * \mathbf{RZ})$; that is, $\mathbf{SG}(\mathbf{H}) * \mathbf{RZ} = (\mathbf{Sl} \vee \mathbf{H}) * \mathbf{RZ} \subseteq (\mathbf{Sl} * \mathbf{RZ}) \vee (\mathbf{H} * \mathbf{RZ})$. Of course, the reverse inclusion is immediate so $\mathbf{SG}(\mathbf{H}) * \mathbf{RZ} = (\mathbf{Sl} * \mathbf{RZ}) \vee (\mathbf{H} * \mathbf{RZ})$.

Let C_2 be the 5 element 0-simple semigroup over the trivial group, with sandwich matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

By [**13**; Theorem 20.4] the variety $\mathbf{V}(C_2)$ has (3) with $n = 1$ for a basis of identities; as well by Proposition 3.3 these form a basis of identities for $\mathbf{CS}^0(\mathbf{T})$ so $\mathbf{V}(C_2) = \mathbf{CS}^0(\mathbf{T})$. It is easy to check that $\mathbf{Sl}*\mathbf{RZ}$ satisfies the identities (3), with $n = 1$, so $\mathbf{CS}^0(\mathbf{T}) \supseteq \mathbf{Sl} * \mathbf{RZ}$. By Theorem 3.6, $\mathcal{CS}^0(\mathcal{T})$ has (3) with $n = 1$ for a basis of pseudo-identities, while by [**1**; Corollary 10.8.3], $\mathcal{S}\updownarrow * \mathcal{RZ}$ also has this basis whence $\mathcal{CS}^0(\mathcal{T}) = \mathcal{S}\updownarrow * \mathcal{RZ}$ (in [**1**], $\mathcal{S}\updownarrow$ is called Com_{11} and $\mathcal{RZ}$ is called D_1). Since $C_2 \in \mathcal{CS}^0(\mathcal{T}) \subseteq \mathbf{Sl} * \mathbf{RZ}$ then $\mathbf{CS}^0(\mathbf{T}) \subseteq \mathbf{Sl} * \mathbf{RZ}$; that is, $\mathbf{CS}^0(\mathbf{T}) = \mathbf{Sl} * \mathbf{RZ}$. Since by [**7**], $\mathbf{H}*\mathbf{RZ} = \mathbf{CS}(\mathbf{H})$ then by Lemma 3.13 $\mathbf{CS}^0(\mathbf{H}) = \mathbf{SG}(\mathbf{H}) * \mathbf{RZ}$.

In the corresponding proof for pseudovarieties, note that by [**1**; Corollary 10.6.8], $\mathcal{CS}(\mathcal{H}) = \mathcal{H} * \mathcal{RZ}$. We get $\mathcal{CS}^0(\mathcal{H}) = \mathcal{SG}(\mathcal{H}) * \mathcal{RZ}$.

Remark 3.16 *The pseudovariety version of Theorem 3.15 allows us to prove Corollary 3.8 by a very different method to that we used previously. Since we already have the Corollary we present below only an outline of the alternative proof. The outline should, however, indicate the potential of this alternative proof technique.*

Alternative proof for Corollary 3.8.

A semigroupoid $\mathcal{C}$ is defined in the same way a category is, except that each hom set, $\hom(a, a)$, from an object a to itself is a semigroup rather than a monoid; $\hom(a, a)$ is the *local semigroup* at a. Each semigroup is the hom set of some one object semigroupoid; we can therefore think of semigroups as being one object semigroupoids. A pseudovariety of semigroups $\mathcal{V}$ is *local* if every semigroupoid $\mathcal{C}$ whose local semigroups are in $\mathcal{V}$ *divides* a member of $\mathcal{V}$; that is, if for each $\mathcal{C}$ there is a semigroupoid $\mathcal{D}$, $V \in \mathcal{V}$ and functors $\phi : \mathcal{D} \to \mathcal{C}$, $\psi : \mathcal{D} \to V$ such that ϕ is bijective on objects and surjective on hom sets, while ψ is injective on hom sets.

We first check that $\mathcal{SG}(\mathcal{H})$ is local. Let $\mathcal{M}$ be the pseudovariety of all monoids. By [**6:** Corollary 8.2], $\mathcal{SG}(\mathcal{H}) \cap \mathcal{M}$ is local. Any semigroupoid $\mathcal{D}$ extends to a category $\mathcal{C}^1$

divides $\mathcal{SG}(\mathcal{H}) \cap \mathcal{M}$. It follows that $\mathcal{C}$ divides $\mathcal{SG}(\mathcal{H})$, whence $\mathcal{SG}(\mathcal{H})$ is local.

By [**15**; Theorem 8.2 and Appendix B] the membership of a semigroup S in a pseudovariety $\mathcal{V} * \mathcal{W}$ is decidable if $\mathcal{V}$ is local and if there is a relational morphism ϕ of S to $T \in \mathcal{W}$ whose derived semigroupoid is locally in $\mathcal{V}$. In particular, if $\mathcal{V}$ is local, if membership of $\mathcal{V}$ is decidable and if there are only finitely many non-equivalent relational morphisms of S into $\mathcal{W}$, then membership of $\mathcal{V} * \mathcal{W}$ is decidable. These conditions are fulfilled when $\mathcal{V} = \mathcal{SG}(\mathcal{H})$ and $\mathcal{W} = \mathcal{RZ}$; by the last paragraph $\mathcal{SG}(\mathcal{H})$ is local, while for any locally finite pseudovariety $\mathcal{W}$ there is a finite free object F of rank $|S|$ in $\mathcal{W}$ and any relational morphism of S into $\mathcal{W}$ factors through one of the finite number of relational morphisms of S to F.

As an immediate consequence of Theorems 3.6 and 3.10 and Corollary 3.7 and 3.11 we get the following.

Theorem 3.17 *Let $\mathcal{H}$ be a pseudovariety of groups. Then $\mathcal{B}(\mathcal{H})$ consists of those semigroups from $\mathcal{CS}^0(\mathcal{H})$ that have commuting idempotents.*

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