Continuum varieties of groups and verbal embeddings of groups

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To Professor Alexander Yurievich Ol'shanskii on his 60'th birthday

1. VERBAL EMBEDDINGS AND A CONSTRUCTION OF OL'SHANSKII

The aim of this talk is to give an example of an application of the method built by Professor Ol'shanskii in the well known paper

A. Yu. Ol'shanskii, *On the problem of finite base of identities in groups*, Izv. AN SSSR, ser. matem. 34 (1970), 376–384.

(in which the continual cardinality of the set of the varieties of groups was proved) in combination with the methods of verbal embeddings of groups.

The embedding $\varphi : H \to G$ of the group H into the group G is V*-verbal* for the given word set $V \subseteq F_\infty$ if the isomorphic copy $\varphi(H)$ lies in the verbal subgroup $V(G)$. If there is no misunderstanding with V, we will just term "verbal embedding φ". The verbal embedding constructions can have various properties, such as:

1. the embedding can be normal or subnormal, which means that the isomorphic image $\varphi(H)$ of H is normal or subnormal in G;

2. the embedding can preserve the properties of H, which means that G is a nilpotent, soluble, ordered, finitely generated, etc. group if the group H has that propert(ies).

3. also, the embedding can "depend on the identities" of the group H, which means that if the groups H_1 and H_2 do not have the

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same identities, that is, if $\mathrm{var}\,(H_1) \neq \mathrm{var}\,(H_2)$, then the corresponding overgoups G_1 and G_2 also do not have the same identities: $\mathrm{var}\,(G_1) \neq \mathrm{var}\,(G_2)$.

Roughly, here is the main point of how we are going to use Ol'shanskii's idea with verbal embeddings. In the mentioned paper he constructed a countably infinite set of groups $\Theta = \{S_1, S_2, \dots, S_n, \dots\}$ with the following properties:

1. $\Theta \subseteq \mathfrak{W} = \mathfrak{G}_5 \cap \mathfrak{B}_{24p}$, where $p > 3$ is a prime number, $\mathfrak{G}_l$ is the variety of soluble groups of length at most l, and $\mathfrak{B}_e$ is the Burnside variety of groups of exponents dividing e;
2. $S_i \notin \mathrm{var}\,(\Theta\backslash\{S_i\})$ for any $i = 1, 2, \dots$.

The first condition guarantees, that the variety generated by any subset of Θ still is inside the variety $\mathfrak{W}$, and the second condition guarantees, that the varieties generated by any two distinct subsets Θ_1 and Θ_2 of Θ are distinct:

$$\text{if } \Theta_1 \neq \Theta_2 \text{ then } \mathrm{var}\,(\Theta_1) \neq \mathrm{var}\,(\Theta_2)$$

Thus, to get continuum varieties of groups one only needs to take the varieties generated by all subsets of Θ.

We are using this idea with verbal embeddings in the following way: we take an infinite subset $\Theta_1 \subseteq \Theta$ such that the set $\Theta_1' = \Theta\backslash\Theta_1$ aso is infinite, and build a V-verbal embedding construction for:

$$H = \prod_{S_i \in \Theta_1} S_i$$

and for V is the word set corresponding to the variety:

$$\mathrm{var}\,(S_i | S_i \in \Theta_1')\,.$$

2. THREE THEOREMS

The first result is that we show that soluble finitely generated non-Hopfian groups are 'many'.

THEOREM 1. *There exists a continuum of* 3*-generator soluble non-Hopfian groups which generate pairwise distinct varieties of groups. The solubility length of these groups is bounded by* 9.

Let us recall that a group G is a *Hopfian group* if every ependomorphism on G is an automorphism, that is, if G is not isomorphic to its proper factor group. Every finitely generated residually finite group (in particular, every free group of finite rank or every free polynilpotent group of finite rank) is a Hopfian group. On the other hand it is easy to see that the free group of infinite rank or the free abelien group of infinite rank are non-Hopfian groups. This contrast makes the finitely generated non-Hopfian groups important (in fact, the original problem of Heinz Hopf was posed by him in 1930's on existence of such groups.

Further:

THEOREM 2. *Each countable group N is embeddable into a* 3*-generator non-Hopfian group K. Moreover, this embedding can be subnormal, and if N is a soluble group of length l, then K is a soluble group of length $l+4$.*

Another property that we can add to our embedding is *verbality*:

THEOREM 3. *For any non-trivial word set $V \subseteq F_\infty$ each countable group N is embeddable into a* 3*-generator non-Hopfian group K, and this embedding is V-verbal. Moreover, this embedding can be subnormal, and if N is a soluble group of length l, then K is a soluble group of length $l + c + 4$, where c is the smallest integer such that the variety $\mathfrak{N}_c$ of nilpotent groups of class c is not contained in the variety $\mathfrak{V}$ corresponding to V.*

Actually all three theorems are results of one large embedding construction built to proof Theorem 3.

3. SOME OF THE PROOF STEPS

The first step is verbal embedding of countable groups:

LEMMA 1. *Let H be any countable group, and V be any non-trivial word set. Then there is a group $R(H,V)$ and embedding $\Psi : H \to R(H,V)$ with the following properties:*

i. *if H generates the variety $\mathfrak{U}$, then $R(H,V)$ generates the variety $\mathfrak{U}\mathfrak{N}_c\mathfrak{A}$;*
ii. *in particular, if* $\mathrm{var}(H_1) \neq \mathrm{var}(H_2)$ *for the groups H_1 and H_2, then also* $\mathrm{var}(R(H_1,V)) \neq \mathrm{var}(R(H_2,V))$;
iii. *the embedding Ψ is subnormal: $\Psi(H) \lhd\lhd R(H,V)$;*
iv. *the embedding Ψ is V_1-verbal: $\Psi(H) \subseteq V_1(R(H,V))$, where V_1 is the word set corresponding to the product variety $\mathfrak{V}_1 = \mathfrak{V}\mathfrak{A}$.*

The second step is the embedding into a 2-generator group:

LEMMA 2. *Let R be a countable group not generating the variety of all groups. Then there is a group $K(R)$ and an isomorphic embedding $\Delta : R \to K(R)$ with the following properties:*

i. *if R generates the variety $\mathfrak{W}$, then $K(R)$ generates the variety $\mathfrak{W}\mathfrak{A}$;*
ii. *in particular, if* $\mathrm{var}(R_1) \neq \mathrm{var}(R_2)$ *for the groups R_1 and R_2, then also* $\mathrm{var}(K(R_1)) \neq \mathrm{var}(K(R_2))$;
iii. *the embedding Δ is subnormal: $\Delta(R) \lhd\lhd K(R)$.*

The third step is embedding of each 2-generator group into a 3-generator non-Hopfian group:

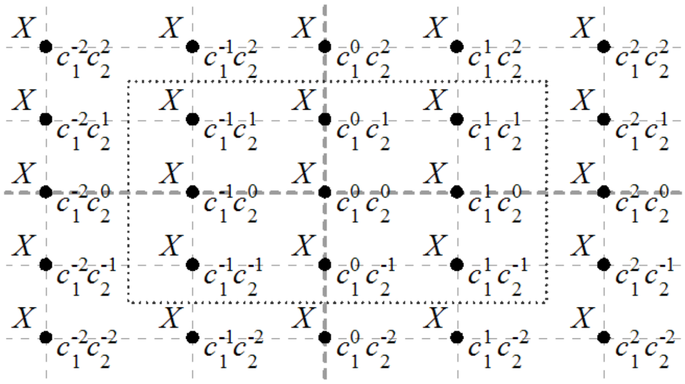
LEMMA 3. *For every 2-generator group K there exists a 3-generator group $G(K)$, and an isomorphic embedding $\Phi : K \to G(K)$ with the following properties:*

i. *$G(K)$ is a non-Hopfian group;*
ii. *if* $\mathrm{var}\,(K_1) \neq \mathrm{var}\,(K_2)$, *then also* $\mathrm{var}\,(G(K_1)) \neq \mathrm{var}\,(G(K_2))$;
iii. *the embedding Φ is subnormal: $\Phi(K) \lhd\lhd G(K)$.*

The proofs are long, and we refer to the article for details. Here we restrict us with consideration of a new notion of *N'-similarity of the elements of nested (Cartesian) wreath products* of type

$$(X \operatorname{Wr} Y) \operatorname{Wr} Z$$

introduced in this work for technical reasons. When the "active group" Y of a wreath product $X \operatorname{Wr} Y$ is a direct product of finitely many copies of the free abelian group, then the base subgroup of that wreath product has some "geometrical" meaning. For example if $Y = \langle c_1\rangle\langle c_2\rangle$ is 2-generator, then the base subgroup X^Y is the Cartesian product of the copies of X indexed by elements of Y. It can be understood as copies of X standing on a "net" build by elements of Y (Picture 1):



Definition. *Let X be any group and Y, Z be any finite direct powers of the infinite cycle C:*

$$Y = \langle c_1\rangle\langle c_2\rangle \cdots \langle c_u\rangle \text{ and } Z = \langle c_1\rangle\langle c_2\rangle \cdots \langle c_v\rangle, \text{ with } \langle c_i\rangle \cong \langle c\rangle$$

for all $i = 1, \dots, \max\{u, v\}$. Then:

1. *For the given positive integer N the elements $y_1\psi_1$ and $y_2\psi_2$ of the Cartesian wreath product $X \operatorname{Wr} Y$ (with $y_1, y_2 \in Y$ and $\psi_1, \psi_2 \in X^Y$) are called N-similar in $X \operatorname{Wr} Y$ if $y_1 = y_2$ and if $\psi_1(y) = \psi_2(y)$ for any $y = c_1^{l_1} \cdots c_u^{l_u} \in Y$ such that $|l_i| \le N, \quad 1 = 1, \dots, u$.*

2. *For the given positive integer N the elements $z_1\varphi_1$ and $z_2\varphi_2$ of the Cartesian wreath product $(X \operatorname{Wr} Y) \operatorname{Wr} Z$ (with $z_1, z_2 \in Z$ and $\varphi_1, \varphi_2 \in (X \operatorname{Wr} Y)^Z$) are called N'-similar in $(X \operatorname{Wr} Y) \operatorname{Wr} Z$ if $z_1 = z_2$ and if $\varphi_1(y)$ is N-similar to $\varphi_2(y)$ for any $z = c_1^{l_1} \cdots c_v^{l_v} \in Z$ such that $|l_i| \le N, \quad 1 = 1, \dots, v$.*