Polynomial maps over fields and residually finite groups

Mark Sapir

The talk is based on the following three papers:

Alexander Borisov, Mark Sapir, Polynomial maps over finite fields and residual finiteness of mapping tori of group endomorphisms. Invent. Math. 160 (2005), no. 2, 341–356.

Alexander Borisov, Mark Sapir, Polynomial maps over p-adics and residual properties of mapping tori of group endomorphisms, preprint, arXiv, math0810.0443, 2008.

Iva Kozáková, Mark Sapir, Almost all one-relator groups with at least three generators are residually finite. preprint, arXiv math0809.4693, 2008.

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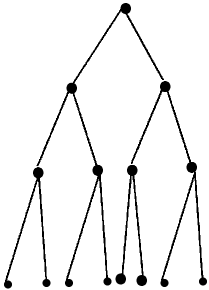
Examples. $\mathbb{Z}$, F_k, linear groups are residually finite. $\mathbb{Q}$, infinite simple groups, free Burnside groups of sufficiently large exponents are not residually finite. Groups acting faithfully on rooted locally

finite trees are residually finite.

Rooted trees



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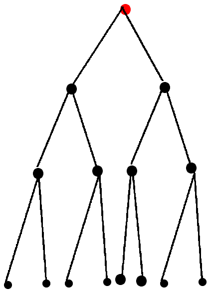

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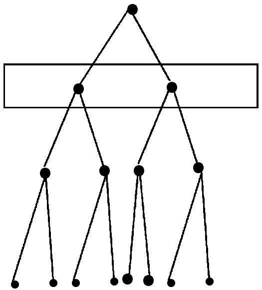

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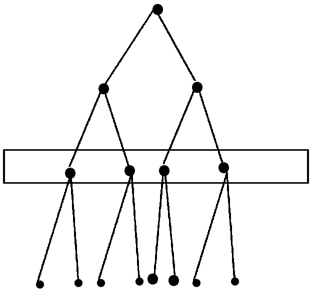

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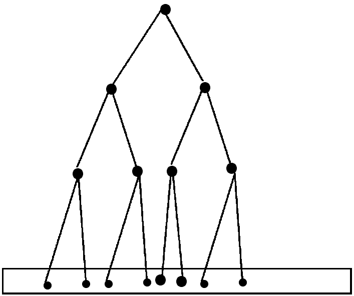

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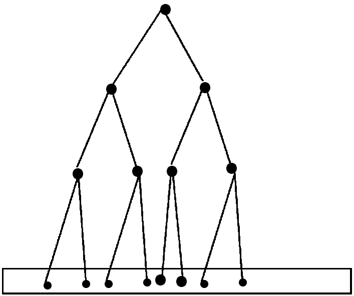

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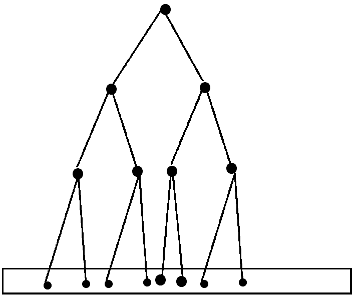

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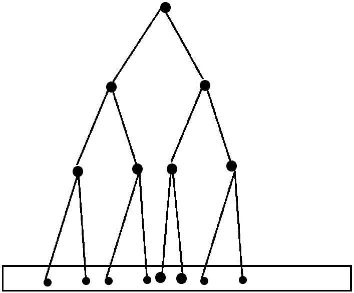

Conversely every finitely generated residually finite group acts faithfully on a locally finite rooted tree.

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Example 2. $BS(1,2)$ $\langle a, t \mid tat^{-1} = a^2\rangle$ is metabelian, and linear, so it is residually finite.

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Problem. (Moldavanskii, Kapovich, Wise) Are ascending HNN extensions of free groups residually finite?
These three problems are related.

Hyperbolic groups, 1-related groups, and mapping tori of free groups

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Example (Magnus procedure). Consider the group $\langle a, b \mid aba^{-1} \cdot b^{-1} \cdot aba^{-1} \cdot b^{-1} \cdot a^{-1}b^{-1}a = 1\rangle$. Replace $a^i ba^{-i}$ by b_i. The index i is called *the Magnus a-index* of that letter.

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$\langle a, b_{-1}, b_0, b_1 \mid b_1 b_0^{-1} b_1 b_0^{-1} b_{-1}^{-1} = 1, a^{-1} b_0 a = b_{-1}, a^{-1} b_1 a = b_0 \rangle$.
So we have a new presentation of the same group.

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Note that b_{-1} appears only once in $b_1 b_0^{-1} b_1 b_0^{-1} b_{-1}^{-1} = 1$.

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So we can replace b_{-1} by $b_1 b_0^{-1} b_1 b_0^{-1}$, remove this generator, and get a new presentation of the same group.

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- Coherent (that is all finitely generated subgroups are finitely presented).

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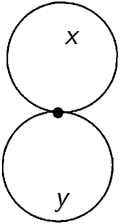

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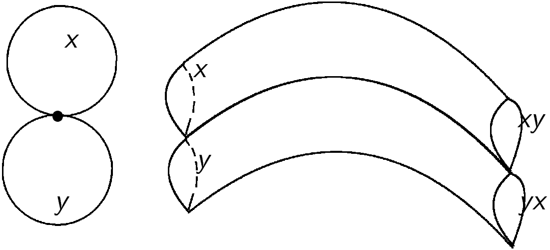

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- (Wise-S.) An ascending HNNextension of a residually finite group can be non-residually finite (example - Grigorcuk's group and its Lysenok extension).

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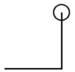

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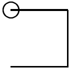

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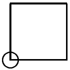

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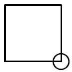

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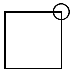

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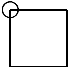

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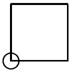

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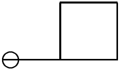

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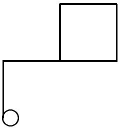

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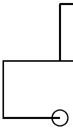

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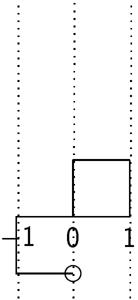

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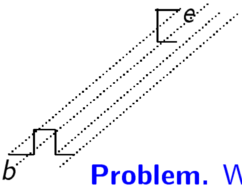

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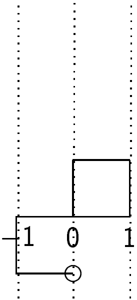

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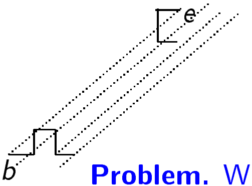

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Dunfield and Thurston proved recently that this probability is strictly between 0 and 1.

Ken Brown's results

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- If $k = 2$ and one of the two support lines of w that is parallel to $\vec{OM}$ intersects w in a single vertex or a single edge, then G is an ascending HNN extension of a free group.
- If $k > 2$ then G is never an ascending HNN extension of a free group.

Some small cancelation theory, embedding into 2-generated groups

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That is Brownian motion is a continuous Markov stationary process with normally distributed increments.

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We are using Rivin's Central Limit Theorem for cyclically reduced walks.

Convex hull of Brownian motion and maximal Magnus indices

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Convex hulls and maximal indices, continued

Step 1. We prove that the number of vertices of Δ is growing (a.s.) with the length of w (here it is used that $k \geq 3$).

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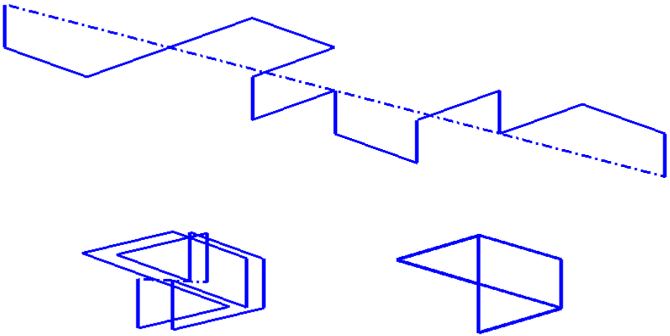
Step 2. For every vertex of Δ for any 'bad" walk w' or length r we construct (in a bijective manner) a "good" walk w' of length $r + 4$. This implies that the number of vertices of "bad" walks is bounded if the probability of a "bad" walk is > 0.

Illustration of Step 2

Here is the walk in $\mathbb{Z}^3$ corresponding to the word

$$cb^{-1}acac^{-1}b^{-1}caca^{-1}b^{-1}aab^{-1}c.$$

and its projection onto $\mathbb{R}^2$



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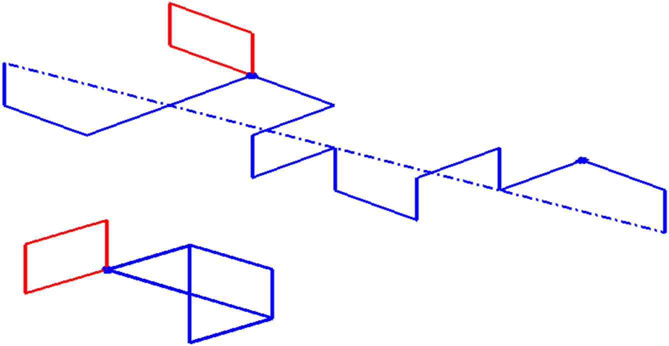

Algebraic geometry

Consider the following example

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on the "space" $V \times V$.

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Key observation. The converse statement is also true (the number of generators and the choice of ϕ do not matter).

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Example continued

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Thus the point (A, B) is periodic in $SL_2(\mathbb{Z}/5\mathbb{Z})$ with period 6.

Example continued. Dynamics of polynomial maps over local fields

Replace 5 by 25, 125, etc. It turned out that (A, B) is periodic in $SL_2(\mathbb{Z}/25\mathbb{Z})$ with period 30, in $\mathrm{SL}_2(\mathbb{Z}/125\mathbb{Z})$ with period 150, etc.

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The general idea

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In fact one is not able to find these matrices in $SL_2(\mathbb{Z})$. One needs a bigger ring.

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Theorem (Borisov-S.) Suppose Φ is an $n-$variable polynomial map with integer coefficients, and V is the Zariski closure of the image of Φ^n (over $\mathbb{Z}$). Suppose W is a proper subscheme of V. Then for every sufficiently large prime p there exist $q = p^l$ and a point $x \in V(\mathbb{F}_q) \setminus W(\mathbb{F}_q)$ such that for every $X \in V(\mathbb{Z}_q)$ with $\pi(X) = x$ we have

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In particular, the point X is uniformly recurrent for Φ in the $p-$adic topology on $V(\mathbb{Z}_q)$.

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Part 2: $n > 1$. We need to lift the periodic point mod p to a uniformly recurrent point over p-adics.

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We have a polynomial map P with integer coefficients on $V(\mathbb{F}_p^{alg})$.
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Conjecture: No.